INEQUALITY, WELFARE AND ORDER STATISTICS

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5th ECINEQ Meeting, Bari. July 2013

Introduction

- Inequality measures ⇔ SWF (Yaari approach (1987, 1988)

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  Preference functions → Normative criteria; Inequality aversion

- Order statistics ⇒ Distribution F. ⇒ Preference F. ⇒ SWF ⇒ Inequality Measures → Diverse normative criteria; Different response to progressive transfers

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  Generalized Gini
  Aaberge (2000)
  Other measures (intermediate positions wrt aversion)

Introduction

- Use of order statistics is rare:
  - Generalized Gini ⇨ first-order statistics (Lambert 2001) ⇨ mean values of these statistics characterizes any income distribution with finite mean (Kleiber y Kotz 2002)
  - Our proposal extends and generalizes this analysis
- Advantages of the use of order statistics:
  1. Alternative characterization of distributions (empirical)
  2. Provides diverse distributive criteria in assessing welfare and inequality
  3. Clear interpretation of each measure in terms of the statistics computed from a random sample drawn from the population. Identification of unbiased estimators of both the SWFs and their associated inequality indices.

Inequality

- Common and intuitive way to assess inequality: weight the deviations between the income perceived by each individual and the mean income (or relative to the mean), using a weight function (value judgments):

$$I = \frac{1}{\mu}\int_0^{\infty}(x-\mu)\omega(x)dF(x) = \frac{1}{\mu}\int_0^{1}(F^{-1}(p)-\mu)\lambda(p)dp$$

Inequality

- Geometric interpretation from the Lorenz curve:

$$I = \int_0^1 (p - L(p))\pi(p)dp, \ \ \pi(p) = \lambda'(p), \ \ 0 < p < 1.$$

  - Gini: $w(x)=2F(x) \circ \pi(p)=2$
  - Generalized Gini $\pi(p)=n\ (n-1)(1-p)^{n-2}$
  - Aaberge (2000) $\pi(p)=n\ p^{n-2}$, $n \geq 2$

Welfare and Inequality

- Yaari approach (1987, 1988) YSWF is given by:

$$W_\phi(F) = \int_{R^+} x d\phi(F(x)) = \int_0^1 F^{-1}(p) d\phi(p) = \int_0^1 \phi'(p) F^{-1}(p) dp$$

Yaari shows that $W_\phi(F)$ presents aversion to inequality if and only if ϕ is concave

Welfare and Inequality

- If μ is the mean income of the distribution and L(p) its Lorenz curve, the YSWF can be expressed as a social welfare function associated to a linear measure of inequality:

$$W_\phi(F) = \mu[1 - I_\phi(F)]$$

where $$I_\phi(F) = \int_0^1 (p - L(p))\pi_\phi(p)dp, \quad \pi_\phi(p) = -\phi''(p)$$

relationship between SWF and inequality measure

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relationship between weighting scheme of the Lorenz differences and distribution of preference

Welfare and Inequality

- Fulfillment of the Principles of Transfers (Necessary and sufficient condition):
  - PDPT: concavity of ϕ
  - PPTS (given difference in ranks): $\phi'''(p) > 0$
  - PDT (given difference in incomes): $-\frac{\phi'''(F(x))}{\phi''(F(x))} > \frac{F''(x)}{(F'(x))^2}$, $x > 0$ or

    $\phi''(F(x))F'(x)$ strictly increasing

Only the properties of its preference distribution matter

Depends on the properties of its preference distribution and on the shape of the income distribution

Order statistics

- Order statistics. Definition

  Let $X_1, X_2, \ldots, X_n$ be a sample of size n, from a distribution F(.) , and define the order statistics $X_{k:n},\ k \in \{1, 2, \ldots, n\}$ in the ascending order by,

  $$X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$$

  The variable that assigns the value at position k-th to each sample.

- Distribution function of $X_{k:n}$, $F_{k:n}(\cdot)$

  $$F_{k:n}(x) = \sum_{j=k}^{n} \binom{n}{j} (F(x))^j (1 - F(x))^{n-j}$$

- The mean values of the order statistics

  $$E(X_{k:n}) = \int_0^{\infty} x dF_{k:n}(x) = k \binom{n}{k} \int_0^1 F^{-1}(p) p^{k-1} (1-p)^{n-k} dp$$

Order statistics

- From the mean of the order statistics:

$$E|X_{k:n}| \leq cE|X|,$$

  - If the distribution has a finite mean, the existence of the first moment of any order statistic is assured.
  - It is important for those distributions, such as heavy-tailed income distributions, for which only a few potential moments exist, and therefore no characterization in terms of (ordinary) moments is feasible.
  - It is interesting to analyze whether the distribution can be characterized by the moments of the order statistics.
  - Recurrence relation between the first moments of order statistics (David, 1981)

$$(n-k)E(X_{k:n}) + kE(X_{k+1:n}) = nE(X_{k:n-1})$$

Order statistics

- **Proposition 2.**

Let X be a random variable with finite mean and k(n) a positive integer, $1 \leq k(n) \leq n$, the distribution F(.) is uniquely determined by the sequence $\{E(X_{k(n):n})\}_{n \in N}$.

Welfare functions and inequality measures generated through mean values of order statistics.

- Order statistics ⇒ Distribution F. ⇒ Preference F. SWF ⇒ Inequality measures

Welfare functions and inequality measures generated through mean values of order statistics.

- **First order statistics and generalized Gini coefficients.**

$$F_{1:n}(x) = 1 - (1 - F(x))^n$$

if $x = F^{-1}(p)$

$$\phi_{1:n}(p) = F_{1:n}(F^{-1}(p)) = 1 - (1-p)^n, \; 0 \le p \le 1, \; n \ge 2$$ ⇨ Can be interpreted as distribution of preferences

$$W_{1:n}(F) = \int_0^1 F^{-1}(p) d\phi_{1:n}(p) = E(X_{1:n}) = n(n-1)\mu \int_0^1 (1-p)^{n-2} L(p) dp, \; n \ge 2.$$

$$I_{1:n}(F) = 1 - \frac{W_{1:n}(F)}{\mu} = 1 - \frac{E(X_{1:n})}{E(X)}, \; n \ge 2$$ ⇨ Inequality measure

$$I_{1:n}(F) = 1 - n(n-1)\int_0^1 (1-p)^{n-2} L(p) dp = n(n-1)\int_0^1 (p - L(p))(1-p)^{n-2} dp, \; n \ge 2$$

⇩ Generalized Gini of order n

Welfare functions and inequality measures generated through mean values of order statistics.

- **First order statistics and generalized Gini coefficients.**

The absolute indices:

$$\mu I_{1:n}(F) = \mu - W_{1:n}(F) = E\left(\overline{X}_n - X_{1:n}\right)$$

if we take random samples of size n, n≥2, from the income distribution and the welfare associated to each sample is identified with the minimum income, the mean value that is obtained when considering all possible samples of the given size is the welfare that the underlying SWF assigns to the generalized Gini coefficient of parameter n, $W_{1:n}(F) = E\left(X_{1:n}\right)$

As a consequence of Proposition 2, we can ensure that any distribution is characterized by the succession of SWFs $\{W_{1:n}(F)\}_n \equiv \{E(X_{1:n})\}_n$ ⇨ any F(.) is characterized by the sequence of the generalized absolute Gini coefficients

Welfare functions and inequality measures generated through mean values of order statistics.

- **General case**
- The distribution function of the order statistics are increasing, but not necessarily concave over the whole range ⇨ SWFs and indices of inequality that would not meet the PDPT.
- However, if for fixed sample size n, we calculate consecutively the arithmetic mean of the functions $\{F_{k:n}(F^{-1}(\cdot))\}_{1\leq k\leq n}$ we obtain a sequence of functions which have an appropriate behavior to be considered distributions of social preferences.

Welfare functions and inequality measures generated through mean values of order statistics.

- **General case**
- **Definition.** For each (n, k), $n \geq 2$, $k = 1, 2, ..., n$, we consider the function

$$\phi_{k:n}(p) = \frac{1}{k}\sum_{i=1}^{k} F_{i:n}(F^{-1}(p)),\ 0 \leq p \leq 1,\ \ k = 1, 2, ..., n.$$

- **Proposition 3.** Each of the functions $\{\phi_{k:n}(\cdot)\}$, defined in the interval [0,1], shows the properties required of a distribution of social preferences (increasing and concave).

Welfare functions and inequality measures generated through mean values of order statistics.

- **General case**

As a consequence, $W_{k:n}(F)=\int_0^1 F^{-1}(p)d\phi_{k:n}(p)=E\left(\frac{1}{k}(X_{1:n}+X_{2:n}+\cdots+X_{k:n})\right), k=1,2,..,n.$

If the level of welfare assigned to any sample of n incomes from F(.) is identified with the mean of their k lower incomes, the welfare of the population is the expectation of those values when considering all possible samples of size n.

Inequality measures:

$$I_{k:n}(F)=1-\frac{W_{k:n}(F)}{\mu}=1-\frac{1}{k\mu}E(X_{1:n}+X_{2:n}+\cdots+X_{k:n}), n\geq 2$$

$$I_{k:n}(F)=(n-k)\binom{n}{k}\int_0^1 (p-L(p))p^{k-1}(1-p)^{n-k-1}dp,\ n\geq 2$$

Welfare functions and inequality measures generated through mean values of order statistics.

- **General case**
- The welfare loss due to inequality is measured by the corresponding absolute indices:

$$\mu I_{k:n}(F) = \mu - W_{k:n}(F) = E\left(\overline{X}_n - \frac{1}{k}(X_{1:n} + X_{2:n} + \cdots + X_{k:n}) \right)$$

- Therefore, $\overline{X}_n - (X_{1:n} + X_{2:n} + \cdots + X_{k:n})/k$ is an unbiased estimator of $\mu I_{k:n}(F)$.

Welfare functions and inequality measures generated through mean values of order statistics.

- **Particular cases**
  - For k=1 we get the Generalized Gini
  - For k= n-1 we get the family of indices proposed by Aaberge (2000).
  - For k=n, the SWF shows no aversion to inequality. It identifies the welfare of each income distribution with its average income, and the associated inequality index is zero for any distribution. This does not imply the absence of inequality, but that both the SWF and its corresponding index are indifferent to inequality.

Some additional policy considerations

Given n
When k increases

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The distribution of preferences reduce their concavity

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The SWFs show less aversion to inequality, from the corresponding to the Generalized Gini until indiference

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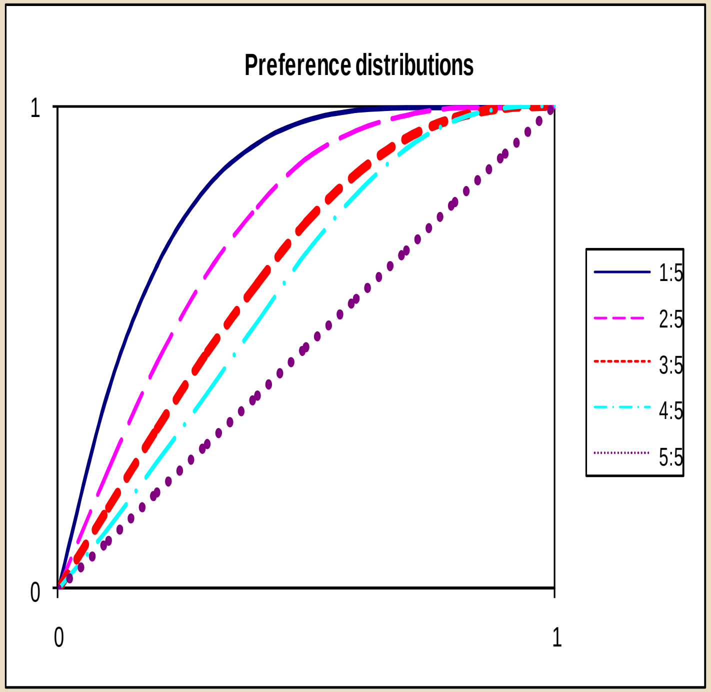
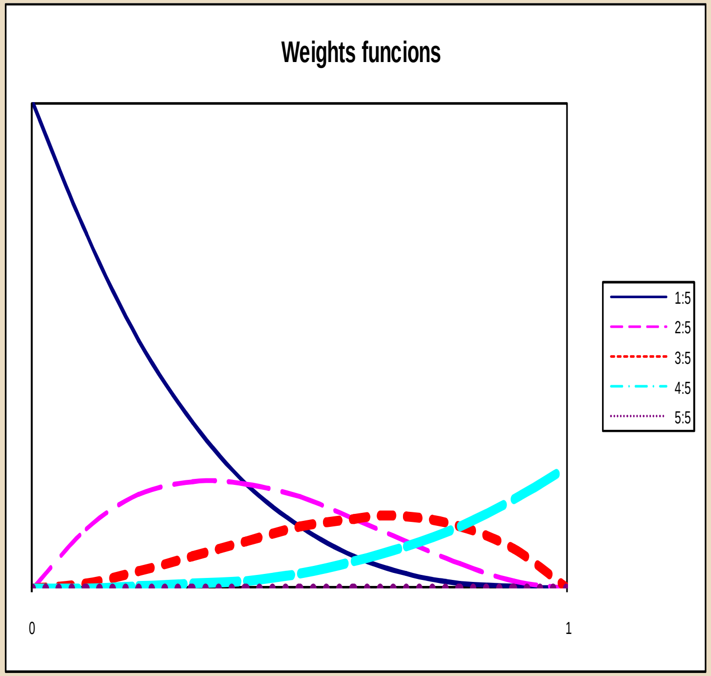
The associated inequality measures assign less weight to the inequality corresponding to low incomes and greater weight to the inequality corresponding to high incomes

Some additional policy considerations





The Genelralized Gini indices and the indices of the family proposed by Aaberge (2000) weight local inequality through monotonic functions along the distribution so that the greater weight is assigned to one of its ends.

However, the weights for the Lorenz differences for the indices of the family, $\{I_{k:n}(F)\}_{(k,n)}$, 1<k<n-1 are not monotonic.

This allows for measures with different attitudes in assessing inequality and welfare, as they pay more attention to different parts of the distribution.

Some additional policy considerations

- **Proposition 4.**
- The indices of the family $\{I_{k:n}(F)\}_{(k,n)}$ $n \geq 2,\ 1 \leq k \leq n$, satisfy the PPTS if and only if

$$T(n,k,p) = \left[(n-2)p - (k-1)\right] > 0 \quad \text{for any} \quad p \in (0,1)$$

- The index $I_{k:n}(F)$, which is applied over the distribution function F, satisfies the DTP if and only if:

$$\frac{(n-2)F(x) - (k-1)}{F(x)(1-F(x))} > \frac{F''(x)}{(F'(x))^2}\ ,\ x > 0$$

Some additional policy considerations

- Regarding the PPTS:
  - The Gini index does not satisfy the PPST.
  - Generalized Gini indices satisfy the PPST (except the Gini index)
  - Other indices exhibit a behavior opposite to the PPTS, as the indices of the family proposed by Aaberge.
  - There are also indices whose behavior with respect to this principle is not uniform.

Some additional policy considerations

- Regarding the DTP:
  - If an index has aversion towards inequality ($\phi''(p)<0$) and its preference function has a non-negative positive third derivative , it will satisfy the PDT for any concave income distribution
  - This is the case of the Gini coefficient.
  - If an inequality measure satisfies the PDT in a certain range, any other measure with greater inequality aversion also verifies that principle on that interval and possibly on others of greater amplitude
  - In our case, the smaller k and the greater the inequality aversion of the index, the wider the set of income distributions for which the PDT is satisfied.

Conclusions

- The use of order statistics in the definition of SWFs and indices of inequality provides a joint treatment of measures that share common features, but differ from and complement each other from the normative standpoint.

- The approach adopted allows us to proving that, given the mean income, certain families of indices characterize the income distribution, and provides a clear statistical interpretation to each SWF and its corresponding index of inequality.

Conclusions

- The appropriate selection of various elements of the set of indices or welfare functions , permits applying very different distributional judgments when comparing levels of inequality or welfare associated with different income distributions

- Hence, the conclusion in a particular application may be interesting either when a robust result is obtained, or if the outcome is different depending on the index considered, as the properties of the different measures are taken into account.

- Thank you