A SPARSE APPROACH TO MIXED WEAK TYPE INEQUALITIES

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Abstract. In this paper we provide some quantitative mixed weak-type estimates assuming conditions that imply that $uv \in A_\infty$ for Calderón-Zygmund operators, rough singular integrals and commutators. The main novelty of this paper lies in the fact that we rely upon sparse domination results, pushing an approach to endpoint estimates that was introduced in [8] and extended in [23] and [24].

1. Introduction and Main Results

In [27], Muckenhoupt and Wheeden, introduced a new type of weak type inequality, that we call mixed weak-type inequality, that consists in considering a perturbation of the Hardy-Littlewood maximal operator with an A_p weight. Their result was the following

Theorem A. *Let $w \in A_1$ then*

$$|\{x \in \mathbb{R} : w(x)Mf(x) > t\}| \leq c_w \frac{1}{t} \int_{\mathbb{R}} |f| w(x)dx.$$

Although this kind of estimate may seem not very different to the standard one, the perturbation caused by having the weight inside the level set makes it way harder to be settled, in contrast with the analogous case of strong type estimates. Furthermore, $w \in A_1$ is no longer a necessary condition for this endpoint estimate to hold (see [27, Section 5]).

Later on, Sawyer [33], motivated by the possibility of providing a new proof of Muckenhoupt's theorem, obtained the following result.

Theorem B. *Let $u, v \in A_1$ then*

$$uv\left(\left\{x \in \mathbb{R} : \frac{M(fv)(x)}{v(x)} > t\right\}\right) \leq c_{u,v} \frac{1}{t} \int_{\mathbb{R}} |f| u(x)v(x)dx. \tag{1.1}$$

Sawyer also conjectured that (1.1) should hold as well for the Hilbert transform. Cruz-Uribe, Martell and Pérez, [7] generalized (1.1) to higher dimensions and actually proved that Sawyer's conjecture holds for Calderón-Zygmund operators via the following extrapolation argument.

Theorem C. *Assume that for every $w \in A_\infty$ and some $0 < p < \infty$,*

$$\|Tf\|_{L^p(w)} \leq c_w \|Gf\|_{L^p(w)}.$$

Then for every $u \in A_1$ and every $v \in A_\infty$

$$\|Tf\|_{L^{1,\infty}(uv)} \leq c_{u,v,n} \|Gf\|_{L^{1,\infty}(uv)}.$$

The conditions on the weights in that extrapolation result lead them to conjecture that (1.1), and consequently the corresponding estimate for Calderón-Zygmund operators should hold as well with $u \in A_1$ and $v \in A_\infty$. That conjecture was positively answered recently in [25] where several quantitative estimates were provided as well. At this point we would like to mention, as well, a recent generalization provided for Orlicz maximal operators in [2].

In [7], besides the aforementioned results, it was shown that (1.1) holds if $u \in A_1$ and $v \in A_\infty(u)$ (see Section 2.2 for the precise definition of $A_p(u)$). The advantage of that condition is that the product uv is an A_∞ weight. Over the past few years, there have been new contributions under

The second author is supported by CONICET PIP 11220130100329CO.

those assumptions such as [4] for the case of fractional integrals and related operators, [28, 29] for related quantitative estimates and [26] for multilinear extensions.

The case of commutators of Calderón-Zygmund operators was settled in [3]. Recall that given T a Calderón-Zygmund operator, $b \in Osc_{\exp L^r} \subset BMO$ (see Section 2.2 for the precise definition) and a positive integer m, we define the higher order commutator $T_b^m f$ by

$$T_b^m f(x) = b(x) T_b^{m-1} f(x) - T_b^{m-1}(bf)(x)$$

where $T_b^1 f(x) = b(x)Tf(x) - T(bf)(x)$.

Now we turn to our contribution. Our approach exploits sparse domination and ideas from [24] that can be traced back to [8]. In the case of commutators our approach is inspired by [23] as well. The main novelty of our proofs is precisely that, in contrast with the techniques used up until now to deal with this kind of questions, we heavily rely upon sparse domination. Our first result is the following.

Theorem 1.1. *Let $u \in A_1$ and $v \in A_p(u)$ for some $1 < p < \infty$.*

1. *If T is a Calderón-Zygmund operator,*

$$\left\| \frac{T(fv)(x)}{v(x)} \right\|_{L^{1,\infty}(uv)} \leq c_{n,p} [uv]_{A_\infty} [u]_{A_1} \log\left(e + [uv]_{A_\infty} [u]_{A_1} [v]_{A_p(u)}\right) \|f\|_{L^1(uv)}$$

and if m is a positive integer, $r > 1$ and $b \in Osc_{\exp L^r}$ then

$$uv\left(\left\{x \in \mathbb{R}^n \,:\, \left|\frac{T_b^m(fv)}{v}\right| > t\right\}\right) \leq c_{n,p} \Gamma_{u,v}^m \int_{\mathbb{R}^n} \Phi_{\frac{m}{r}}\left(\frac{|f| \|b\|_{Osc_{\exp L^r}}^m}{t}\right) uv \tag{1.2}$$

where

$$\Gamma_{u,v}^m = \sum_{h=0}^m [u]_{A_1} [uv]_{A_\infty}^{1+\frac{h}{r}} [u]_{A_\infty}^{\frac{m-h}{r}} \log\left(e + [u]_{A_1} [uv]_{A_\infty}^{1+\frac{h}{r}} [u]_{A_\infty}^{\frac{m-h}{r}} [v]_{A_p(u)}\right)^{1+\frac{h}{r}}$$

and $\Phi_\rho(t) = t\left(1 + \log^+(t)\right)^\rho$.

2. *If $\Omega \in L^\infty(\mathbb{S}^{n-1})$ then*

$$\left\| \frac{T_\Omega(fv)(x)}{v(x)} \right\|_{L^{1,\infty}(uv)} \leq c_{n,p} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} [uv]_{A_\infty} [u]_{A_1} [u]_{A_\infty} \log\left(e + [uv]_{A_\infty} [u]_{A_1} [u]_{A_\infty} [v]_{A_p(u)}\right) \|f\|_{L^1(uv)}.$$

We would like to note that for $u = 1$, in the case of Calderón-Zygmund operators, the estimate above reduces to

$$\left\| \frac{T(fv)(x)}{v(x)} \right\|_{L^{1,\infty}(v)} \leq c_{n,p} [v]_{A_\infty} \log\left(e + [v]_{A_p}\right) \|f\|_{L^1(v)} \qquad p \geq 1.$$

That estimate improves the bound provided in [29, Theorems 1.16 and 1.17], namely,

$$\left\| \frac{T(fv)(x)}{v(x)} \right\|_{L^{1,\infty}(v)} \leq c_{n,p} [v]_{A_p} \log\left(e + [v]_{A_p}\right) \|f\|_{L^1(v)} \qquad p \geq 1.$$

In the case of the commutator our approach provides a new proof of [3, Theorem 2] obtaining a quantitative estimate as well. An arguable drawback of the estimates above is that in neither of them we recover the best known dependence in the case $v = 1$. We wonder whether the factor $[uv]_{A_\infty}$ in each of them can be removed.

In our following result we assume that $v \in A_1$ and $u \in A_1(v)$. It is not hard to check that those conditions are equivalent to assume that $u \in A_1$ and $v \in A_1(u)$, so there is no gain in terms of the size of the class of weights considered. However, in this case, if $v = 1$ we recover the best known estimates for $u \in A_1$.

Theorem 1.2. *Let $v \in A_1$ and $u \in A_1(v)$.*

1. *If T is a Calderón-Zygmund operator*

$$\left\|\frac{T(fv)(x)}{v(x)}\right\|_{L^{1,\infty}(uv)} \leq c_{n,T}[v]_{A_1}[v]_{A_\infty}[u]_{A_1(v)}\log\left(e+[uv]_{A_\infty}[v]_{A_1}\right)\|f\|_{L^1(uv)}$$

and if m is a positive integer, $r>1$ and $b\in Osc_{\exp L^r}$ then

$$uv\left(\left\{x\in\mathbb{R}^n\,:\,\left|\frac{T_b^m(fv)}{v}\right|>t\right\}\right)\leq c_{n,p}\Gamma_{u,v}^m\int_{\mathbb{R}^n}\Phi_{\frac{m}{r}}\left(\frac{|f|\|b\|_{Osc_{\exp L^r}}^m}{t}\right)uv \tag{1.3}$$

where

$$\Gamma_{u,v}^m=\sum_{h=0}^m[v]_{A_1}[v]_{A_\infty}^{\frac{h}{r}}[uv]_{A_\infty}^{\frac{m-h}{r}}[u]_{A_1(v)}[v]_{A_\infty}\log\left(e+[v]_{A_1}[v]_{A_\infty}^{\frac{h}{r}}[uv]_{A_\infty}^{\frac{m-h}{r}}\right)^{1+\frac{h}{r}}$$

and $\Phi_\rho(t)=t\left(1+\log^+(t)\right)^\rho$.

2. *If $\Omega\in L^\infty(\mathbb{S}^{n-1})$ then*

$$\left\|\frac{T_\Omega(fv)(x)}{v(x)}\right\|_{L^{1,\infty}(uv)}\leq c_n\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[uv]_{A_\infty}[v]_{A_1}[u]_{A_1(v)}[v]_{A_\infty}\log\left(e+[uv]_{A_\infty}[v]_{A_1}\right)\|f\|_{L^1(uv)}.$$

As we pointed out above, we would like to note that this result recovers the best dependences known obtained in [21, 22, 24, 23, 14] in the case, $v=1$. Furthermore, for commutators we obtain the following estimate

$$u\left(\{x\in\mathbb{R}^n\,:\,|T_b^m(f)|>t\}\right)\leq c_{n,p}[u]_{A_1}[u]_{A_\infty}^{\frac{m}{r}}\log\left(e+[u]_{A_\infty}\right)\int_{\mathbb{R}^n}\Phi_{\frac{m}{r}}\left(\frac{|f|\|b\|_{Osc_{\exp L^r}}^m}{t}\right)u. \tag{1.4}$$

Observe that (1.4) contains as a particular case the endpoint estimate obtained in [14] and provides a precise quantitative bound for the case in which the symbol has better local decay properties than BMO functions. We recall that in [1], it was shown that if a commutator of a certain singular integral satisfies a weak-type $(1,1)$ estimate then $b\in L^\infty$ and that the $L\log L$ estimate, introduced in [31], implies that $b\in BMO$. Bearing those results in mind we wonder whether $b\in Osc_{\exp L^r}$ should be a neccesary condition for (1.4), at least in the case $u=1$, to hold.

The rest of the paper is organized as follows. Section 2 is devoted to provide some basic results and to fix notation that will be used throughout the remainder of the paper and in Section 3 we provide the proofs of the main results.

2. Preliminaries

2.1. **Sparse domination results.**
In this section we begin borrowing some definitions from [20].

Given a cube Q we denote by $\mathcal{D}(Q)$ the standard dyadic grid relative to Q.

We say that a family of cubes $\mathcal{D}$ is a dyadic lattice if it satisfies the following conditions.

1. If $Q\in\mathcal{D}$ then $\mathcal{D}(Q)\subset\mathcal{D}$.
2. If $P,Q\in\mathcal{D}$ then there exists $R\in\mathcal{D}$ such that $P,Q\in\mathcal{D}(R)$.
3. For every compact set $K\subset\mathbb{R}^d$ there exists some $Q\in\mathcal{D}$ such that $K\subset Q$.

We recall that $\mathcal{S}$ is a η-sparse family if for every $Q\in\mathcal{S}$ there exists $E_Q\subset Q$ such that

1. $\eta|Q|\leq|E_Q|$.
2. The sets E_Q are pairwise disjoint.

In some situations it is useful to approximate arbitrary cubes by dyadic cubes. For that purpose, one dyadic lattice is not enough, however 3^n are. That fact follows from the following Lemma that we borrow as well from [20].

Lemma 2.1. *For every dyadic lattice $\mathcal{D}$ there exist 3^n dyadic lattices $\mathcal{D}_j$ such that*

$$\{3Q\,:\,Q\in\mathcal{D}\}=\bigcup_{j=1}^{3^n}\mathcal{D}_j$$

and for every cube $Q\in\mathcal{D}$ and $j=1,\dots,3^n$, there exists a unique cube $R\in\mathcal{D}_j$ of sidelenght $l_R=3l_Q$ containing Q.

In the last years, and after Lerner's simplification the proof of the A_2 theorem [18] that had been settled earlier by Hytönen [10], the sparse domination approach has been widely and succesfully applied in the theory of weights. The philosophy behind that approach consists in controlling, in some sense, the operator that we want to study by suitable sparse operators and providing estimates for the latter ones, which are in general easier to settle.

In the following Theorem we gather the sparse domination results that we will rely upon in the main results of the paper.

Theorem 2.2. *Let $f\in\mathcal{C}_c^\infty$.*

[6, 20, 16, 13, 19]**:** *If T is a Calderón-Zygmund operator there exist 3^n ε-sparse families contained in 3^n dyadic lattices $\mathcal{D}_j$ such that*

$$|Tf(x)|\leq c_{n,T,\varepsilon}\sum_{j=1}^{3^n}A_{\mathcal{S}}\left(|f|\right)(x)$$

where $A_{\mathcal{S}}f(x)=\sum_{Q\in\mathcal{S}}\frac{1}{|Q|}\int_Q f(y)dy\chi_Q(x)$.

[14, 23]**:** *If T is a Calderón-Zygmund operator and $b\in BMO$ then there exist 3^n ε-sparse families contained in 3^n dyadic lattices $\mathcal{D}_j$ such that*

$$|T_b^m f(x)|\leq c_{n,T,\varepsilon}\sum_{j=1}^{3^n}\sum_{h=0}^{m}A_{\mathcal{S}}^{m,h}(b,f)(x)$$

where $h=0,\dots,m$ and

$$A_{\mathcal{S}}^{m,h}(b,f)(x)=\sum_{Q\in\mathcal{S}}|b(x)-b_Q|^{m-h}\frac{1}{|Q|}\int_Q|b-b_Q|^h f\chi_Q(x).$$

[5, 17]**:** *If $\Omega\in L^\infty(\mathbb{S}^{n-1})$ then there exists a sparse family $\mathcal{S}$ such that*

$$\left|\int_{\mathbb{R}^n}T_\Omega fg\right|\leq c_{n,\Omega}r'\Lambda_{\mathcal{S}}^r(f,g)\quad r>1 \tag{2.1}$$

where $f\in L^r$ and $g\in L^1_{loc}$

$$\Lambda_{\mathcal{S}}^r(f,g)=\sum_{Q\in\mathcal{S}}\frac{1}{|Q|}\int_Q|f|\left(\frac{1}{|Q|}\int_Q|g|^r\right)^{\frac{1}{r}}|Q|.$$

Remark 2.3. Note that the 3^n-dyadic lattices trick (Lemma 2.1) allows us to show that for every dyadic lattice $\mathcal{D}$,

$$\Lambda_{\mathcal{S}}^r(f,g)\leq\sum_{j=1}^{3^n}\Lambda_{\mathcal{S}_j}^r(f,g)$$

where each $\mathcal{S}_j\subset\mathcal{D}_j$ and the choice of the dyadic lattices $\mathcal{D}_j$ is independent of f,g.

2.2. A_p **weights and Orlicz maximal functions.** We recall that given a weight u, $v \in A_p(u)$ if

$$[v]_{A_p(u)} = \sup_Q \frac{1}{u(Q)} \int_Q vu \left(\frac{1}{u(Q)} \int_Q v^{-\frac{1}{p-1}} u \right)^{p-1} < \infty$$

in the case $1 < p < \infty$ and

$$[v]_{A_1(u)} = \left\| \frac{M_u v}{v} \right\|_{L^\infty} < \infty$$

where $M_u v = \sup_Q \frac{1}{u(Q)} \int_Q vu$. If $u = 1$ we recover the classical Muckenhoupt's condition.

We would like also to recall that

$$A_\infty = \bigcup_{p \geq 1} A_p.$$

This class of weights is characterized in terms of the following condition

$$w \in A_\infty \iff [w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w) < \infty.$$

This characterization was discovered by Fujii [9] and rediscovered by Wilson [34]. Up until now $[w]_{A_\infty}$ is the smallest constant characterizing the A_∞ class (see Pérez and Hytönen [11]). A result that we will use as well is the following reverse Hölder inequality that was obtained in [11] (see [12] for another proof).

Lemma 2.4. *There exists τ_n such that for every $w \in A_\infty$*

$$\left(\frac{1}{|Q|} \int_Q w^{r_w} \right)^{\frac{1}{r_w}} \leq \frac{2}{|Q|} \int_Q w$$

where $r_w = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$.

We recall that given a Young function $A : [0, \infty) \to [0, \infty)$, namely a convex, non-decreasing function such that $A(0) = 0$ and $\frac{A(t)}{t} \to \infty$ when $t \to \infty$ we can define

$$\|f\|_{A(u),Q} = \|f\|_{A(L)(u),Q} = \inf \left\{ \lambda > 0 \,:\, \frac{1}{u(Q)} \int_Q A\left(\frac{|f(x)|}{\lambda} \right) u(x)dx \leq 1 \right\}.$$

It is possible to provide a definition of the norm equivalent to the latter (see [15]), namely

$$\|f\|_{A(u),Q} \simeq \inf_{\mu > 0} \left\{ \mu + \frac{\mu}{u(Q)} \int_Q A\left(\frac{|f(x)|}{\mu} \right) u(x)dx \right\}.$$

Associated to each Young A function there exists another Young function $\overline{A}$ such that

$$\frac{1}{u(Q)} \int_Q |fg|u \leq 2\|f\|_{A(u),Q} \|g\|_{\overline{A}(u),Q}.$$

We shall drop u in the notation in the case of Lebesgue measure. Some particular cases of interest for us will be $A(t) = t \log(e + t)^{\frac{1}{r}}$ and $\overline{A}(t) = \exp(t^r) - 1$ for $r > 1$.

Let u a weight and A a Young function. We define the maximal operator $M^{\mathcal{F}}_{A(u)}$ by

$$M^{\mathcal{F}}_{A(u)} f(x) = \sup_{x \in Q \in \mathcal{F}} \|f\|_{A(u),Q}.$$

where the supremum is taken over all the cubes in the family $\mathcal{F}$. We shall drop the superscript in case the context makes clear the family of cubes considered. If we choose $A(t) = t$ and $u = 1$ and $\mathcal{F}$ is the family of all cubes we recover the classical Hardy-Littlewood operator.

Now we recall if $b \in BMO$, then

$$\sup_Q \|b - b_Q\|_{\exp L,Q} \leq c_n \|b\|_{BMO}.$$

It is possible to define classes of symbols with even better properties of integrability than BMO symbols. Given $r>1$ we say that $b\in Osc_{\exp L^r}(w)$ if

$$\|b\|_{Osc_{\exp L^r}(w)}=\sup_Q \|b-b_Q\|_{\exp L^r(w),Q}<\infty.$$

Note that $Osc_{\exp L^r}\subsetneq BMO$ for every $r>1$. It is not hard to prove that for those classes of functions the following estimates hold.

Lemma 2.5. *Let $w\in A_\infty$ and $b\in Osc_{\exp L^r}$. Then*

$$\|b-b_Q\|_{\exp L^r(w)}\leq c[w]_{A_\infty}^{\frac{1}{r}}\|b\|_{Osc_{\exp L^r}}.$$

Furthermore, if $j>0$ then

$$\left\||b-b_Q|^j\right\|_{\exp L^{\frac{r}{j}}(w)}\leq c[w]_{A_\infty}^{\frac{j}{r}}\|b\|^j_{Osc_{\exp L^r}}.$$

We end up this section with a result that allows us to change the underlying weight of Orlicz averages.

Lemma 2.6. *Let u a weight, $v\in A_p(u)$, and Φ a Young function. Then, for every cube Q,*

$$\|f\|_{\Phi(u),Q}\leq \|f\|_{[v]_{A_p(u)}\Phi^p(uv),Q}.$$

Proof. The proof of the preceding estimate is fairly standard. We argue as follows. Let $\lambda>0$. Then

$$\begin{aligned}
&\frac{1}{u(Q)}\int_Q\Phi\left(\frac{|f(x)|}{\lambda}\right)u(x)dx\\
&=\frac{1}{u(Q)}\int_Q\Phi\left(\frac{|f(x)|}{\lambda}\right)v^{\frac{1}{p}}(x)v^{-\frac{1}{p}}(x)u^{\frac{1}{p}+\frac{1}{p'}}(x)dx\\
&\leq\frac{1}{u(Q)}\left(\int_Q\Phi\left(\frac{|f(x)|}{\lambda}\right)^p v(x)u(x)dx\right)^{\frac{1}{p}}\left(\int_Q v^{-\frac{p'}{p}}udx\right)^{\frac{1}{p'}}\\
&\leq\left(\frac{uv(Q)}{u(Q)}\right)^{\frac{1}{p}}\left(\frac{1}{u(Q)}\int_Q v^{-\frac{p'}{p}}udx\right)^{\frac{1}{p'}}\left(\frac{1}{uv(Q)}\int_Q\Phi\left(\frac{|f(x)|}{\lambda}\right)^p v(x)u(x)dx\right)^{\frac{1}{p}}\\
&\leq[v]_{A_p(u)}^{\frac{1}{p}}\left(\frac{1}{uv(Q)}\int_Q\Phi\left(\frac{|f(x)|}{\lambda}\right)^p v(x)u(x)dx\right)^{\frac{1}{p}}\\
&\leq\left(\frac{1}{uv(Q)}\int_Q[v]_{A_p(u)}\Phi\left(\frac{|f(x)|}{\lambda}\right)^p v(x)u(x)dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Choosing $\lambda=\|f\|_{[v]_{A_p(u)}\Phi^p(L)(uv),Q}$ we have then that

$$\frac{1}{u(Q)}\int_Q\Phi\left(\frac{|f(x)|}{\lambda}\right)u(x)dx\leq 1$$

from which the desired conclusion follows. □

We recommend the reader to consult [30, 32] for more information about Young functions and Orlicz spaces.

3. Proofs of the main results

3.1. **Scheme of the proofs of the main results.** In this section we briefly outline the scheme that we are going to follow for each of the proofs of the estimates in the main results. As we mentioned in the introduction, the scheme can be traced back to [8, 23, 24]. Let T be a linear operator, possibly a sparse operator and let $\tilde{M}_{uv}f$ maximal and dyadic, in some sense, operator such that

$$uv\left(\left\{x\in\mathbb{R}^d\,:\,|\tilde{M}_{uv}f(x)|>t\right\}\right)\lesssim\frac{1}{t}\int A\left(\frac{|f|}{t}\right)uv$$

where A is a Young function. First we note that by homogeneity it suffices to show that

$$uv\left(\left\{x\in\mathbb{R}^d\,:\,\frac{|T(fv)(x)|}{v}>1\right\}\right)\lesssim c_{n,T}\kappa_{u,v}\int A\left(|f|\right)uv \tag{3.1}$$

where $\kappa_{u,v}\geq 1$ is the constant given by the dependence on the weights involved. Taking that into account we could proceed as follows

$$\begin{aligned}uv\left(\left\{x\in\mathbb{R}^d\,:\,\frac{|T(fv)(x)|}{v}>1\right\}\right)&\leq uv\left(\left\{x\in\mathbb{R}^d\,:\,\frac{|T(fv)(x)|}{v}>1,\tilde{M}_{uv}f(x)\leq\frac{1}{2}\right\}\right)\\&+uv\left(\left\{x\in\mathbb{R}^d\,:\,|\tilde{M}_{uv}f(x)|>\frac{1}{2}\right\}\right).\end{aligned}$$

Since the desired estimate holds for the second term it suffices to control the first one. Let us call

$$G=\left\{x\in\mathbb{R}^d\,:\,\frac{|T(fv)(x)|}{v}>1,\tilde{M}_{uv}f(x)\leq\frac{1}{2}\right\}.$$

Then it suffices to prove

$$uv\left(\left\{x\in\mathbb{R}^d\,:\,\frac{|T(fv)(x)|}{v}>1,\tilde{M}_{uv}f(x)\leq\frac{1}{2}\right\}\right)\leq c_{n,T}\kappa_{u,v}\int A\left(|f|\right)uv+\frac{1}{2}uv(G) \tag{3.2}$$

since this this yields

$$uv\left(\left\{x\in\mathbb{R}^d\,:\,\frac{|T(fv)(x)|}{v}>1,\tilde{M}_{uv}f(x)\leq\frac{1}{2}\right\}\right)\leq 2c_{n,T}\kappa_{u,v}\int A\left(|f|\right)uv$$

and, consequently, then (3.1) holds.

Bearing this scheme in mind, the proofs of the main results boil down to settle (3.2) for each operator and choice of weights described there. The following subsections will be devoted to that task.

3.2. **Lemmatta.** Before starting with the proofs of the main results we provide some technical lemmas.

Lemma 3.1. *Let $\gamma_1,\gamma_2>1$. For every j,k non negative integers let*

$$\alpha_{k,j}=\min\{\gamma_1 2^{-k}j^{\rho_1},\beta\gamma_2 2^{-j}2^{-k}2^{\delta k}k^{\rho_2}\},$$

where $\rho_1,\rho_2,\delta\geq 0$ and $\beta>0$. Then

$$\sum_{j,k\geq 0}\alpha_{k,j}\leq c_{\rho_1,\rho_2,\gamma,\delta}\gamma_1\log_2\left(e+\gamma_2\right)^{1+\rho_1}+\frac{1}{2\gamma}\beta,$$

where $\gamma\geq 1$.

Proof. We start writing

$$\sum_{j,k\geq 0}\alpha_{k,j}=\sum_{k=0}^{\infty}\sum_{j\geq\lceil\log_2((e+\gamma_2)8\gamma)\rceil+(\lceil\delta+\rho_2\rceil+1)k}\alpha_{k,j}+\sum_{k=0}^{\infty}\sum_{j<\lceil\log_2((e+\gamma_2)8\gamma)\rceil+(\lceil\delta+\rho_2\rceil+1)k}\alpha_{k,j}$$

For the first term, note that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{j\geq\lceil\log_2((e+\gamma_2)8\gamma)\rceil+(\lceil\delta+\rho_2\rceil+1)k}\alpha_{k,j}\\
&\leq\beta\gamma_2\sum_{k=0}^{\infty}2^{-k}2^{\delta k}k^{\rho_2}\sum_{j\geq\lceil\log_2((e+\gamma_2)8\gamma)\rceil+(\lceil\delta+\rho_2\rceil+1)k}2^{-j}\\
&=\beta\gamma_2\sum_{k=0}^{\infty}2^{-k}2^{\delta k}k^{\rho_2}2^{-\lceil\log_2((e+\gamma_2)8\gamma)\rceil+(\lceil\delta+\rho_2\rceil+1)k}\\
&\leq\frac{\beta\gamma_2}{(e+\gamma_2)8\gamma}\sum_{k=0}^{\infty}2^{-k-\rho_2 k}k^{\rho_2}\\
&=\frac{\beta\gamma_2}{(e+\gamma_2)8\gamma}\sum_{k=1}^{\infty}2^{-(1+\rho_2)k}2^{\rho_2\log k}\\
&\leq\frac{\beta\gamma_2}{(e+\gamma_2)8\gamma}\sum_{k=0}^{\infty}2^{-k}\\
&\leq\frac{2\gamma_2\beta}{(e+\gamma_2)8\gamma}\\
&\leq\frac{\gamma_2}{(e+\gamma_2)4\gamma}\beta\leq\frac{1}{2\gamma}\beta
\end{aligned}$$

For the second term, we observe that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{j<\lceil\log_2((e+\gamma_2)8\gamma)\rceil+(\lceil\delta+\rho_2\rceil+1)k}\alpha_{k,j}\\
&\leq\gamma_1\sum_{k=0}^{\infty}2^{-k}\sum_{1\leq j<\lceil\log_2((e+\gamma_2)8\gamma)\rceil+(\lceil\delta+\rho_2\rceil+1)k}j^{\rho_1}\\
&\leq\gamma_1\sum_{k=0}^{\infty}\left(\lceil\log_2\left((e+\gamma_2)8\gamma\right)\rceil+\left(\lceil\delta+\rho_2\rceil+1\right)k\right)^{1+\rho_1}2^{-k}\\
&\leq c2\left(\delta+\rho_2\right)\gamma_1\log_2\left((e+\gamma_2)8\gamma\right)^{1+\rho_1}\\
&\leq c_{\rho_1,\rho_2,\gamma,\delta}\gamma_1\log\left(e+\gamma_2\right)^{1+\rho_1}
\end{aligned}$$

and we are done. ∎

The second result we will rely upon is the following.

Lemma 3.2. *Let A Young function such that $A(xy)\leq\kappa_A A(x)A(y)$ for some $\kappa_A\geq 1$ and $\mathcal{S}$ a $\frac{\kappa_A 8A(2)}{1+\kappa_A 8A(2)}$-sparse family. Let $f\in\mathcal{C}_c^{\infty}$ and $w\in A_\infty$ and assume that for every $Q\in\mathcal{S}$*

$$2^{-j-1}\leq\|f\|_{A(w),Q}\leq 2^{-j}.$$

Then for every $Q\in\mathcal{S}$ there exists $\tilde{E}_Q\subseteq Q$ such that

$$\sum_{Q\in\mathcal{S}}\chi_{\tilde{E_Q}}(x)\leq c_n[w]_{A_\infty}$$

and

$$w(Q)\|f\|_{A(w),Q}\leq 4\kappa_A\frac{A(2^{j+1})}{2^{j+1}}\int_{\tilde{E_Q}}A\left(|f|\right)w.$$

Proof. We split the family $\mathcal{S}$ in the following way

$$\begin{aligned}\mathcal{S}^0 &= \{\text{Maximal in } \mathcal{S}\}\\ \mathcal{S}^1 &= \{\text{Maximal in } \mathcal{S}\setminus\mathcal{S}^0\}\\ &\dots\\ \mathcal{S}^i &= \{\text{Maximal in } \mathcal{S}\setminus\cup_{r=0}^{i-1}\mathcal{S}^r\}\end{aligned}$$

Note that since $w\in A_\infty$ we have that, for each cube Q and each measurable subset $E\subset Q$,

$$w(E)\leq 2\left(\frac{|E|}{|Q|}\right)^{\frac{1}{c_n[w]_{A_\infty}}}w(Q) \tag{3.3}$$

In particular if $Q\in\mathcal{S}^i$ and $J_1=\bigcup_{P\in\mathcal{S}^{i+1},\,P\subsetneq Q}P$ then

$$|J_1|=\left|\bigcup_{P\in\mathcal{S}^{i+1},\,P\subsetneq Q}P\right|\leq\left(\frac{1+\kappa_A 8A(2)}{\kappa_A 8A(2)}-1\right)|Q|=\frac{1}{\kappa_A 8A(2)}|Q|.$$

Furthermore, arguing by induction, if we denote $J_\nu=\bigcup_{P\in\mathcal{S}^{i+\nu},\,P\subsetneq Q}P$, we have that

$$|J_\nu|\leq\left(\frac{1}{\kappa_A 8A(2)}\right)^\nu|Q|.$$

Combining that with (3.3) yields

$$w(J_\nu)\leq 2\left(\frac{1}{\kappa_A 8A(2)}\right)^{\frac{\nu}{c_n[w]_{A_\infty}}}w(Q),$$

and particular, if we choose $\nu=\lceil c_n[w]_{A_\infty}\rceil$, then

$$w(J_\nu)\leq\frac{1}{\kappa_A 4A(2)}w(Q).$$

Let $Q\in\mathcal{S}^i$. and let $\tilde{E_Q}=Q\setminus\bigcup_{P\in\mathcal{S}^{i+\lceil c_n[w]_{A_\infty}\rceil}}P$.

$$\begin{aligned}&w(Q)\|f\|_{A(w),Q}\\ &\leq w(Q)\left\{2^{-j-1}+\frac{2^{-j-1}}{w(Q)}\int_Q A\left(2^{j+1}|f|\right)w\right\}\\ &\leq w(Q)2^{-j-1}+\frac{1}{2^{j+1}}\int_Q A\left(2^{j+1}|f|\right)w\\ &\leq w(Q)2^{-j-1}+\frac{1}{2^{j+1}}\int_{\tilde{E_Q}}A\left(2^{j+1}|f|\right)w+\frac{1}{2^{j+1}}\sum_{P\in\mathcal{S}_{j,k}^{i+\lceil c_n[w]_{A_\infty}\rceil}}\int_P A\left(2^{j+1}|f|\right)w\\ &\leq w(Q)2^{-j-1}+\kappa_A\frac{A(2^{j+1})}{2^{j+1}}\int_{\tilde{E_Q}}A\left(|f|\right)w+\frac{1}{2^{j+1}}\sum_{P\in\mathcal{S}_{j,k}^{i+\lceil c_n[w]_{A_\infty}\rceil}}\int_P A\left(2^{j+1}|f|\right)w.\end{aligned}$$

Taking into account that

$$\|f\|_{A(w),P}\leq 2^{-j}\Rightarrow\frac{1}{w(P)}\int_P A\left(2^j|f|\right)w\leq 1$$

we can bound the sum in the last term as follows.

$$\begin{aligned}
&\sum_{P\in\mathcal{S}_{j,k}^{i+\lceil c_n[w]_{A_\infty}\rceil}} \int_P A\left(2^{j+1}|f|\right) w \\
&\leq \kappa_A A(2) \sum_{P\in\mathcal{S}_{j,k}^{i+\lceil c_n[w]_{A_\infty}\rceil}} w(P)\frac{1}{w(P)}\int_P A\left(2^{j}|f|\right) w \\
&\leq \kappa_A A(2) \sum_{P\in\mathcal{S}_{j,k}^{i+\lceil c_n[w]_{A_\infty}\rceil}} w(P) \\
&\leq \frac{\kappa_A A(2)}{\kappa_A 4A(2)} w(Q) \leq \frac{1}{4} w(Q).
\end{aligned}$$

Hence

$$\begin{aligned}
w(Q)\|f\|_{A(w),Q} &\leq \frac{1}{2^{j+1}}w(Q) + \frac{A(2^{j+1})}{2^{j+1}}\kappa_A \int_{\tilde{E_Q}} A(|f|)\, w + \frac{1}{2^{j+1}}\frac{1}{4}w(Q) \\
&\leq \left(\frac{1}{2}+\frac{1}{4}\right) w(Q)\|f\|_{A(w),Q} + \frac{A(2^{j+1})}{2^{j+1}}\kappa_A \int_{\tilde{E_Q}} A(|f|)\, w \\
&= \frac{3}{4} w(Q)\|f\|_{A(w),Q} + \frac{A(2^{j+1})}{2^{j+1}}\kappa_A \int_{\tilde{E_Q}} A(|f|)\, w,
\end{aligned}$$

from which readily follows the desired conclusion. □

The following lemma will be also used repeatedly.

Lemma 3.3. *Let $w \in A_\infty$ and $\mathcal{S}$ a η-sparse family of cubes. Then*

$$\sum_{Q\in\mathcal{S}} w(Q) \leq c_n[w]_{A_\infty} w\left(\bigcup_{Q\in\mathcal{S}} Q\right).$$

Proof. We can assume that $\mathcal{S} = \bigcup_{k=0}^\infty \mathcal{S}_k$ where $\{\mathcal{S}_k\}$ is an increasing sequence of finite sparse families. Now we fix k and consider $\mathcal{S}_k^*$ the family of maximal cubes of $\mathcal{S}_k$ with respect to the inclusion. Then

$$\sum_{Q\in\mathcal{S}_k} w(Q) = \sum_{Q\in\mathcal{S}_k^*}\sum_{P\in\mathcal{S}_k,\, P\subset Q} w(P).$$

Note that

$$\begin{aligned}
\sum_{P\in\mathcal{S}_k,\, P\subset Q} w(P) &\leq \frac{1}{\eta}\sum_{P\in\mathcal{S}_k,\, P\subset Q}\frac{1}{|P|}w(P)|E_P| \leq \frac{1}{\eta}\sum_{P\in\mathcal{S}_k,\, P\subset Q}\inf_{z\in P} M(w\chi_Q)(z)|E_P| \\
&\leq \frac{1}{\eta}\sum_{P\in\mathcal{S}_k,\, P\subset Q}\int_{E_P} M(w\chi_Q) \leq \frac{1}{\eta}\int_Q M(w\chi_Q) \\
&= \frac{1}{\eta}\frac{1}{w(Q)}\int_Q M(w\chi_Q)w(Q) \leq \frac{1}{\eta}[w]_{A_\infty} w(Q).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{Q\in\mathcal{S}^*}\sum_{P\in\mathcal{S}_k,\, P\subset Q} w(P) &\leq \frac{1}{\eta}[w]_{A_\infty}\sum_{Q\in\mathcal{S}_k^*} w(Q) = \frac{1}{\eta}[w]_{A_\infty} w\left(\bigcup_{Q\in\mathcal{S}_k^*} Q\right) \\
&= \frac{1}{\eta}[w]_{A_\infty} w\left(\bigcup_{Q\in\mathcal{S}_k} Q\right) \leq \frac{1}{\eta}[w]_{A_\infty} w\left(\bigcup_{Q\in\mathcal{S}} Q\right)
\end{aligned}$$

Consequently

$$\sum_{Q\in\mathcal{S}_k} w(Q)\leq \frac{1}{\eta}[w]_{A_\infty} w\left(\bigcup_{Q\in\mathcal{S}} Q\right)$$

and letting $k\to\infty$ we are done. □

To end the section we provide a result on weighted maximal functions.

Lemma 3.4. *Let A a Young function such that $A(st)\leq \kappa A(s)A(t)$. Let $\mathcal{D}_j$ $j=1,\dots,k$ be dyadic grids and let w a weight. Then*

$$w\left(\left\{x\in\mathbb{R}^n : M^{\mathcal{F}}_{A(w)}f(x)>t\right\}\right)\leq \kappa c_n\int_{\mathbb{R}^d} A\left(\frac{|f(x)|}{t}\right)w(x)dx$$

where $\mathcal{F}=\bigcup_{j=1}^{3^n}\mathcal{D}_j$.

Proof. Let $t>0$. Note that

$$M^{\mathcal{F}}_{A(w)}f(x)\leq \sum_{j=1}^{3^n} M^{\mathcal{D}_j}_{A(w)}f(x).$$

Then taking into account that for any dyadic lattice $\mathcal{D}$ it is not hard to check that

$$w\left(\left\{x\in\mathbb{R}^n : M^{\mathcal{D}}_{A(w)}f(x)>\lambda\right\}\right)\leq \int_{\mathbb{R}^d} A\left(\frac{|f(x)|}{\lambda}\right)w(x)dx,$$

(see [20, Section 15] for the case $A(t)=t$, the general case is analogous), we have that

$$\begin{aligned}
&w\left(\left\{x\in\mathbb{R}^n : M^{\mathcal{F}}_{A(w)}f(x)>t\right\}\right)\\
&\leq w\left(\left\{x\in\mathbb{R}^n : \sum_{j=1}^{3^n} M^{\mathcal{D}_j}_{A(w)}f(x)>t\right\}\right)\\
&\leq \sum_{j=1}^{3^n} w\left(\left\{x\in\mathbb{R}^n : M^{\mathcal{D}_j}_{A(w)}f(x)>\frac{t}{3^n}\right\}\right)\\
&\leq \sum_{j=1}^{3^n}\int_{\mathbb{R}^d} A\left(\frac{3^n|f(x)|}{t}\right)w(x)dx\\
&\leq c_n\kappa\int_{\mathbb{R}^d} A\left(\frac{|f(x)|}{t}\right)w(x)dx
\end{aligned}$$

and we are done. □

3.3. **Proof of Theorem 1.1.**

3.3.1. *Calderón-Zygmund operators.* Using pointwise sparse domination it suffices to settle the result for a sparse operator $A_{\mathcal{S}}$ where $\mathcal{S}$ is a sparse family contained in a dyadic lattice $\mathcal{D}$.

Let $G=\left\{\frac{A_{\mathcal{S}}(fv)(x)}{v(x)}>1\right\}\setminus\left\{M^{\mathcal{D}}_{uv}(f)>\frac{1}{2}\right\}$ and assume that $\|f\|_{L^1(uv)}=1$. Then it suffices to prove that

$$uv(G)\leq c_{n,p}[uv]_{A_\infty}[u]_{A_1}\log\left(e+[uv]_{A_\infty}[u]_{A_1}[v]_{A_p(u)}\right)+\frac{1}{2}uv(G).$$

If we denote $g=\chi_G$ then

$$\begin{aligned}
uv(G)&\leq \sum_{Q\in\mathcal{S}_j}\langle fv\rangle_{Q,1}\langle g\rangle^u_{Q,1}u(Q)\\
&\leq [u]_{A_1}\sum_{Q\in\mathcal{S}}\langle f\rangle^{uv}_{Q,1}\langle g\rangle^u_{Q,1}uv(Q)
\end{aligned}$$

and it suffices to prove that

$$
\begin{aligned}
&[u]_{A_1}\sum_{Q\in\mathcal{S}}\langle f\rangle_{Q,1}^{uv}\langle g\rangle_{Q,1}^{u}uv(Q)\\
&\leq c_{n,p}[uv]_{A_\infty}[u]_{A_1}\log\left(e+[uv]_{A_\infty}[u]_{A_1}[v]_{A_p(u)}\right)+\frac{1}{2}uv(G).
\end{aligned}
$$

We split the sparse family as follows. Let $Q\in\mathcal{S}_{k,j}$, $k,j\geq 0$ if

$$
\begin{aligned}
2^{-j-1}&<\langle f\rangle_{Q,1}^{uv}\leq 2^{-j},\\
2^{-k-1}&<\langle g\rangle_{Q,1}^{u}\leq 2^{-k}.
\end{aligned}
$$

Let us call

$$s_{k,j}=\sum_{Q\in\mathcal{S}_{k,j}}\langle f\rangle_{Q,1}^{uv}\langle g\rangle_{Q,1}^{u}uv(Q).$$

We claim that

$$s_{k,j}\leq\begin{cases}c_n2^{-k}[uv]_{A_\infty},\\ c_{n,p}[uv]_{A_\infty}[v]_{A_p(u)}2^{-j}2^{k(p-1)}uv\left(G\right).\end{cases}$$

For the top estimate we argue as follows. Using Lemma 3.2 we have that there exist sets $\tilde{E}_Q\subset Q$ such that

$$\sum_{Q\in\mathcal{S}_{k,j}}\chi_{\tilde{E_Q}}(x)\leq\lceil c_n[uv]_{A_\infty}\rceil$$

and

$$\int_Q fuv\leq 4\int_{\tilde{E_Q}}fuv.$$

Then

$$
\begin{aligned}
s_{k,j}&\leq 2^{-k}\sum_{Q\in\mathcal{S}_{j,k}}\int_Q fuv\\
&\leq 4\cdot 2^{-k}\sum_{Q\in\mathcal{S}_{j,k}}\int_{\tilde{E_Q}}fuv \qquad (3.4)\\
&\leq c_n[uv]_{A_\infty}2^{-k}\int_{\mathbb{R}^n}fuv=c_n[uv]_{A_\infty}2^{-k}.
\end{aligned}
$$

For the lower estimate, using Lemma 3.3,

$$
\begin{aligned}
s_{k,j}&\leq 2^{-j}2^{-k}\sum_{Q\in\mathcal{S}_{j,k}}uv(Q)\\
&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\bigcup_{Q\in\mathcal{S}_{j,k}}Q\right)\\
&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n\,:\,M_ug>2^{-k-1}\right\}\right).
\end{aligned}
$$

Now note that since $v\in A_p(u)$, Lemma 2.6 yields

$$\frac{1}{u(Q)}\int_Q gu\leq\left(\frac{[v]_{A_p(u)}}{uv(Q)}\int_Q guv\right)^{\frac{1}{p}}.$$

Taking that into account, by Lemma 3.4

$$
\begin{aligned}
&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n\,:\,M_ug>2^{-k-1}\right\}\right)\\
&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n\,:\,M_{uv}g>2^{-kp-p}[v]_{A_p(u)}^{-1}\right\}\right)\\
&\leq c_{n,p}[uv]_{A_\infty}[v]_{A_p(u)}2^{-j}2^{k(p-1)}uv\left(G\right).
\end{aligned}
$$

Combining the estimates above

$$\begin{aligned} uv(G) &\leq [u]_{A_1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}s_{k,j} \\ &\leq \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\min\left\{c_n 2^{-k}[u]_{A_1}[uv]_{A_\infty}, c_{n,p}[u]_{A_1}[uv]_{A_\infty}[v]_{A_p(u)}2^{-j}2^{-k}2^{kp}uv(G)\right\} \end{aligned}.$$

Now we are left with estimating the double sum. Applying Lemma 3.1 with

$$\gamma_1 = c_n[u]_{A_1}[uv]_{A_\infty}$$
$$\gamma_2 = c_{n,p}[u]_{A_1}[uv]_{A_\infty}[v]_{A_p(u)}$$

$\delta = p$, $\beta = uv(G)$, $\rho_1 = \rho_2 = 0$ and $\gamma = 1$ we are done.

3.3.2. *Commutators.* Using pointwise sparse domination it suffices to settle the result for the corresponding sparse operators. Let

$$G = \left\{\frac{\sum_{Q\in\mathcal{S}}|b-b_Q|^{m-h}\chi_Q\frac{1}{|Q|}\int_Q|b-b_Q|^h fv}{v(x)} > 1\right\}\setminus\left\{M^{\mathcal{D}}_{L(\log L)^{\frac{h}{r}}(uv)}(f) > \frac{1}{2}\right\}.$$

Assume that $\|b\|_{Osc_{\exp L^r}} = 1$. It suffices to prove that

$$uv(G) \leq c\varphi_{m,h}(u,v)\int_{\mathbb{R}^n}\Phi_{\frac{h}{r}}(|f(x)|)dx + \frac{1}{2}uv(G)$$

where

$$\varphi_{m,h}(u,v) = [u]_{A_1}[uv]_{A_\infty}^{1+\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}\log\left(e + [u]_{A_1}[uv]_{A_\infty}^{1+\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}[v]_{A_p(u)}\right)^{1+\frac{h}{r}}.$$

If we denote $g = \chi_G$ then

$$\begin{aligned} uv(G) &\leq \sum_{Q\in\mathcal{S}}\frac{1}{|Q|}\int_Q|b-b_Q|^h fv\int_Q|b-b_Q|^{m-h}u \\ &\leq \sum_{Q\in\mathcal{S}}\frac{u(Q)}{|Q|}\int_Q|b-b_Q|^h fv\frac{1}{u(Q)}\int_Q|b-b_Q|^{m-h}u \\ &\leq [u]_{A_1}\sum_{Q\in\mathcal{S}}\frac{1}{uv(Q)}\int_Q|b-b_Q|^h fuv\frac{1}{u(Q)}\int_Q|b-b_Q|^{m-h}uuv(Q) \\ &\leq [u]_{A_1}\sum_{Q\in\mathcal{S}}\Big(\||b-b_Q|^h\|_{\exp L^{r/h}(uv),Q}\|f\|_{L(\log L)^{\frac{h}{r}}(uv),Q} \\ &\quad\times\||b-b_Q|^{m-h}\|_{\exp L^{r/(m-h)}(uv),Q}\|g\|_{L(\log L)^{\frac{m-h}{r}}(u),Q}uv(Q)\Big) \\ &\leq c\|b\|^m_{Osc_{\exp L^r}}[uv]_{A_\infty}^{\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}\sum_{Q\in\mathcal{S}}\|f\|_{L(\log L)^{\frac{h}{r}}(uv),Q}\|g\|_{L(\log L)^{\frac{m-h}{r}}(u),Q}uv(Q) \\ &= c[uv]_{A_\infty}^{\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}\sum_{Q\in\mathcal{S}}\|f\|_{L(\log L)^{\frac{h}{r}}(uv),Q}\|g\|_{L(\log L)^{\frac{m-h}{r}}(u),Q}uv(Q). \end{aligned}$$

We split the sparse family as follows. $Q\in\mathcal{S}_{k,j}$, $k,j\geq 0$ if

$$2^{-j-1} < \|f\|_{L(\log L)^{\frac{h}{r}}(uv),Q} \leq 2^{-j},$$
$$2^{-k-1} < \|g\|_{L(\log L)^{\frac{m-h}{r}}(u),Q} \leq 2^{-k}.$$

Then

$$\sum_{Q\in\mathcal{S}}\|f\|_{L(\log L)^{\frac{h}{r}}(uv),Q}\|g\|_{L(\log L)^{\frac{m-h}{r}}(u),Q}uv(Q) = \sum_{k,j\geq 0}s_{k,j}$$

where

$$s_{k,j}=\sum_{Q\in\mathcal{S}_{k,j}}\|f\|_{L(\log L)^{\frac{h}{r}}(uv),Q}\|g\|_{L(\log L)^{\frac{m-h}{r}}(u),Q}uv(Q).$$

Now we observe that

$$s_{k,j}\leq\begin{cases}c_n[uv]_{A_\infty}2^{-k}j^{\frac{h}{r}},\\ c_{n,p,m}[uv]_{A_\infty}[v]_{A_p(u)}2^{-j}2^{k(p-1)}k^{\frac{m-h}{r}p}uv(G)uv\left(G\right).\end{cases}$$

For the top estimate we use Lemma 3.2 with $w=uv$ and $A(t)=\Phi_{\frac{h}{r}}(t)$, and we have that

$$uv(Q)\|f\|_{L(\log L)^{\frac{h}{r}}(uv),Q}\leq cj^{\frac{h}{r}}\int_{\tilde{E_Q}}\Phi_{\frac{h}{r}}\left(|f|\right)uv.$$

with

$$\sum_{Q\in\mathcal{S}_{k,j}}\chi_{\tilde{E_Q}}(x)\leq\lceil c_n[uv]_{A_\infty}\rceil\,.$$

Then

$$\begin{aligned}s_{k,j}&\leq 2^{-k}j^{\frac{h}{r}}\sum_{Q\in\mathcal{S}_{j,k}}\int_{\tilde{E_Q}}\Phi_{\frac{h}{r}}\left(|f|\right)uv.\\&\leq 2\cdot 2^{-k}j^{\frac{h}{r}}\sum_{Q\in\mathcal{S}_{j,k}}\int_{\tilde{E_Q}}\Phi_{\frac{h}{r}}\left(|f|\right)uv.\\&\leq c_n[uv]_{A_\infty}2^{-k}j^{\frac{h}{r}}\int_{\mathbb{R}^n}\Phi_{\frac{h}{r}}\left(|f|\right)uv.\end{aligned}$$

For the lower estimate, by Lemma 3.3

$$\begin{aligned}s_{k,j}&\leq 2^{-j}2^{-k}\sum_{Q\in\mathcal{S}_{j,k}}uv(Q)\\&=c[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\bigcup_{Q\in\mathcal{S}_{j,k}}Q\right)\\&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n\,:\,M_{L(\log L)^{\frac{m-h}{r}}(u)}g>2^{-k-1}\right\}\right).\end{aligned}$$

Taking into account Lemma 2.6

$$\|g\|_{L(\log L)^{\frac{m-h}{r}}(u)}\leq\|g\|_{[v]_{A_p(u)}L^p(\log L)^{p\frac{m-h}{r}}(uv)}.$$

That estimate combined with Lemma 3.4 allows us to argue as follows

$$\begin{aligned}&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n\,:\,M_{[v]_{A_p(u)}L^p(\log L)^{p\frac{m-h}{r}}(uv)}g>2^{-k-1}\right\}\right)\\&\leq c_{n,p}[uv]_{A_\infty}2^{-j}2^{-k}\int_{\mathbb{R}^d}[v]_{A_p(u)}\Phi_{\frac{m-h}{r}}\left(2^{k+1}g\right)^p.\\&\leq c_{n,p}[uv]_{A_\infty}[v]_{A_p(u)}2^{-j}2^{-k}\Phi_{\frac{m-h}{r}}\left(2^{k+1}\right)^p uv(G).\\&\leq c_{n,p}[uv]_{A_\infty}[v]_{A_p(u)}2^{-j}2^{-k}2^{kp+p}\log\left(e+2^{k+1}\right)^{\frac{m-h}{r}p}uv(G).\\&\leq c_{n,p,m}[uv]_{A_\infty}[v]_{A_p(u)}2^{-j}2^{k(p-1)}k^{\frac{m-h}{r}p}uv(G).\end{aligned}$$

Combining the estimates above

$$\begin{aligned}uv(G) &\leq c[uv]_{A_\infty}^{\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}[u]_{A_1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}s_{k,j}\\ &\leq \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\min\Big\{cc_{n,p,m}[uv]_{A_\infty}^{1+\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}[u]_{A_1}[v]_{A_p(u)}2^{-j}2^{k(p-1)}k^{\frac{m-h}{r}p}uv(G),\\ &\quad cc_n[uv]_{A_\infty}^{1+\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}[u]_{A_1}2^{-k}j^{\frac{h}{r}}\int_{\mathbb{R}^n}\Phi_{\frac{h}{r}}(|f|)\,uv\Big\}.\end{aligned}$$

We end up the proof applying Lemma 3.1, with $\gamma_1 = cc_n[u]_{A_1}[uv]_{A_\infty}^{1+\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}\int_{\mathbb{R}^n}\Phi_{\frac{h}{r}}(|f|)\,uv$, $\gamma_2 = cc_{n,p,m}[u]_{A_1}[uv]_{A_\infty}^{1+\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}[v]_{A_p(u)}$, $\beta = uv(G)$, $\delta = p-1$, $\gamma = 1$, $\rho_1 = \frac{h}{r}$ and $\rho_2 = \frac{m-h}{r}p$.

3.3.3. *Rough singular integrals.* Let us fix a dyadic lattice $\mathcal{D}$ and let $\mathcal{D}_j$, $j = 1, \ldots, 3^n$, be the lattices obtained using the 3^n dyadic lattices trick (Lemma 2.1). Now let

$$G = \left\{\frac{T_\Omega(fv)(x)}{v(x)} > 1\right\}\setminus\left\{M_{uv}^{\mathcal{F}}(f) > \frac{1}{2}\right\}$$

where $\mathcal{F} = \bigcup_{j=1}^{3^n}\mathcal{D}_j$ and assume that $\|f\|_{L^1(uv)} = \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} = 1$.

Then it suffices to prove that

$$uv(G) \leq c_{n,p}[uv]_{A_\infty}[u]_{A_\infty}[u]_{A_1}\log\left(e + [uv]_{A_\infty}[u]_{A_\infty}[u]_{A_1}[v]_{A_p(u)}\right) + \frac{1}{2}uv(G).$$

Note that

$$uv(G) \leq \left|\int_{\mathbb{R}^n}\frac{T_\Omega(fv)}{v}uvg\right| = \left|\int_{\mathbb{R}^n}T_\Omega(fv)ug\right|$$

where $g \simeq \chi_G$. Then for $s = 1 + \frac{1}{2\tau_n[u]_{A_\infty}}$, note that, arguing as in [24]

$$\frac{u^s(G\cap Q)}{u^s(Q)} \leq c_n\left(\frac{u(G\cap Q)}{u(Q)}\right)^{\frac{1}{2}}.$$

Taking into account (2.1) and Remark 2.3 we have that

$$\begin{aligned}uv(G) &\leq c_n s'\sum_{m=1}^{3^n}\sum_{Q\in\mathcal{S}_m}\langle fv\rangle_{Q,1}\langle\chi_G u\rangle_{Q,s}\\ &= c_n s'\sum_{m=1}^{3^n}\sum_{Q\in\mathcal{S}_m}\langle fv\rangle_{Q,1}\langle\chi_G\rangle_{Q,s}^{u^s}\langle u\rangle_{Q,s}\\ &\leq c_n[u]_{A_\infty}[u]_{A_1}\sum_{m=1}^{3^n}\sum_{Q\in\mathcal{S}_m}\langle f\rangle_{Q,1}^{uv}\langle g\rangle_{Q,2s}^{u}uv(Q)\end{aligned}$$

where $g = \chi_G$ and each $\mathcal{S}_j \subset \mathcal{D}_j$.

Before continuing with the proof one remark is in order.

Remark 3.5. Note that, since pointwise domination is not available, we need to remove the cubes where the maximal function is large from the sparse family using just one maximal function. By Lemma 3.4, choosing $M_{uv}^{\mathcal{F}}$, we avoid the dependence on the doubling constant of the measure $uvdx$ that M_{uv} would have introduced.

After that remark we continue with the proof. Note that it suffices to prove that for each m,

$$\begin{aligned}&c_n[u]_{A_\infty}[u]_{A_1}\sum_{Q\in\mathcal{S}_m}\langle f\rangle_{Q,1}^{uv}\langle g\rangle_{Q,2s}^{u}uv(Q)\\ &\leq c_{n,p}[uv]_{A_\infty}[u]_{A_\infty}[u]_{A_1}\log\left(e + [uv]_{A_\infty}[u]_{A_\infty}[u]_{A_1}[v]_{A_p(u)}\right) + \frac{1}{2\cdot 3^n}uv(G).\end{aligned}$$

Let us fix m and, for the sake of simplicity, denote $\mathcal{S}_m = \mathcal{S}$. Taking into account the definition of G, since we removed the set where $M_{uv}^{\mathcal{F}}(f) > \frac{1}{2}$ we can split the sparse $\mathcal{S}$ family as follows $Q \in \mathcal{S}_{k,j}$, $k, j \geq 0$ if

$$2^{-j-1} < \langle f \rangle_{Q,1}^{uv} \leq 2^{-j},$$
$$2^{-k-1} < \langle g \rangle_{Q,2s}^{u} \leq 2^{-k}.$$

Let us define

$$s_{k,j} = \sum_{Q \in \mathcal{S}_{k,j}} \langle f \rangle_{Q,1}^{uv} \langle g \rangle_{Q,2s}^{u} uv(Q).$$

Now we observe that

$$s_{k,j} \leq \begin{cases} c_n 2^{-k} [uv]_{A_\infty}, \\ c_{n,p} [uv]_{A_\infty} [v]_{A_p(u)} 2^{-j} 2^{(2ps-1)k} uv\left(G\right). \end{cases}$$

For the top estimate we argue as we did in (3.4). For the lower estimate, using Lemma 3.3,

$$\begin{aligned} s_{k,j} &\leq 2^{-j} 2^{-k} \sum_{Q \in \mathcal{S}_{j,k}} uv(Q) \\ &\leq c_n [uv]_{A_\infty} 2^{-j} 2^{-k} uv\left(\bigcup_{Q \in \mathcal{S}_{j,k}} Q \right) \\ &\leq c_n [uv]_{A_\infty} 2^{-j} 2^{-k} uv\left(\left\{ x \in \mathbb{R}^n \,:\, (M_u g)^{\frac{1}{2s}} > 2^{-k-1} \right\} \right). \end{aligned}$$

Since $v \in A_p(u)$, taking into account Lemmas 2.6 and 3.4

$$\begin{aligned} &\leq c_n [uv]_{A_\infty} 2^{-j} 2^{-k} uv\left(\left\{ x \in \mathbb{R}^n \,:\, (M_u g)^{\frac{1}{2s}} > 2^{-k-1} \right\} \right) \\ &\leq c_n [uv]_{A_\infty} 2^{-j} 2^{-k} uv\left(\left\{ x \in \mathbb{R}^n \,:\, \left([v]_{A_p(u)} M_{uv} g \right)^{\frac{1}{2sp}} > 2^{-k-1} \right\} \right) \\ &\leq c_n [uv]_{A_\infty} 2^{-j} 2^{-k} uv\left(\left\{ x \in \mathbb{R}^n \,:\, M_{uv} g > 2^{-2spk-2sp} [v]_{A_p(u)}^{-1} \right\} \right) \\ &\leq c_{n,p} [uv]_{A_\infty} [v]_{A_p(u)} 2^{-j} 2^{(2ps-1)k} uv\left(G\right). \end{aligned}$$

Combining the estimates above

$$\begin{aligned} c_n [u]_{A_\infty} [u]_{A_1} \sum_{Q \in \mathcal{S}} \langle f \rangle_{Q,1}^{uv} \langle g \rangle_{Q,2s}^{u} uv(Q) &= c_n [u]_{A_\infty} [u]_{A_1} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} s_{k,j} \\ \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \min \left\{ c_n \kappa_{u,v} 2^{-k}, c_{n,p} \kappa_{u,v} [v]_{A_p(u)} 2^{-j} 2^{(2ps-1)k} uv\left(G\right) \right\} \end{aligned}$$

where $\kappa_{u,v} = [uv]_{A_\infty} [u]_{A_\infty} [u]_{A_1}$. We end the proof using Lemma 3.1, with $\gamma_1 = c_n \kappa_{u,v}$, $\gamma_2 = c_{n,p} \kappa_{u,v} [v]_{A_p(u)}$, $\beta = uv\left(G\right)$, $\delta = 2ps - 1$, $\gamma = 3^n$ and $\rho_1 = \rho_2 = 0$.

3.4. **Proof of Theorem 1.2.**

3.4.1. *Calderón-Zygmund operators.* Using pointwise sparse domination it suffices to settle the result for a sparse operator $A_{\mathcal{S}}$.

Let $G = \left\{ \frac{A_{\mathcal{S}}(fv)(x)}{v(x)} > 1 \right\} \setminus \left\{ M_v^{\mathcal{D}}(f) > \frac{1}{2} \right\}$ and assume that $f \geq 0$ and $\|f\|_{L^1(uv)} = 1$. Then it suffices to prove that

$$uv(G) \leq c_{n,p} [v]_{A_1} [v]_{A_\infty} [u]_{A_1(v)} \log\left(e + [uv]_{A_\infty} [v]_{A_1} [u]_{A_1(v)} \right) + \frac{1}{2} uv(G).$$

If we denote $g = \chi_G$ then

$$\begin{aligned} uv(G) &\leq \sum_{Q\in\mathcal{S}} \langle fv\rangle_{Q,1} \int_Q gu \\ &= \sum_{Q\in\mathcal{S}_j} \langle f\rangle^v_{Q,1} \frac{v(Q)}{|Q|} \int_Q gu \\ &\leq [v]_{A_1} \sum_{Q\in\mathcal{S}_j} \langle f\rangle^v_{Q,1} \int_Q guv. \\ &= [v]_{A_1} \sum_{Q\in\mathcal{S}_j} \langle f\rangle^v_{Q,1} \langle g\rangle^{uv}_{Q,1} uv(Q) \end{aligned}$$

and it suffices to prove that

$$\begin{aligned} &[v]_{A_1} \sum_{Q\in\mathcal{S}} \langle f\rangle^v_{Q,1} \langle g\rangle^{uv}_{Q,1} uv(Q) \\ &\leq c_{n,p}[v]_{A_1}[v]_{A_\infty}[u]_{A_1(v)} \log\left(e + [uv]_{A_\infty}[v]_{A_1}[u]_{A_1(v)}\right) + \frac{1}{2}uv(G). \end{aligned}$$

We split the sparse family as follows. Let $Q \in \mathcal{S}_{k,j}$, $k, j \geq 0$ if

$$\begin{aligned} 2^{-j-1} &< \langle f\rangle^v_{Q,1} \leq 2^{-j}, \\ 2^{-k-1} &< \langle g\rangle^{uv}_{Q,1} \leq 2^{-k}. \end{aligned}$$

Let us call

$$s_{k,j} = \sum_{Q\in\mathcal{S}_{k,j}} \langle f\rangle^v_{Q,1} \langle g\rangle^{uv}_{Q,1} uv(Q).$$

Now we observe that

$$s_{k,j} \leq \begin{cases} c_n 2^{-k}[u]_{A_1(v)}[v]_{A_\infty}, \\ c_n[uv]_{A_\infty} 2^{-j}2^{-k} uv(G). \end{cases}$$

For the top estimate we argue as follows. Using Lemma 3.2 we have that

$$\int_Q fv \leq 4\int_{\tilde{E_Q}} fv$$

where $\tilde{E}_Q \subset Q$ and

$$\sum_{Q\in\mathcal{S}_{k,j}} \chi_{\tilde{E_Q}}(x) \leq \lceil c_n[v]_{A_\infty} \rceil .$$

Then

$$\begin{aligned} s_{k,j} &\leq 2^{-k} \sum_{Q\in\mathcal{S}_{j,k}} \frac{uv(Q)}{v(Q)} \int_Q fv \\ &\leq 2\cdot 2^{-k} \sum_{Q\in\mathcal{S}_{j,k}} \frac{uv(Q)}{v(Q)} \int_{\tilde{E_Q}} fv \\ &\leq 2\cdot 2^{-k}[u]_{A_1(v)} \sum_{Q\in\mathcal{S}_{j,k}} \int_{\tilde{E_Q}} fuv \qquad (3.5) \\ &\leq c_n 2^{-k}[u]_{A_1(v)}[v]_{A_\infty} \int_{\mathbb{R}^n} fuv. \\ &= c_n 2^{-k}[u]_{A_1(v)}[v]_{A_\infty}. \end{aligned}$$

For the lower estimate, using Lemma 3.3,

$$\begin{aligned}
s_{k,j} &\leq 2^{-j}2^{-k}\sum_{Q\in\mathcal{S}_{j,k}} uv(Q)\\
&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\bigcup_{Q\in\mathcal{S}_{j,k}} Q\right)\\
&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n\,:\,M^{\mathcal{D}}_{uv}(g)>2^{-k-1}\right\}\right).
\end{aligned}$$

Now using the weak-type $(1,1)$ of M_{uv} (Lemma 3.4)

$$\begin{aligned}
&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n\,:\,M^{\mathcal{D}}_{uv}(g)>2^{-k-1}\right\}\right)\\
&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv(G).
\end{aligned}$$

Combining the estimates above,

$$\begin{aligned}
uv(G) &\leq [v]_{A_1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} s_{k,j}\\
&\leq \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\min\left\{c_n2^{-k}[v]_{A_1}[v]_{A_\infty}[u]_{A_1(v)}, c_n[v]_{A_1}[uv]_{A_\infty}2^{-j}2^{-k}uv\,(G)\right\}.
\end{aligned}$$

An application of Lemma 3.1 with

$$\gamma_1 = c_n[v]_{A_1}[v]_{A_\infty}[u]_{A_1(v)} \qquad \gamma_2 = c_n[v]_{A_1}[uv]_{A_\infty}$$

$\delta=0$, $\beta=uv(G)$, $\gamma=1$ and $\rho_1=\rho_2=0$ ends the proof.

3.4.2. *Commutators.* Using pointwise sparse domination it suffices to settle the result for the corresponding sparse operators. Let

$$G=\left\{\frac{\sum_{Q\in\mathcal{S}}|b(x)-b_Q|^{m-h}\chi_Q(x)\frac{1}{|Q|}\int_Q|b-b_Q|^h fv}{v(x)}>1\right\}\setminus\left\{M^{\mathcal{D}}_{L(\log L)^{\frac{h}{r}}(v)}(f)>\frac{1}{2}\right\}$$

Assume that $\|b\|_{Osc_{\exp L^r}}=1$. It suffices to prove that

$$uv(G)\leq c\varphi_{m,h}(u,v)\int_{\mathbb{R}^n}\Phi_{\frac{h}{r}}(|f(x)|)dx+\frac{1}{2}uv(G).$$

where$\gamma_1=cc_n[u]_{A_1}[uv]_{A_\infty}^{1+\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}\int_{\mathbb{R}^n}\Phi_{\frac{h}{r}}(|f|)\,uv$, $\gamma_2=cc_{n,p,m}[u]_{A_1}[uv]_{A_\infty}^{1+\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}[v]_{A_p(u)}$,

$$\varphi_{m,h}(u,v)=[u]_{A_1}[uv]_{A_\infty}^{1+\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}\log\left(e+[u]_{A_1}[uv]_{A_\infty}^{1+\frac{h}{r}}[u]_{A_\infty}^{\frac{m-h}{r}}[v]_{A_p(u)}\right)^{1+\frac{h}{r}}$$

If we denote $g = \chi_G$ then

$$\begin{aligned}
uv(G) &\leq \sum_{Q\in\mathcal{S}} \frac{1}{|Q|}\int_Q |b-b_Q|^h fv \int_Q |b-b_Q|^{m-h}u \\
&\leq \sum_{Q\in\mathcal{S}} \frac{1}{v(Q)}\int_Q |b-b_Q|^h fv \frac{v(Q)}{|Q|}\int_Q |b-b_Q|^{m-h}u \\
&\leq [v]_{A_1}\sum_{Q\in\mathcal{S}} \frac{1}{v(Q)}\int_Q |b-b_Q|^h fv \frac{1}{uv(Q)}\int_Q |b-b_Q|^{m-h}uvuv(Q) \\
&\leq [v]_{A_1}\sum_{Q\in\mathcal{S}} \Big(\||b-b_Q|^h\|_{\exp L^{r/h}(v),Q}\|f\|_{L(\log L)^{\frac{h}{r}}(v),Q} \\
&\quad \times \||b-b_Q|^{m-h}\|_{\exp L^{r/(m-h)}(uv),Q}\|g\|_{L(\log L)^{\frac{m-h}{r}}(uv),Q} uv(Q)\Big) \\
&\leq c\|b\|^m_{Osc_{\exp L^r}}[v]_{A_1}[v]^{\frac{h}{r}}_{A_\infty}[uv]^{\frac{m-h}{r}}_{A_\infty}\sum_{Q\in\mathcal{S}}\|f\|_{L(\log L)^{\frac{h}{r}}(v),Q}\|g\|_{L(\log L)^{\frac{m-h}{r}}(uv),Q} uv(Q) \\
&= c[v]_{A_1}[v]^{\frac{h}{r}}_{A_\infty}[uv]^{\frac{m-h}{r}}_{A_\infty}\sum_{Q\in\mathcal{S}}\|f\|_{L(\log L)^{\frac{h}{r}}(v),Q}\|g\|_{L(\log L)^{\frac{m-h}{r}}(uv),Q} uv(Q)
\end{aligned}$$

Let us split the sparse family as follows. Let $Q \in \mathcal{S}_{k,j}$, $k, j \geq 0$ if

$$\begin{aligned}
2^{-j-1} &< \|f\|_{L(\log L)^{\frac{h}{r}}(v),Q} \leq 2^{-j} \\
2^{-k-1} &< \|g\|_{L(\log L)^{\frac{m-h}{r}}(uv),Q} \leq 2^{-k}.
\end{aligned}$$

Let us call

$$s_{k,j} = \sum_{Q\in\mathcal{S}_{k,j}}\|f\|_{L(\log L)^{\frac{h}{r}}(v),Q}\|g\|_{L(\log L)^{\frac{m-h}{r}}(uv),Q} uv(Q).$$

Then

$$\sum_{Q\in\mathcal{S}}\|f\|_{L(\log L)^{\frac{h}{r}}(v),Q}\|g\|_{L(\log L)^{\frac{m-h}{r}}(uv),Q} uv(Q) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} s_{k,j}.$$

Now we observe that

$$s_{k,j} \leq \begin{cases} c_n[u]_{A_1(v)}[v]_{A_\infty}2^{-k}j^{\frac{h}{r}}\int_{\mathbb{R}^n}\Phi_{\frac{h}{r}}(|f|)\,uv \\ c_{n,p,m}c[uv]_{A_\infty}2^{-j}2^{k(p-1)}k^{\frac{m-h}{r}p}uv(G). \end{cases}$$

For the top estimate we use Lemma 3.2 with $w = v$ and $A(t) = \Phi_{\frac{h}{r}}(t)$, and we have that

$$v(Q)\|f\|_{L(\log L)^{\frac{h}{r}}(v),Q} \leq cj^{\frac{h}{r}}\int_{\tilde{E_Q}}\Phi_{\frac{h}{r}}(|f|)\,uv.$$

with

$$\sum_{Q\in\mathcal{S}_{k,j}}\chi_{\tilde{E_Q}}(x) \leq \lceil c_n[v]_{A_\infty}\rceil.$$

Then

$$\begin{aligned}
s_{k,j} &\leq 2^{-k}j^{\frac{h}{r}}\sum_{Q\in\mathcal{S}_{j,k}}\frac{uv(Q)}{v(Q)}\int_{\tilde{E_Q}}\Phi_{\frac{h}{r}}(|f|)\,v. \\
&\leq 2[u]_{A_1(v)}2^{-k}j^{\frac{h}{r}}\sum_{Q\in\mathcal{S}_{j,k}}\int_{\tilde{E_Q}}\Phi_{\frac{h}{r}}(|f|)\,uv. \\
&\leq c_n[u]_{A_1(v)}[v]_{A_\infty}2^{-k}j^{\frac{h}{r}}\int_{\mathbb{R}^n}\Phi_{\frac{h}{r}}(|f|)\,uv.
\end{aligned}$$

For the lower estimate, by Lemma 3.4

$$\begin{aligned} s_{k,j} &\leq 2^{-j}2^{-k}\sum_{Q\in\mathcal{S}_{j,k}} uv(Q) \\ &= c[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\bigcup_{Q\in\mathcal{S}_{j,k}} Q\right) \\ &\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n\,:\,M^{\mathcal{D}}_{L(\log L)^{\frac{m-h}{r}}(uv)}g>2^{-k-1}\right\}\right) \\ &\leq c_{n,p}[uv]_{A_\infty}2^{-j}2^{-k}\Phi_{\frac{m-h}{r}}\left(2^{k+1}\right)^p uv(G). \\ &\leq c_{n,p}[uv]_{A_\infty}2^{-j}2^{-k}2^{kp+p}\log\left(e+2^{k+1}\right)^{\frac{m-h}{r}p} uv(G). \\ &\leq c_{n,p,m}[uv]_{A_\infty}2^{-j}2^{k(p-1)}k^{\frac{m-h}{r}p}uv(G). \end{aligned}$$

Combining the estimates above

$$\begin{aligned} uv(G) &\leq c[v]_{A_1}[v]_{A_\infty}^{\frac{h}{r}}[uv]_{A_\infty}^{\frac{m-h}{r}}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}s_{k,j} \\ &\leq \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\min\left\{cc_{n,p,m}[v]_{A_1}[v]_{A_\infty}^{\frac{h}{r}}[uv]_{A_\infty}^{\frac{m-h}{r}}2^{-j}2^{k(p-1)}k^{\frac{m-h}{r}p}uv(G),\right. \\ &\qquad\left. cc_n[v]_{A_1}[v]_{A_\infty}^{\frac{h}{r}}[uv]_{A_\infty}^{\frac{m-h}{r}}[u]_{A_1(v)}[v]_{A_\infty}2^{-k}j^{\frac{h}{r}}\int_{\mathbb{R}^n}\Phi_{\frac{h}{r}}(|f|)\,uv\right\} \end{aligned}$$

We end up the proof applying Lemma 3.1, with

$$\begin{aligned} \gamma_1 &= cc_n[v]_{A_1}[v]_{A_\infty}^{\frac{h}{r}}[uv]_{A_\infty}^{\frac{m-h}{r}}[u]_{A_1(v)}[v]_{A_\infty}2^{-k}j^{\frac{h}{r}}\int_{\mathbb{R}^n}\Phi_{\frac{h}{r}}(|f|)\,uv, \\ \gamma_2 &= cc_{n,p,m}[v]_{A_1}[v]_{A_\infty}^{\frac{h}{r}}[uv]_{A_\infty}^{\frac{m-h}{r}}2^{-j}2^{k(p-1)}k^{\frac{m-h}{r}p}, \end{aligned}$$

$\beta = uv(G)$, $\delta = p-1$, $\gamma = 1$, $\rho_1 = \frac{h}{r}$ and $\rho_2 = \frac{m-h}{r}p$.

3.4.3. *Rough singular integrals.* Let us fix a dyadic lattice $\mathcal{D}$ and let $\mathcal{D}_j$, $j = 1,\dots,3^n$, be the lattices obtained using the 3^n dyadic lattices trick. Now let

$$G = \left\{\frac{T_\Omega(fv)(x)}{v(x)} > 1\right\} \setminus \left\{M^{\mathcal{F}}_{uv}(f) > \frac{1}{2}\right\}$$

where $\mathcal{F} = \bigcup_{j=1}^{3^n}\mathcal{D}_j$ and assume that $\|f\|_{L^1(uv)} = \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} = 1$. Here again we choose $M^{\mathcal{F}}_{uv}$ instead of M_{uv} (see Remark 3.5). To settle the desired estimate it suffices to prove that

$$uv(G) \leq c_{n,p}[uv]_{A_\infty}[v]_{A_1}[u]_{A_1(v)}[v]_{A_\infty}\log\left(e+[uv]_{A_\infty}[v]_{A_1}\right) + \frac{1}{2}uv(G).$$

Note that

$$uv(G) \leq \left|\int_{\mathbb{R}^n}\frac{T_\Omega(fv)}{v}uvg\right| = \left|\int_{\mathbb{R}^n}T_\Omega(fv)ug\right|$$

where $g \simeq \chi_G$. Then for $s = 1+\frac{1}{2\tau_n[uv]_{A_\infty}}$, note that, arguing as in [24]

$$\frac{(uv)^s(G\cap Q)}{(uv)^s(Q)} \leq c_n\left(\frac{uv(G\cap Q)}{uv(Q)}\right)^{\frac{1}{2}}.$$

Taking that into account we have that

$$\begin{aligned}
uv(G) &\leq c_n[uv]_{A_\infty}\sum_{m=1}^{3^n}\sum_{Q\in\mathcal{S}_m}\langle fv\rangle_{Q,1}\langle\chi_G u\rangle_{Q,s}|Q| \\
&\leq c_n[uv]_{A_\infty}\sum_{m=1}^{3^n}\sum_{Q\in\mathcal{S}_m}\langle f\rangle_{Q,1}^{v}\frac{v(Q)}{|Q|}\langle\chi_G u\rangle_{Q,s}|Q| \\
&\leq c_n[uv]_{A_\infty}\sum_{m=1}^{3^n}\sum_{Q\in\mathcal{S}_m}\langle f\rangle_{Q,1}^{v}\langle\chi_G uv\rangle_{Q,s}|Q| \\
&\leq c_n[uv]_{A_\infty}[v]_{A_1}\sum_{m=1}^{3^n}\sum_{Q\in\mathcal{S}_m}\langle f\rangle_{Q,1}^{v}\langle\chi_G\rangle_{Q,s}^{(uv)^s}\left(\frac{(uv)^s(Q)}{|Q|}\right)^{\frac{1}{s}} \\
&\leq c_n[uv]_{A_\infty}[v]_{A_1}\sum_{m=1}^{3^n}\sum_{Q\in\mathcal{S}_m}\langle f\rangle_{Q,1}^{v}\left(\langle g\rangle_{Q,1}^{uv}\right)^{\frac{1}{2}}\left(\frac{(uv)^s(Q)}{|Q|}\right)^{\frac{1}{s}}|Q| \\
&\leq c_n[uv]_{A_\infty}[v]_{A_1}\sum_{m=1}^{3^n}\sum_{Q\in\mathcal{S}_m}\langle f\rangle_{Q,1}^{v}\left(\langle g\rangle_{Q,1}^{uv}\right)^{\frac{1}{2}}uv(Q).
\end{aligned}$$

Hence it suffices to prove that for every sparse family $\mathcal{S}_m$,

$$\begin{aligned}
&c_n[uv]_{A_\infty}[v]_{A_1}\sum_{Q\in\mathcal{S}_m}\langle f\rangle_{Q,1}^{v}\left(\langle g\rangle_{Q,1}^{uv}\right)^{\frac{1}{2}}uv(Q) \\
&\leq c_{n,p}[uv]_{A_\infty}[v]_{A_1}[u]_{A_1(v)}[v]_{A_\infty},\log\left(e+[uv]_{A_\infty}[v]_{A_1}\right)+\frac{1}{2\cdot 3^n}uv(G).
\end{aligned}$$

Let us fix m and, for the sake of simplicity, denote $\mathcal{S}_m=\mathcal{S}$. We split the sparse family $\mathcal{S}$ as follows $Q\in\mathcal{S}_{k,j}$, $k,j\geq 0$ if

$$\begin{aligned}
2^{-j-1}&<\langle f\rangle_{Q,1}^{v}\leq 2^{-j}, \\
2^{-k-1}&<\langle g\rangle_{Q,1}^{uv}\leq 2^{-k}.
\end{aligned}$$

We define

$$s_{k,j}=\sum_{Q\in\mathcal{S}_{k,j}}\langle f\rangle_{Q,1}^{v}\left(\langle g\rangle_{Q,1}^{uv}\right)^{\frac{1}{2}}uv(Q).$$

Now we observe that

$$s_{k,j}\leq\begin{cases}c_n 2^{-k}[u]_{A_1(v)}[v]_{A_\infty}\\ c_n[uv]_{A_\infty}2^{-j}2^{k}uv\left(G\right).\end{cases}$$

For the top estimate we argue as we did to get (3.5). For the lower estimate, using Lemma 3.3,

$$\begin{aligned}
s_{k,j}&\leq 2^{-j}2^{-k}\sum_{Q\in\mathcal{S}_{j,k}}uv(Q) \\
&=c[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\bigcup_{Q\in\mathcal{S}_{j,k}}Q\right) \\
&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n\,:\,(M_{uv}^{\mathcal{D}}g)^{\frac{1}{2}}>2^{-k-1}\right\}\right).
\end{aligned}$$

Since $v\in A_1(u)$, taking into account Lemmas 2.6 and 3.4,

$$\begin{aligned}
&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}uv\left(\left\{x\in\mathbb{R}^n\,:\,M_{uv}^{\mathcal{D}}g>2^{-2(k+1)}\right\}\right) \\
&\leq c_n[uv]_{A_\infty}2^{-j}2^{-k}2^{2(k+1)}uv\left(G\right) \\
&\leq c_n[uv]_{A_\infty}2^{-j}2^{k}uv\left(G\right).
\end{aligned}$$

Combining the estimates above

$$c_n[uv]_{A_\infty}[v]_{A_1}\sum_{Q\in\mathcal{S}}\langle f\rangle_{Q,1}^{v}\left(\langle g\rangle_{Q,1}^{uv}\right)^{\frac{1}{2}}uv(Q)=c_n[uv]_{A_\infty}[v]_{A_1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}s_{k,j}$$
$$\leq\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\min\left\{c_n[uv]_{A_\infty}[v]_{A_1}[u]_{A_1(v)}[v]_{A_\infty}2^{-k},c_n[uv]_{A_\infty}^2[v]_{A_1}2^{-j}2^{k}uv\left(G\right)\right\}.$$

We end the proof using Lemma 3.1, with

$$\gamma_1=c_n[uv]_{A_\infty}[v]_{A_1}[u]_{A_1(v)}[v]_{A_\infty},\qquad\gamma_2=c_n[uv]_{A_\infty}^2[v]_{A_1},$$

$\beta=uv\left(G\right)$, $\delta=1$ and $\gamma=3^n$.

Acknowledgment

The authors would like to thank Sheldy Ombrosi for his comments on an earlier version of this manuscript and for some enlightening discussions on this topic.

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