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Bergman projection induced by radial weight acting on growth spaces

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Abstract

Let ω be a radial weight on the unit disc of the complex plane $\mathbb{D}$ and denote by $\widehat{\omega}(r) = \int_r^1 \omega(s)\,ds$ the tail integrals. A radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if satisfies the upper doubling condition

$$\sup_{0<r<1} \frac{\widehat{\omega}(r)}{\widehat{\omega}\left(\frac{1+r}{2}\right)} < \infty.$$

If ν or ω belongs to $\widehat{\mathcal{D}}$, we describe the boundedness of the Bergman projection P_ω induced by ω on the growth space $L^\infty_{\widehat{\nu}} = \{f : \|f\|_{\infty,\nu} = \text{ess sup}_{z\in\mathbb{D}} |f(z)|\widehat{\nu}(z) < \infty\}$ in terms of neat conditions on the moments and/or the tail integrals of ω and ν. Moreover, we solve the analogous problem for P_ω from $L^\infty_{\widehat{\nu}}$ to the Bloch type space $\mathcal{B}^\infty_{\widehat{\nu}} = \{f \text{ analytic in} \mathbb{D} : \|f\|_{\mathcal{B}^\infty_{\widehat{\nu}}} = \sup_{z\in\mathbb{D}} (1-|z|)\widehat{\nu}(z)|f'(z)| < \infty\}$. Similar questions for exponentially decreasing radial weights will also be studied.

Keywords Bergman projection · Boundedness · Weighted sup-norm · Radial weight · Doubling weight · Exponential weight · Weighted Hardy space · Szegö projection

Mathematics Subject Classification 30H20 · 47G10

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1 Introduction and main results

The question of when the Bergman projection P_ω, induced by a radial weight ω on the unit disc $\mathbb{D}$ of the complex plane, is a bounded operator in given function spaces is fundamental in the theory of spaces of analytic functions on $\mathbb{D}$. This is not only due to the mathematical difficulties the question raises, but also to its numerous applications in operator theory. Indeed, bounded analytic projections can be used to establish duality relations and to obtain useful equivalent norms in spaces of analytic functions [6, 14, 15, 21]. Recently, radial weights ω such that P_ω is bounded from L^∞ to the classical Bloch space $\mathcal{B}$ have been described in [15]. In this paper we are also interested in the natural limit case $p = \infty$, and our main results provide complete characterizations of the radial weights ω (respectively, v) such that P_ω is bounded on the growth space induced by v, when v (resp. ω) satisfies an upper doubling condition. In addition, we will consider bounded projections onto weighted Bloch spaces in the setting of doubling weights, and we also obtain new results on the boundedness of P_ω, when ω and v belong to certain classes of exponentially decreasing weights. In order to present the precise statements of our results, some definitions are needed.

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in $\mathbb{D}$. For a nonnegative function $\omega \in L^1([0,1))$, the extension to $\mathbb{D}$, defined by $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, is called a radial weight. For $0 < p < \infty$ and such an ω, the weighted Lebesgue space L^p_ω consists of complex-valued measurable functions f on $\mathbb{D}$ such that

$$\|f\|^p_{L^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. The corresponding weighted Bergman space is $A^p_\omega = L^p_\omega \cap \mathcal{H}(\mathbb{D})$. Throughout this paper we assume $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\, ds > 0$ for all $z \in \mathbb{D}$, for otherwise $A^p_\omega = \mathcal{H}(\mathbb{D})$. We also consider the Lebesgue space L^∞ of complex-valued measurable functions f on $\mathbb{D}$ such that $\|f\|_\infty = \operatorname{ess\,sup}_{z\in\mathbb{D}}|f(z)| < \infty$ and the Hardy space $H^\infty = L^\infty \cap \mathcal{H}(\mathbb{D})$.

For a radial weight ω, the orthogonal Bergman projection P_ω from L^2_ω to A^2_ω is

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\, \omega(\zeta) dA(\zeta),$$

where B^ω_z are the reproducing kernels of A^2_ω. As usual, A^p_α stands for the classical weighted Bergman space induced by the standard radial weight $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, B^α_z are the kernels of A^2_α, and P_α denotes the corresponding Bergman projection.

One of the main obstacles throughout this work is the lack of explicit expressions for the Bergman reproducing kernel B^ω_z. For a radial weight ω, the kernel has the representation $B^\omega_z(\zeta) = \sum \overline{e_n(z)} e_n(\zeta)$ for each orthonormal basis $\{e_n\}$ of A^2_ω, and therefore we are basically forced to work with the formula $B^\omega_z(\zeta) = \sum_{n=0}^\infty \frac{(\overline{z}\zeta)^n}{2\omega_{2n+1}}$ induced by the normalized monomials. Here ω_{2n+1} are the odd moments of ω, and from now on we write $\omega_x = \int_0^1 r^x \omega(r)\, dr$ for all $x \ge 0$. Therefore the influence of the weight to the kernel is transmitted by its moments through this infinite sum, and nothing more than that can be said in general. This is in stark contrast with the neat expression $(1-\overline{z}\zeta)^{-(2+\alpha)}$ of the standard Bergman kernel B^α_z which is easy to work with. It is well known that for $1 < p < \infty$ and $\alpha > -1$, the Bergman projection P_α acts as a bounded operator from L^p_α to A^p_α [21, Section 4]. However, P_α is never bounded from L^∞ to H^∞. In fact, this is a general phenomenon (known e.g. in the isomorphic theory of Banach spaces) rather than a particular case.

Theorem A *There does not exist any bounded projection from L^∞ to H^∞. In particular, P_ω is not bounded from L^∞ to H^∞ for any radial weight.*

In view of the previous result it is natural to look for a substitute of H^∞ in the target space when the Bergman projection P_ω acts on L^∞. As for this question, it is known that the standard Bergman projection P_α is bounded and onto from L^∞ to the Bloch space $\mathcal{B}$, which consists of functions $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) + |f(0)| < \infty$ [21, Theorem 5.2]. Furthermore, it has been recently proved that the Bergman projection P_ω induced by a radial weight ω acts as a bounded operator from the space L^∞ to $\mathcal{B}$ if and only if $\omega \in \widehat{\mathcal{D}}$, and $P_\omega : L^\infty \to \mathcal{B}$ is bounded and onto if and only if $\omega \in \mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$ [15, Theorems 1-2-3]. Let us recall the reader that a radial weight ω belongs to $\widehat{\mathcal{D}}$, and it is called upper doubling, if there exists $C = C(\omega) > 0$ such that

$$\widehat{\omega}(r) \le C\widehat{\omega}\left(\frac{1+r}{2}\right), \quad r \to 1^-,$$

and $\omega \in \check{\mathcal{D}}$ if there exists $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\widehat{\omega}(r) \ge C\widehat{\omega}\left(1-\frac{1-r}{K}\right), \quad r \to 1^-.$$

We say that ω is a doubling weight if $\omega \in \mathcal{D}$. Standard weights belong to $\widehat{\mathcal{D}}$ but exponentially decreasing weights do not belong to $\widehat{\mathcal{D}}$. Moreover, the containment in $\widehat{\mathcal{D}}$ or $\check{\mathcal{D}}$ does not require differentiability, continuity or strict positivity. In fact, the weights in these classes may vanish on a relatively large part of each outer annulus $\{z : r \le |z| < 1\}$ of $\mathbb{D}$. For basic properties of the aforementioned classes, concrete nontrivial examples and further details, see [12, 13, 15] and the relevant references therein.

Next we study the properties of radial weights ω which are determinative so that P_ω is bounded on the growth space $L^\infty_v = \{f \text{ measurable} : \|f\|_{\infty,v} = \operatorname{ess\,sup}_{z\in\mathbb{D}}|f(z)|v(z) < \infty\}$, where v is a radial, continuous and decreasing weight with $\lim_{r\to 1^-} v(r) = 0$. We denote $H^\infty_v = L^\infty_v \cap \mathcal{H}(\mathbb{D})$. Sections 2–4 deal with cases of doubling weights, and in the previous definitions of spaces, v will be equal to $\widehat{\nu}(z) = \int_{|z|}^1 \nu(s)\,ds$ for a radial weight ν. In Sect. 4.1 we will study the boundedness of P_ω from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$ under some hypothesis on ν. In particular, assuming $\nu \in \widehat{\mathcal{D}}$ we will prove the following characterization of the pairs of radial weights (ω, ν) such that this fact happens.

Theorem 1 *Let ω be a radial weight and $\nu \in \widehat{\mathcal{D}}$. Then, the following conditions are equivalent:*

(i) $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ *is bounded;*

(ii) $\nu \in \check{\mathcal{D}}$ *and*

$$\sup_{x\ge 0} \frac{\nu_x}{\omega_{2x}} \left(\frac{\omega}{\widehat{\nu}}\right)_x < \infty; \tag{1.1}$$

(iii) $\omega \in \widehat{\mathcal{D}}$, $\nu \in \check{\mathcal{D}}$ *and*

$$\sup_{0\le r<1} \frac{\widehat{\nu}(r)}{\widehat{\omega}(r)} \int_r^1 \frac{\omega(s)}{\widehat{\nu}(s)}\,ds < \infty; \tag{1.2}$$

(iv) $\frac{\omega}{\widehat{\nu}} \in \mathcal{D}$ *and* $\nu \in \check{\mathcal{D}}$.

It is worth mentioning that $\int_r^1 \frac{\omega(s)}{\widehat{\nu}(s)}\,ds \geq \frac{\widehat{\omega}(r)}{\widehat{\nu}(r)}$ holds for all radial weights and that (1.2) means nothing else but that the function $\frac{1}{\widehat{\nu}}$ satisfies a Bekollé-Bonami type condition $B_{1,\omega}$, see [17, Theorem 3] for related results. As for the proof of Theorem 1, the condition (1.1) is deduced from (i) using the the reformulation

$$\sup_{a\in\mathbb{D}} \widehat{\nu}(a) \int_{\mathbb{D}} |B_a^\omega(\zeta)| \frac{\omega(\zeta)}{\widehat{\nu}(\zeta)} dA(\zeta) < \infty, \tag{1.3}$$

together with a Hausdorff-Young inequality [4, Theorem 6.1]. Then, it is proved that condition (1.1) implies that $\omega \in \widehat{\mathcal{D}}$, which, together with [14, Theorem 1], allows us to express the integral in (1.3) in terms of $\widehat{\omega}$ and $\widehat{\nu}$. Finally, some technical calculations imply that $\nu \in \check{\mathcal{D}}$ (see Proposition 13 below) and we get (ii). The proofs of (ii)⇔(iii)⇔(iv) involve several descriptions of the classes of radial weights $\widehat{\mathcal{D}}$ and $\check{\mathcal{D}}$. In order to prove (iii)⇒(i), [14, Theorem 1] is employed again.

In the case of standard weights $\omega(r) = (1-r^2)^\beta$, $\nu(r) = (1-r^2)^\alpha$, $\alpha, \beta > -1$, the projection P_ω is bounded on $L^\infty_{\widehat{\nu}}$, if and only if

$$\beta > \alpha. \tag{1.4}$$

This follows from the Forelli-Rudin estimates, see Ref. [21], Lemma 3.10 and the proof of Theorem 3.12. Since both ω and ν belong to $\mathcal{D}$, we conclude that (1.4) is an equivalent formulation of (1.1) and (1.2) in this simple case.

As a byproduct of Theorem 1 we deduce the next result which might be expected in view of Theorem A, because roughly speaking the spaces $L^\infty_{\widehat{\nu}}$ are "small" growth spaces for $\nu \in \widehat{\mathcal{D}} \setminus \check{\mathcal{D}}$.

Corollary 2 *If $\nu \in \widehat{\mathcal{D}} \setminus \check{\mathcal{D}}$ and ω is a radial weight, P_ω is not bounded from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$. Moreover, if $\nu \in \mathcal{D}$, there is a radial weight ω such that $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded.*

Theorem 1 is also related with the boundedness of the Szegö projection. In Sect. 3 we recall some literature references on the proof of Theorem A and its relations to the boundedness of the Szegö projection, since these can be used to give an alternative proof for one of the statements in Corollary 2.

In Sect. 4.2 we will study the boundedness of P_ω from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$ assuming some hypothesis on ω. The following main results will be proved using techniques similar to those employed in the proof of Theorem 1.

Theorem 3 *Let ν be a radial weight and $\omega \in \widehat{\mathcal{D}}$. Then, the following conditions are equivalent:*

(i) $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ *is bounded;*
(ii) $\nu \in \check{\mathcal{D}}$ *and* (1.1) *holds;*
(iii) $\nu \in \check{\mathcal{D}}$ *and* (1.2) *holds;*
(iv) $\frac{\omega}{\widehat{\nu}} \in \mathcal{D}$ *and* $\nu \in \mathcal{D}$.

Corollary 4 *If $\omega \in \widehat{\mathcal{D}} \setminus \check{\mathcal{D}}$ and ν is a radial weight, P_ω is not bounded from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$. Moreover, if $\omega \in \mathcal{D}$, there is a radial weight ν such that $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded.*

Bearing in mind Theorem A, [15, Theorem 1] and our previous results, it is also interesting to study the boundedness from L^∞_v to the Bloch type space

$$\mathcal{B}^\infty_v = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathcal{B}^\infty_v} = \sup_{z\in\mathbb{D}} (1-|z|)v(z)|f'(z)| < \infty \right\},$$

where v is a continuous and decreasing weight with $\lim_{r\to 1^-} v(r) = 0$. In Sect. 4.3 we will prove the following results for $v = \widehat{\nu}$.

Theorem 5 *Let ω be a radial weight and $\nu \in \widehat{\mathcal{D}}$. Then, the following conditions are equivalent:*

(i) $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ *is bounded;*
(ii) (1.1) *holds;*
(iii) $\omega \in \widehat{\mathcal{D}}$ *and* (1.2) *holds.*

Theorem 6 *Let ν be a radial weight and $\omega \in \widehat{\mathcal{D}}$. Then, the following conditions are equivalent:*

(i) $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ *is bounded;*
(ii) (1.1) *holds;*
(iii) (1.2) *holds.*

We point out that on the contrary to the boundedness of $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$, the boundedness of $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$, $\omega, \nu \in \widehat{\mathcal{D}}$, does not require that $\omega, \nu \in \check{\mathcal{D}}$.

Section 5 is devoted to the study of the cases with exponentially decreasing weights, where we use techniques different from those employed to prove Theorems 1–6. We postpone there the detailed definitions and the formulation of the results.

Finally, concerning the notation in this paper, we denote by C (respectively $C(\omega)$ etc.) generic positive constants independent of the variables involved (resp. depending only on ω), the values of which may vary from place to place. As usual, the notation $A \lesssim B$ or $B \gtrsim A$, where A and B are nonnegative functions, means that $A \le C\,B$, for some constant C. Furthermore, we write $A \asymp B$ when $A \lesssim B$ and $A \gtrsim B$.

2 Preliminary results on radial weights

Throughout the paper we will employ different descriptions of the weight classes $\widehat{\mathcal{D}}$ and $\check{\mathcal{D}}$. The next result gathers several characterizations of $\widehat{\mathcal{D}}$ proved in [12, Lemma 2.1].

Lemma B *Let ω be a radial weight. Then, the following statements are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;
(ii) *There exist* $C = C(\omega) \geqslant 1$ *and* $\alpha_0 = \alpha_0(\omega) > 0$ *such that*

$$\widehat{\omega}(s) \leqslant C \left(\frac{1-s}{1-t}\right)^\alpha \widehat{\omega}(t), \quad 0 \leqslant s \leqslant t < 1,$$

for all $\alpha \geqslant \alpha_0$;
(iii)

$$\omega_x = \int_0^1 s^x \omega(s)ds \asymp \widehat{\omega}\left(1 - \frac{1}{x}\right), \quad x \in [1, \infty);$$

(iv) *There exists* $C(\omega) > 0$ *such that* $\omega_x \leq C\omega_{2x}$, *for any* $x \geq 1$.

In particular, we will repeatedly use Lemma B(iii) and the fact that $\sup_{x\geq 1} \frac{\widehat{\omega}\left(1-\frac{1}{x}\right)}{\omega_x} < \infty$ for any radial weight ω.

We will also use the following descriptions of the class $\check{\mathcal{D}}$, see [11, Lemma 4] and [16, Lemma B].

Lemma C *Let* ω *be a radial weight. Then, the following statements are equivalent:*

(i) $\omega \in \check{\mathcal{D}}$;

(ii) *There exist* $C = C(\omega) > 0$ *and* $\beta = \beta(\omega) > 0$ *such that*

$$\widehat{\omega}(s) \leqslant C\left(\frac{1-s}{1-t}\right)^{\beta} \widehat{\omega}(t), \quad 0 \leqslant t \leqslant s < 1;$$

(iii) *For each (or some)* $\gamma > 0$ *there exists* $C = C(\gamma, \omega) > 0$ *such that*

$$\int_0^r \frac{ds}{\widehat{\omega}(s)^{\gamma}(1-s)} \leqslant \frac{C}{\widehat{\omega}(r)^{\gamma}}, \quad 0 \leqslant r < 1;$$

(iv) *There exist* $K(\omega) > 1$ *and* $C = C(\omega) > 0$ *such that*

$$\int_r^{1-\frac{1-r}{K}} \omega(s)\,ds \geq C \int_r^1 \omega(s)\,ds, \quad 0 \leq r < 1.$$

We will also use the following results.

Lemma 7 *Let* ω *be a radial weight and let* φ *be a mapping* $\varphi : [0, 1) \to (0, \infty)$.

(i) *If* $\omega \in \widehat{\mathcal{D}}$ *and* φ *is a non-decreasing function such that* $\omega\varphi$ *is a weight, then* $\omega\varphi \in \widehat{\mathcal{D}}$.

(ii) *If* $\omega \in \check{\mathcal{D}}$ *and* φ *is a non-increasing function, then* $\omega\varphi \in \check{\mathcal{D}}$.

Proof (i). Observe that $\omega \in \widehat{\mathcal{D}}$ if and only if there exists $C(\omega) > 0$ such that

$$\int_r^{\frac{1+r}{2}} \omega(s)ds \leqslant C \int_{\frac{1+r}{2}}^1 \omega(s)ds, \quad 0 \leqslant r < 1.$$

Applying this and the fact that φ is non-decreasing, we obtain that

$$\begin{aligned}\frac{1}{\varphi(\frac{1+r}{2})} \int_r^{\frac{1+r}{2}} \omega(s)\varphi(s)ds &\leqslant \int_r^{\frac{1+r}{2}} \omega(s)ds \\ &\leqslant C \int_{\frac{1+r}{2}}^1 \omega(s)ds \leqslant C \frac{1}{\varphi(\frac{1+r}{2})} \int_{\frac{1+r}{2}}^1 \omega(s)\varphi(s)ds, \quad 0 \leqslant r < 1,\end{aligned}$$

thus,

$$\int_r^{\frac{1+r}{2}} \omega(s)\varphi(s)ds \leqslant C \int_{\frac{1+r}{2}}^1 \omega(s)\varphi(s)ds, \quad 0 \leqslant r < 1,$$

so $\omega\varphi \in \widehat{\mathcal{D}}$.

(ii). By Lemma C(iv), $\omega \in \check{\mathcal{D}}$ if and only if there are $K > 1$ and $C(\omega) > 0$ such that

$$\int_r^{1-\frac{1-r}{K}} \omega(s)ds \geq C \int_{1-\frac{1-r}{K}}^1 \omega(s)ds, \quad 0 \leqslant r < 1.$$

So, since φ is a non-increasing function,

$$\int_r^{1-\frac{1-r}{K}} \omega(s)\varphi(s)\,ds \geq \varphi\left(1-\frac{1-r}{K}\right)\int_r^{1-\frac{1-r}{K}} \omega(s)\,ds$$
$$\geq C\varphi\left(1-\frac{1-r}{K}\right)\int_{1-\frac{1-r}{K}}^1 \omega(s)ds \geq \int_{1-\frac{1-r}{K}}^1 \omega(s)\varphi(s)\,ds, \quad 0 \leqslant r < 1.$$

Hence $\omega\varphi \in \check{\mathcal{D}}$. This finishes the proof. □

Lemma 8 *Let $\omega \in \mathcal{D}$ and $\nu \in \mathcal{D}$. Then $\omega\widehat{\nu} \in \mathcal{D}$ and*

$$\int_r^1 \omega(s)\widehat{\nu}(s)ds \asymp \widehat{\omega}(r)\widehat{\nu}(r), \quad 0 \leq r < 1. \tag{2.1}$$

Proof By Lemma 7(ii), we have $\omega\widehat{\nu} \in \check{\mathcal{D}}$. On the other hand, by Lemma C(iv) there are $K > 1$ and $C(\omega) > 0$ such that

$$\int_r^{1-\frac{1-r}{K}} \omega(s)ds \geq C \int_r^1 \omega(s)ds, \quad 0 \leqslant r < 1.$$

So, Lemma B(ii), together with the above inequality, yields

$$\int_r^1 \omega(s)\widehat{\nu}(s)\,ds \geq \widehat{\nu}\left(1-\frac{1-r}{K}\right)\int_{1-\frac{1-r}{K}}^1 \omega(s)ds \gtrsim \widehat{\nu}(r)\widehat{\omega}(r), \quad 0 \leq r < 1.$$

Therefore, using that $\omega, \nu \in \widehat{\mathcal{D}}$ we get

$$\int_r^1 \omega(s)\widehat{\nu}(s)ds \leqslant \widehat{\omega}(r)\widehat{\nu}(r) \lesssim \widehat{\omega}\left(\frac{1+r}{2}\right)\widehat{\nu}\left(\frac{1+r}{2}\right) \lesssim \int_{\frac{1+r}{2}}^1 \omega(s)\widehat{\nu}(s)ds,$$

hence, $\omega\widehat{\nu} \in \widehat{\mathcal{D}}$ and (2.1) holds. This finishes the proof. □

3 Preliminary results on the boundedness of $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$

3.1 On the boundedness of the Szegö projection in growth spaces

Some known results on the boundedness of the Szegö projection R, also known as the Riesz projection, will be relevant for our studies on Bergman projections. Recall that if f is a complex valued harmonic function on $\mathbb{D} \ni z = re^{i\theta}$ with a series representation $f(re^{i\theta}) = \sum_{m=-\infty}^{\infty} f_m r^{|m|} e^{im\theta}$ converging at least uniformly on every compact subset of $\mathbb{D}$, then the Szegö projection R by definition maps f to the analytic function $\sum_{m=0}^{\infty} f_m r^m e^{im\theta}$, where the series also converges at least uniformly on compact subsets. See [21, Section 9.1].

As for the known proofs of Theorem A, if P were a bounded projection from L^∞ onto H^∞, then one could consider the restriction of P as a bounded projection from $L^\infty(\partial\mathbb{D})$ onto H^∞. But it is well-known that such a projection does not exist. Indeed, by techniques explained in [19, Theorem 5.18, Example 5.19], see also [21, Theorem 9.7], one can show that the existence of such a bounded projection would imply the untrue fact that the Szegö projection $R : L^\infty(\partial\mathbb{D}) \to H^\infty$ is bounded (e.g.[21], p. 258).

Another way to see Theorem A follows from the fact that H^∞ is known not to be isomorphic to the Banach space ℓ^∞ of all bounded sequences (e.g. [9, Theorem 1.1]). Recall that a closed subspace of a Banach space is called *complemented*, if there exists a bounded projection onto it. Now, by [18], [7, p.111], the space L^∞ is isomorphic to the Banach space ℓ^∞, which is a *prime* Banach space ([7, Theorem 2.a.7]), i.e. all of its complemented subspaces are again isomorphic to ℓ^∞. We thus find that H^∞ is not complemented in L^∞.

The next result is probably also known to experts, but we indicate the simple proof for the convenience of the reader.

Proposition 9 *Let ω and ν be radial weights such that $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded. Then, the restriction of P_ω onto the closed subspace $h^\infty_{\widehat{\nu}}$ of $L^\infty_{\widehat{\nu}}$, which consists of harmonic functions, coincides with the Szegö projection and consequently, $R : h^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded.*

Proof Assume that $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded and $f(re^{i\theta}) = \sum_{m=-\infty}^{\infty} f_m r^{|m|} e^{im\theta} \in h^\infty_{\widehat{\nu}} \subset L^\infty_{\widehat{\nu}}$ and $z = re^{i\theta} \in \mathbb{D}$. Bearing in mind that $B^\omega_z(\zeta) = \sum_{n=0}^{\infty} \frac{(\overline{z}\zeta)^n}{2\omega_{2n+1}}$ we obtain

$$
\begin{aligned}
P_\omega f(z) &= \int_{\mathbb{D}} \sum_{n=0}^{\infty} \frac{\left(z\overline{\zeta}\right)^n}{2\omega_{2n+1}} f(\zeta)\omega(\zeta)dA(\zeta) \\
&= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \sum_{n=0}^{\infty} \left(2\omega_{2n+1}\right)^{-1} r^n e^{in\theta} \sum_{m=-\infty}^{\infty} f_m s^{n+|m|+1} e^{i\varphi(m-n)} \omega(s) d\varphi ds \qquad (3.1) \\
&= \sum_{n=0}^{\infty} \left(\omega_{2n+1}\right)^{-1} f_n r^n e^{in\theta} \int_0^1 s^{2n+1}\omega(s)ds = \sum_{n=0}^{\infty} f_n r^n e^{in\theta} = Rf(z).
\end{aligned}
$$

This finishes the proof. □

Now, let us observe that the first part of Corollary 2 can be deduced from Proposition 9 and some results in [9] (see also [8]). In fact, some calculations show that $\nu \in \widehat{\mathcal{D}}$ if and only if

$$
\sup_{n\in\mathbb{N}} \frac{\widehat{\nu}(1-2^{-n})}{\widehat{\nu}(1-2^{-n-1})} < \infty
$$

and $\nu \in \check{\mathcal{D}}$ if and only if

$$\inf_{k\in\mathbb{N}} \limsup_{n\in\mathbb{N}} \frac{\widehat{\nu}(1-2^{-n-k})}{\widehat{\nu}(1-2^{-n})} < \infty.$$

Moreover, Proposition 6.4 and the first lines of p. 26 of [9] show that if $\nu \in \widehat{\mathcal{D}}$, then the Szegö projection R is bounded from $h^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$ if and only if $\nu \in \check{\mathcal{D}}$, which together with Proposition 9 proves that if ω is a radial weight and $\nu \in \widehat{\mathcal{D}} \setminus \check{\mathcal{D}}$ then P_ω is not bounded from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$. Later on we will prove Corollary 2 using Theorem 1.

3.2 Preliminary results on the boundedness of $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$

In this section we will prove some preliminary results of their own interest, which will be used in the proof of Theorem 1.

Proposition 10 *Let ω and ν be radial weights. Then, $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded if and only if*

$$\sup_{a\in\mathbb{D}} \widehat{\nu}(a) \int_{\mathbb{D}} |B^\omega_a(\zeta)| \frac{\omega(\zeta)}{\widehat{\nu}(\zeta)} dA(\zeta) < \infty. \tag{3.2}$$

Proof It is clear that $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded if (3.2) holds. Reciprocally, assume that $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded, and for each $a \in \mathbb{D}$ consider the function

$$f_a(\zeta) = \begin{cases} \frac{B^\omega_a(\zeta)}{\widehat{\nu}(\zeta)|B^\omega_a(\zeta)|} & \text{if } B^\omega_a(\zeta) \neq 0 \\ 0 & \text{if } B^\omega_a(\zeta) = 0 \end{cases}.$$

Then, for each $a \in \mathbb{D}$, we have $\|f_a\|_{L^\infty_{\widehat{\nu}}} = 1$ and

$$\begin{aligned} \widehat{\nu}(a) \int_{\mathbb{D}} |B^\omega_a(\zeta)| \frac{\omega(\zeta)}{\widehat{\nu}(\zeta)} dA(\zeta) &= \widehat{\nu}(a) \int_{\mathbb{D}} f_a(\zeta)\overline{B^\omega_a(\zeta)}\,\omega(\zeta) dA(\zeta) \\ &\leq \sup_{z\in\mathbb{D}} \widehat{\nu}(z) \left| \int_{\mathbb{D}} f_a(\zeta)\overline{B^\omega_z(\zeta)}\,\omega(\zeta) dA(\zeta) \right| \leq \|P_\omega\| \|f_a\|_{L^\infty_{\widehat{\nu}}} = \|P_\omega\|. \end{aligned}$$

Therefore, (3.2) holds. This finishes the proof. □

It is worth noticing that an argument equivalent to that of Proposition 10 provides an alternative proof of the fact that P_ω is not bounded from L^∞ to H^∞ for any radial weight ω. In fact, arguing as in the the proof of Proposition 10 it follows that P_ω is bounded from L^∞ to H^∞ if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |B^\omega_a(\zeta)|\,\omega(\zeta) dA(\zeta) < \infty.$$

Moreover, using Hardy's inequality [4, Theorem 5.1], for each $a \in \mathbb{D} \setminus \{0\}$

$$\begin{aligned}\int_{\mathbb{D}} |B_a^\omega(\zeta)|\,\omega(\zeta)dA(\zeta) &= 2\int_0^1 M_1(B_a^\omega, s)s\omega(s)\,ds\\ &\geq 2\pi \int_0^1 \left(\sum_{n=0}^\infty \frac{s^n|a|^n}{2(n+1)\omega_{2n+1}}\right) s\omega(s)\,ds\\ &= \pi \sum_{n=0}^\infty \frac{|a|^n}{(n+1)\omega_{2n+1}} \int_0^1 s^{n+1}\omega(s)\,ds\\ &= \pi \sum_{n=0}^\infty \frac{|a|^n\omega_{n+1}}{(n+1)\omega_{2n+1}} \geq \pi \sum_{n=0}^\infty \frac{|a|^n}{n+1} = \frac{\pi}{|a|}\log\left(\frac{1}{1-|a|}\right),\end{aligned}$$

Therefore, $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |B_a^\omega(\zeta)|\,\omega(\zeta)dA(\zeta) = \infty$ and consequently P_ω is not bounded from L^∞ to H^∞.

We will use Proposition 10 to obtain a necessary condition for the boundedness of $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ in terms of moments and tails of ω and ν.

Proposition 11 *Let ω and ν radial weights such that $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded. Then, there exists $C(\omega,\nu) > 0$ such that*

$$\left(\frac{\omega}{\widehat{\nu}}\right)_x \leqslant C\frac{\omega_{2x}}{\widehat{\nu}\left(1-\frac{1}{x}\right)}, \quad x \geqslant 1. \tag{3.3}$$

Proof By Proposition 10 there exists $C(\omega,\nu) > 0$, such that

$$\sup_{z\in\mathbb{D}} \widehat{\nu}(z) \int_{\mathbb{D}} |B_z^\omega(\zeta)| \frac{\omega(\zeta)}{\widehat{\nu}(\zeta)} dA(\zeta) \leqslant C,$$

and, by applying [4, Theorem 6.1] we obtain

$$\begin{aligned} C &\geqslant \sup_{z\in\mathbb{D}} \widehat{\nu}(z) \int_{\mathbb{D}} |B_z^\omega(\zeta)| \frac{\omega(\zeta)}{\widehat{\nu}(\zeta)} dA(\zeta) \asymp \sup_{z\in\mathbb{D}} \widehat{\nu}(z) \int_0^1 \frac{\omega(r)}{\widehat{\nu}(r)} M_1(B_z^\omega, r)dr\\ &\gtrsim \sup_{z\in\mathbb{D}, n\in\mathbb{N}} \widehat{\nu}(z) \int_0^1 \frac{\omega(r)}{\widehat{\nu}(r)} \frac{|z|^n r^n}{\omega_{2n+1}} dr \geqslant \widehat{\nu}\left(1-\frac{1}{n}\right) \frac{\left(\frac{\omega}{\widehat{\nu}}\right)_n}{\omega_{2n+1}} \left(1-\frac{1}{n}\right)^n\\ &\gtrsim \widehat{\nu}\left(1-\frac{1}{n}\right) \frac{\left(\frac{\omega}{\widehat{\nu}}\right)_n}{\omega_{2n}}, \quad n \in \mathbb{N}, \quad n \geq 2. \end{aligned} \tag{3.4}$$

For $x \geqslant 1$, take $N \in \mathbb{N}$ such that $N \leqslant x < N+1$. Then, the previous inequality yields

$$\left(\frac{\omega}{\widehat{\nu}}\right)_x \leqslant \left(\frac{\omega}{\widehat{\nu}}\right)_N \leqslant C\frac{\omega_{2N}}{\widehat{\nu}\left(1-\frac{1}{N}\right)} \leqslant C\frac{\omega_{2(N+1)}}{\widehat{\nu}\left(1-\frac{1}{N}\right)} \leqslant C\frac{\omega_{2x}}{\widehat{\nu}\left(1-\frac{1}{x}\right)}.$$

This finishes the proof. □

4 Boundedness of Bergman projection on growth space induced by radial doubling weight.

4.1 Boundedness of $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ assuming conditions on ν.

Theorem 1 and Corollary 2 will be proved in this section. Two preliminary results are needed for their proofs.

Proposition 12 *Let ω be a radial weight and $\nu \in \widehat{\mathcal{D}}$ such that (1.1) holds. Then, $\omega \in \widehat{\mathcal{D}}$ and $\omega\widehat{\nu} \in \widehat{\mathcal{D}}$.*

Proof Let $x \geqslant 1$. By applying Lemma B (iv) and (1.1) we get

$$\begin{aligned}\omega_x &\lesssim \frac{1}{\nu_x}\int_0^1 \omega(s)\widehat{\nu}(s)s^{2x}ds = \frac{1}{\nu_x}\int_0^1 \omega(s)s^{2x}\int_s^1 \nu(t)dt\,ds \\ &\leqslant \frac{1}{\nu_x}\int_0^1 \omega(s)s^{\frac{3x}{2}}\int_s^1 t^{\frac{x}{2}}\nu(t)dt\,ds \leqslant \omega_{\frac{3x}{2}}\frac{\nu_{\frac{x}{2}}}{\nu_x} \lesssim \omega_{\frac{3x}{2}}, \quad x \geq 1.\end{aligned} \tag{4.1}$$

Hence, $\omega_x \lesssim \omega_{\frac{3x}{2}}$. So, iteration of this inequality yields $\omega_x \lesssim \omega_{\frac{3x}{2}} \lesssim \omega_{\frac{9x}{4}} \leq \omega_{2x}$. Therefore, by Lemma B (iv), $\omega \in \widehat{\mathcal{D}}$.

Now, a similar argument, the fact that $\omega, \nu \in \widehat{\mathcal{D}}$ and (1.1) imply

$$(\omega\widehat{\nu})_x \leqslant \omega_{\frac{x}{2}}\nu_{\frac{x}{2}} \lesssim \omega_x\nu_x \lesssim (\omega\widehat{\nu})_{2x}, \quad x \geqslant 1,$$

so $\omega\widehat{\nu} \in \widehat{\mathcal{D}}$ by Lemma B (iv). □

Proposition 13 *Let ν be a radial weight and $\omega \in \widehat{\mathcal{D}}$ such that $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded. Then, $\nu \in \check{\mathcal{D}}$.*

Proof By Proposition 10, there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |B_z^\omega(\zeta)|\frac{\omega(\zeta)}{\widehat{\nu}(\zeta)}dA(\zeta) \leqslant \frac{C}{\widehat{\nu}(z)}, \quad z \in \mathbb{D}.$$

Now, by [14, Theorem 1] and Fubini's theorem, for $\frac{1}{2} \leqslant |z| < 1$ there holds

$$\int_{\mathbb{D}} |B_z^\omega(\zeta)|\frac{\omega(\zeta)}{\widehat{\nu}(\zeta)}dA(\zeta) \asymp \int_0^1 \frac{\omega(s)}{\widehat{\nu}(s)}\int_0^{|z|s}\frac{dt}{\widehat{\omega}(t)(1-t)}ds = \int_0^{|z|}\frac{1}{\widehat{\omega}(t)(1-t)}\int_{\frac{t}{|z|}}^1 \frac{\omega(s)}{\widehat{\nu}(s)}ds\,dt.$$

Applying Lemma B (ii), we obtain that

$$\begin{aligned}\int\limits_{\mathbb{D}}|B_z^\omega(\zeta)|\frac{\omega(\zeta)}{\widehat{\nu}(\zeta)}dA(\zeta) &\gtrsim \int\limits_0^{|z|}\frac{\widehat{\omega}(\frac{t}{|z|})}{\widehat{\omega}(t)}\cdot\frac{dt}{\widehat{\nu}(\frac{t}{|z|})(1-t)} \geqslant \int\limits_0^{2|z|-1}\frac{\widehat{\omega}(\frac{t}{|z|})}{\widehat{\omega}(t)}\cdot\frac{dt}{\widehat{\nu}(t)(1-t)}\\ &\gtrsim \int\limits_0^{2|z|-1}\left(\frac{1-\frac{t}{|z|}}{1-t}\right)^\beta\cdot\frac{dt}{\widehat{\nu}(t)(1-t)}\\ &\geqslant \int\limits_0^{2|z|-1}\left(\frac{1-\frac{2t}{1+t}}{1-t}\right)^\beta\cdot\frac{dt}{\widehat{\nu}(t)(1-t)}\\ &\asymp \int\limits_0^{2|z|-1}\frac{dt}{\widehat{\nu}(t)(1-t)},\quad |z|\geqslant\frac{1}{2}.\end{aligned}$$

Then, we take $|z|=\frac{1+r}{2}$ and observe that there is $C=C(\omega,\nu)>0$ such that

$$\int\limits_0^r\frac{dt}{\widehat{\nu}(t)(1-t)}\leq\frac{C}{\widehat{\nu}\left(\frac{1+r}{2}\right)},\quad 0\leqslant r<1.$$

Therefore, if $0\leqslant r\leqslant s<1$, then

$$\frac{1}{\widehat{\nu}(r)}\log\frac{1-r}{1-s}=\frac{1}{\widehat{\nu}(r)}\int\limits_r^s\frac{dt}{1-t}\leqslant\int\limits_r^s\frac{dt}{\widehat{\nu}(t)(1-t)}\leqslant\frac{C}{\widehat{\nu}\left(\frac{1+s}{2}\right)}.$$

Finally, take $K>2$ such that $\frac{\log\frac{K}{2}}{C}>1$ and choose $s=1-\frac{2}{K}(1-r)>r$. Then, $\frac{1+s}{2}=1-\frac{1-r}{K}$, and

$$\widehat{\nu}(r)\geqslant\frac{\log\frac{K}{2}}{C}\widehat{\nu}\left(1-\frac{1-r}{K}\right),\quad 0\leq r<1.$$

This shows that $\nu\in\check{\mathcal{D}}$ and finishes the proof. □

Proof of Theorem 1. We will prove (i)⇒(ii)⇔(iii)⇔(iv) and finally (iii)⇒(i).

Assume that $P_\omega: L^\infty_{\widehat{\nu}}\to H^\infty_{\widehat{\nu}}$ is bounded. Then, (1.1) holds by Proposition 11 and Lemma B (iii). Next, Proposition 12 implies that $\omega\in\widehat{\mathcal{D}}$ and therefore $\nu\in\check{\mathcal{D}}$ by Proposition 13, so we get (ii). Now, if (ii) holds, then Proposition 12 implies that $\omega\in\widehat{\mathcal{D}}$. So, (1.2) follows from (1.1) and Lemma B (iii). That is, (iii) is satisfied. Reciprocally, if (iii) holds, then $\frac{\omega}{\nu}\in\widehat{\mathcal{D}}$ by Lemma 7, and (1.1) follows from (1.2) and Lemma B (iii). Therefore we get (ii). Now let us prove (iii)⇒(iv). Firstly, $\frac{\omega}{\nu}\in\widehat{\mathcal{D}}$ by Lemma 7. In order to prove that $\frac{\omega}{\nu}\in\check{\mathcal{D}}$ and to make the notation easier, we denote $\mu=\frac{\omega}{\nu}$. Then, (1.2) is equivalent to the existence of $C>1$ such that

$$\widehat{\mu}(r)\leqslant C\frac{\widehat{\omega}(r)}{\widehat{\nu}(r)},\quad 0\leqslant r<1. \tag{4.2}$$

On the other hand, an integration by parts yields

$$\widehat{\omega}(r)=\widehat{\nu}(r)\widehat{\mu}(r)-\int\limits_r^1\nu(s)\widehat{\mu}(s)ds,\quad 0\leqslant r<1,$$

so (4.2) is equivalent to the existence of $C \in (0,1)$ such that

$$\frac{\int_r^1 \nu(s)\widehat{\mu}(s)ds}{\widehat{\nu}(r)} \leqslant C\widehat{\mu}(r), \quad 0 \leqslant r < 1. \tag{4.3}$$

By Lemma C (ii) there exists $C_2 > 0$ and $\alpha > 0$ such that for every $K > 1$

$$\frac{\widehat{\nu}\left(1-\frac{1-r}{K}\right)}{\widehat{\nu}(r)} \leqslant \frac{C_2}{K^\alpha}, \quad 0 \leqslant r < 1.$$

Therefore, taking K such that $\frac{C_2}{K^\alpha} < 1 - C$, we obtain

$$\begin{aligned} C\widehat{\mu}(r) &\geqslant \frac{\int_r^1 \nu(s)\widehat{\mu}(s)ds}{\widehat{\nu}(r)} \geqslant \frac{\int_r^{1-\frac{1-r}{K}} \nu(s)\widehat{\mu}(s)ds}{\widehat{\nu}(r)} \geqslant \widehat{\mu}\left(1-\frac{1-r}{K}\right)\frac{\widehat{\nu}(r)-\widehat{\nu}\left(1-\frac{1-r}{K}\right)}{\widehat{\nu}(r)} \\ &= \widehat{\mu}\left(1-\frac{1-r}{K}\right)\left(1-\frac{\widehat{\nu}\left(1-\frac{1-r}{K}\right)}{\widehat{\nu}(r)}\right) \geqslant \widehat{\mu}\left(1-\frac{1-r}{K}\right)\left(1-\frac{C_2}{K^\alpha}\right), \quad 0 \leqslant r < 1. \end{aligned}$$

Consequently, $\mu \in \check{\mathcal{D}}$ and (iv) holds. Conversely, if (iv) holds, we get $\omega \in \widehat{\mathcal{D}}$ by Lemma 8. Moreover, since $\frac{\omega}{\widehat{\nu}} \in \check{\mathcal{D}}$, Lemma C(iv) implies that there are $K > 1$ and $C > 0$ such that

$$\int_r^1 \frac{\omega(s)}{\widehat{\nu}(s)}ds \leqslant C \int_r^{1-\frac{1-r}{K}} \frac{\omega(s)}{\widehat{\nu}(s)}ds, \quad 0 \leqslant r < 1.$$

This, together with Lemma B(ii), gives

$$\begin{aligned} \widehat{\omega}(r) &= \int_r^1 \omega(s)ds = \int_r^1 \frac{\omega(s)}{\widehat{\nu}(s)}\widehat{\nu}(s)ds \geqslant \int_r^{1-\frac{1-r}{K}} \frac{\omega(s)}{\widehat{\nu}(s)}\widehat{\nu}(s)ds \\ &\geqslant \widehat{\nu}\left(1-\frac{1-r}{K}\right)\int_r^{1-\frac{1-r}{K}} \frac{\omega(s)}{\widehat{\nu}(s)}ds \gtrsim \widehat{\nu}(r)\int_r^1 \frac{\omega(s)}{\widehat{\nu}(s)}ds, \quad 0 \leqslant r < 1, \end{aligned}$$

which yields (iii).

Finally, we assume that (iii) holds and prove that $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded. By Proposition 10, it is sufficient to prove that

$$\sup_{a\in\mathbb{D}, |a|\geqslant\frac{1}{2}.} \widehat{\nu}(a)\int_{\mathbb{D}} |B_a^\omega(z)|\frac{\omega(z)}{\widehat{\nu}(z)}dA(z) < \infty.$$

By [14, Theorem 1] and Fubini's Theorem, this is equivalent with proving that there is $C > 0$ such that

$$\widehat{\nu}(a)\int_0^{|a|} \frac{1}{\widehat{\omega}(t)(1-t)}\int_{\frac{t}{|a|}}^1 \frac{\omega(r)}{\widehat{\nu}(r)}dr\,dt \leqslant C, \quad |a| \geqslant \frac{1}{2}.$$

By applying (1.2) and Lemma C (iii) we get

$$\widehat{\nu}(a)\int_0^{|a|}\frac{1}{\widehat{\omega}(t)(1-t)}\int_{\frac{t}{|a|}}^1\frac{\omega(r)}{\widehat{\nu}(r)}dr\,dt \le \widehat{\nu}(a)\int_0^{|a|}\frac{1}{\widehat{\omega}(t)(1-t)}\int_t^1\frac{\omega(r)}{\widehat{\nu}(r)}dr\,dt$$

$$\lesssim \widehat{\nu}(a)\int_0^{|a|}\frac{dt}{\widehat{\nu}(t)(1-t)}dt \lesssim 1,$$

so $P_\omega : L^\infty_{\widehat{\nu}} \to B^\infty_{\widehat{\nu}}$ is bounded. This finishes the proof. □

Proof of Corollary 2. If $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded then $\nu \in \check{\mathcal{D}}$ by Theorem 1. Reciprocally, if $\nu \in \mathcal{D}$ by Lemma B(ii), there exist $C, \alpha > 0$ such that

$$\frac{1}{\widehat{\nu}(s)} \leqslant C\left(\frac{1-r}{1-s}\right)^\alpha \frac{1}{\widehat{\nu}(r)}, \quad 0 \leqslant r \leqslant s < 1.$$

Therefore, if $\omega(z) = (1-|z|)^\gamma$ with $\gamma > \alpha - 1$, then

$$\int_r^1 \frac{\omega(s)}{\widehat{\nu}(s)}ds \lesssim \frac{(1-r)^\alpha}{\widehat{\nu}(r)}\int_r^1 (1-s)^{\gamma-\alpha}ds \asymp \frac{(1-r)^{\gamma+1}}{\widehat{\nu}(r)} \asymp \frac{\widehat{\omega}(r)}{\widehat{\nu}(r)},$$

which means that (1.2) holds. Thus, P_ω is bounded from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$ by Theorem 1. This finishes the proof. □

4.2 Boundedness of $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ assuming conditions on ω.

This section is dedicated to the proofs of Theorem 3 and Corollary 4 among other results.

Proof of Theorem 3. We will prove (i)⇔(iii)⇔(ii) and (iii)⇔(iv).

If (i) holds, then $\nu \in \check{\mathcal{D}}$ by Proposition 13 and (3.3) holds by Proposition 11. So, from the property $\omega \in \widehat{\mathcal{D}}$ and Lemma B we get (1.2). Hence, (iii) holds. Reciprocally, if (iii) holds, using (1.2) and the fact that $\omega \in \widehat{\mathcal{D}}$ yields

$$\widehat{\nu}(r) \lesssim \frac{\widehat{\omega}(r)}{\int_r^1 \frac{\omega(s)\,ds}{\widehat{\nu}(s)}} \lesssim \frac{\widehat{\omega}\left(\frac{1+r}{2}\right)}{\int_{\frac{1+r}{2}}^1 \frac{\omega(s)\,ds}{\widehat{\nu}(s)}} \leqslant \widehat{\nu}\left(\frac{1+r}{2}\right), \quad 0 < r < 1.$$

That is, $\nu \in \widehat{\mathcal{D}}$. So, (i) follows from Theorem 1. Next, assuming again that (iii) holds we have $\nu \in \widehat{\mathcal{D}}$. Moreover, $\frac{\omega}{\widehat{\nu}} \in \widehat{\mathcal{D}}$ by Lemma 7(i). Therefore (1.1) follows from (1.2) and Lemma B. So (ii) is satisfied. Reciprocally, if (ii) holds, using that $\omega \in \widehat{\mathcal{D}}$, Lemma B and (1.1), we get (1.2) and thus also (iii). Finally, if (iii) holds, arguing as above yields $\nu \in \widehat{\mathcal{D}}$, and so $\frac{\omega}{\widehat{\nu}} \in \mathcal{D}$ by Theorem 1. Hence, (iv) holds. On the other hand, (iv)⇒(iii) follows from Theorem 1. This finishes the proof. □

Proof of Corollary 4. Let $\omega \in \widehat{\mathcal{D}}$. If $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded, then Theorem 3 yields $\frac{\omega}{\widehat{\nu}} \in \mathcal{D}$ and $\nu \in \mathcal{D}$, hence by Lemma 8, $\omega \in \mathcal{D}$. On the other hand, if $\omega \in \mathcal{D}$ then by Lemma B (ii) there exists $0 < \beta$ such that

$$\left(\frac{1-r}{1-s}\right)^\beta \widehat{\omega}(s) \lesssim \widehat{\omega}(r) \quad 0 \leqslant r \leqslant s < 1.$$

So, in particular $\widehat{\omega}(s) \lesssim (1-s)^\beta$, $0 \le s < 1$. Now, take $0 < \gamma < \beta$ and consider the standard weight $\nu(r) = (1-r)^{\gamma-1}$. We have $\widehat{\nu}(r) \asymp (1-r)^\gamma$ and

$$\int_r^1 \frac{\omega(s)}{(1-s)^\gamma} \lesssim \frac{\widehat{\omega}(r)}{(1-r)^\gamma} + \int_r^1 \frac{\gamma\,\widehat{\omega}(s)}{(1-s)^{\gamma+1}} ds$$

$$\lesssim \frac{\widehat{\omega}(r)}{(1-r)^\gamma} + \frac{\widehat{\omega}(r)}{(1-r)^\beta} \int_0^1 (1-s)^{\beta-\gamma-1} ds \lesssim \frac{\widehat{\omega}(r)}{(1-r)^\gamma}, \quad 0 \leqslant r < 1.$$

Consequently, by Theorem 3, P_ω is bounded from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$. □

We finish this section proving that both weights, ω and ν, have to be doubling if the operator $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded and one of them is assumed to be upper doubling.

Corollary 14 *Let ω and ν be radial weights such that $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded and $\omega \in \widehat{\mathcal{D}}$ or $\nu \in \widehat{\mathcal{D}}$. Then, both weights $\omega, \nu \in \mathcal{D}$.*

Proof Assume that $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded and $\omega \in \widehat{\mathcal{D}}$, then $\omega \in \mathcal{D}$ by Corollary 4 and $\nu \in \mathcal{D}$ by Theorem 3. On the other hand, if $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ is bounded and $\nu \in \widehat{\mathcal{D}}$, then $\nu \in \mathcal{D}$ and $\frac{\omega}{\nu} \in \mathcal{D}$ by Theorem 1, so by Lemma 8 $\omega \in \mathcal{D}$. This finishes the proof. □

4.3 Boundedness of $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$.

In this section, we will prove Theorem 5 and Theorem 6 among other results.

Firstly, by mimicking the proof of Proposition 10 we get the analogous result for the boundedness of $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$.

Proposition 15 *Let ω and ν be radial weights. Then, $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} (1-|z|)\widehat{\nu}(z) \int_{\mathbb{D}} |(B^\omega_\zeta)'(z)| \frac{\omega(\zeta)}{\widehat{\nu}(\zeta)} dA(\zeta) < \infty.$$

As a byproduct of Proposition 15 we get the following.

Proposition 16 *Let ω and ν radial weights. If $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ is bounded, then (3.3) holds.*

Proof The proof follows the ideas from the demonstration of Proposition 11. We include the details for the sake of completeness. If $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ is bounded, then Proposition 15 and [4, Theorem 6.1] yield

$$\begin{aligned}
\infty &> \sup_{z\in\mathbb{D}} (1-|z|)\widehat{\nu}(z) \int_{\mathbb{D}} |(B^\omega_\zeta)'(z)| \frac{\omega(\zeta)}{\widehat{\nu}(\zeta)} dA(\zeta) \\
&\gtrsim \sup_{z\in\mathbb{D}, n\in\mathbb{N}} (1-|z|)\widehat{\nu}(z) \frac{n\int_0^1 |z|^{n-1} r^{n+1} \frac{\omega(r)}{\widehat{\nu}(r)}\, dr}{\omega_{2n+1}} \\
&\gtrsim \sup_{n\in\mathbb{N}} \left(1-\frac{1}{n}\right)^{n-1} \widehat{\nu}\left(1-\frac{1}{n}\right) \frac{\left(\frac{\omega}{\widehat{\nu}}\right)_{n+1}}{\omega_{2n+1}} \\
&\gtrsim \widehat{\nu}\left(1-\frac{1}{n}\right) \frac{\left(\frac{\omega}{\widehat{\nu}}\right)_{n+1}}{\omega_{2n+1}}, \quad n \ge 2.
\end{aligned}$$

So, arguing as in the proof of Proposition 11 one gets (3.3). This finishes the proof. □

Proof of Theorem 5. We will prove (i)⇒(ii)⇒(iii)⇒(i).

Assume that $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ is bounded. Then, Proposition 16, together with Lemma B(iii), implies (1.1). Hence, (ii) is satisfied. Next, assuming that (ii) holds, Proposition 12 implies that $\omega \in \widehat{\mathcal{D}}$ and consequently (1.2) follows from (1.1) and Lemma B, so we get (iii). Finally, if (iii) holds, then using that $z(B^\omega_\zeta)'(z) = \overline{\zeta(B^\omega_z)'(\zeta)}$, and [14, Theorem 1], we obtain

$$\int_{\mathbb{D}} |(B^\omega_\zeta)'(z)| \frac{\omega(\zeta)}{\widehat{\nu}(\zeta)} dA(\zeta) \lesssim \int_0^{|z|} \frac{1}{\widehat{\omega}(t)(1-t)^2} \int_t^1 \frac{\omega(s)}{\widehat{\nu}(s)} ds\, dt$$
$$\lesssim \int_0^{|z|} \frac{dt}{\widehat{\nu}(t)(1-t)^2} \lesssim \frac{1}{\widehat{\nu}(z)(1-|z|)}, \quad |z| \geqslant \frac{1}{2}.$$

Consequently, (i) holds by Proposition 15 and thus the proof is complete. □

Proof of Theorem 6. We will prove (i)⇒(iiii)⇔(ii) and (iii)⇒(i).

Assuming that $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ is bounded, Proposition 16, together with Lemma B(iii), implies (1.2). Next, if (iii) holds $\frac{\omega}{\widehat{\nu}} \in \widehat{\mathcal{D}}$ by Lemma 7. Moreover, using (1.2) we get

$$\widehat{\nu}(r) \lesssim \frac{\widehat{\omega}(r)}{\int_r^1 \frac{\omega(s)\,ds}{\widehat{\nu}(s)}} \lesssim \frac{\widehat{\omega}\left(\frac{1+r}{2}\right)}{\int_{\frac{1+r}{2}}^1 \frac{\omega(s)\,ds}{\widehat{\nu}(s)}} \leqslant \widehat{\nu}\left(\frac{1+r}{2}\right), \quad 0 < r < 1.$$

That is, $\nu \in \widehat{\mathcal{D}}$. Now, bearing in mind Lemma B(iii) we conclude that (1.1) is satisfied. Reciprocally, if (ii) holds, (1.2) follows from Lemma B(iii). Finally, if (iii) holds and $\omega \in \widehat{\mathcal{D}}$, we get $\nu \in \widehat{\mathcal{D}}$ by arguing as above, hence, (i) follows from Theorem 5. This finishes the proof. □

Next, we will obtain some consequences from Theorem 5 and Theorem 6.

Corollary 17 *Let ω and ν be radial weights such that $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ is bounded. Then, $\omega \in \widehat{\mathcal{D}}$ if and only if $\nu \in \widehat{\mathcal{D}}$.*

Proof Assume that $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ is bounded. If $\nu \in \widehat{\mathcal{D}}$, then $\omega \in \widehat{\mathcal{D}}$ by Theorem 5. On the other hand, if $\omega \in \widehat{\mathcal{D}}$, the proof of (iii)⇒(ii) in Theorem 6 gives that $\nu \in \widehat{\mathcal{D}}$. This finishes the proof. □

Let us observe that Theorem 1, together with Theorem 5, implies that for any radial weight ω and $\nu \in \mathcal{D}$, the operators $P_\omega : L^\infty_{\widehat{\nu}} \to H^\infty_{\widehat{\nu}}$ and $P_\omega : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ are simultaneously bounded. This fact can be also deduced from the following result [5, Theorems 3.2, 3.3].

Theorem D *Let ν be a radial weight. Then, the following statements holds:*

(i) *$H^\infty_{\widehat{\nu}}$ is continously embedded in $\mathcal{B}^\infty_{\widehat{\nu}}$ if and only if $\nu \in \widehat{\mathcal{D}}$;*
(ii) *$H^\infty_{\widehat{\nu}} = \mathcal{B}^\infty_{\widehat{\nu}}$ with equivalence of norms if and only if $\nu \in \mathcal{D}$.*

However, here we are able to prove that the aforementioned result on the boundedness of P_ω does not remain true for $\nu \in \widehat{\mathcal{D}} \setminus \check{\mathcal{D}}$.

Corollary 18 *Let $\nu \in \widehat{\mathcal{D}} \setminus \check{\mathcal{D}}$. Then, for any radial weight ω, P_ω is not bounded from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$. However, there exists a radial weight ω_ν such that $P_{\omega_\nu} : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ is bounded.*

Proof Since $\nu \in \widehat{\mathcal{D}} \setminus \check{\mathcal{D}}$, Corollary 14 implies that P_ω is not bounded from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$ for any radial weight ω. Now, consider $\omega_\nu = \nu\widehat{\nu}$, then $\widehat{\omega}(r) \asymp (\widehat{\nu}(r))^2$, $0 \le r < 1$, and $\omega \in \widehat{\mathcal{D}}$. So, $\int_r^1 \frac{\omega_\nu(s)}{\widehat{\nu}(s)}\,ds = \widehat{\nu}(r) \asymp \frac{\widehat{\omega_\nu}(r)}{\widehat{\nu}(r)}$, $0 \le r < 1$. Therefore, $P_{\omega_\nu} : L^\infty_{\widehat{\nu}} \to \mathcal{B}^\infty_{\widehat{\nu}}$ is bounded, by Theorem 5. This finishes the proof. □

It is also worth mentioning that there exist $\omega \in \mathcal{D}$ such that P_ω is not bounded from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$ but $P_\omega : L^\infty_{\widehat{\nu_\omega}} \to \mathcal{B}^\infty_{\widehat{\nu_\omega}}$ is bounded. Take $\omega = 1$ and $\nu(r) = (1-r)^{-1}\left(\log \frac{e}{1-r}\right)^{-2} \in \widehat{\mathcal{D}}\setminus\check{\mathcal{D}}$, and apply Corollary 14 and Theorem 5

The final result of this section reads as follows.

Corollary 19 *Let $\omega \in \widehat{\mathcal{D}} \setminus \check{\mathcal{D}}$. Then for any radial weight ν, P_ω is not bounded from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$. However there exists a radial weight ν_ω such that $P_\omega : L^\infty_{\widehat{\nu_\omega}} \to \mathcal{B}^\infty_{\widehat{\nu_\omega}}$ is bounded.*

Proof Since $\omega \in \widehat{\mathcal{D}} \setminus \check{\mathcal{D}}$, Corollary 14 yields that P_ω is not bounded from $L^\infty_{\widehat{\nu}}$ to $H^\infty_{\widehat{\nu}}$ for any radial weight ν. Now, for $\nu_\omega(r) = \frac{d}{dr}\left((\widehat{\omega}(r))^{\frac{1}{2}}\right)$ we have $\widehat{\nu_\omega}(r) \asymp (\widehat{\omega}(r))^{\frac{1}{2}}$, $0 \le r < 1$, and $\nu \in \widehat{\mathcal{D}}$. Moreover, $\int_r^1 \frac{\omega(s)}{\widehat{\nu_\omega}(s)}\,ds \asymp (\widehat{\omega}(r))^{\frac{1}{2}} \asymp \frac{\widehat{\omega}(r)}{\widehat{\nu_\omega}(r)}$, $0 \le r < 1$. Therefore, $P_\omega : L^\infty_{\widehat{\nu_\omega}} \to \mathcal{B}^\infty_{\widehat{\nu_\omega}}$ is bounded by Theorem 5. This finishes the proof. □

5 Study of the boundedness of the Bergman projections for exponentially decreasing weights

5.1 Class of exponentially decreasing weights

In this section we consider the boundedness of Bergman projections in the space L^∞_v for a class of exponentially decreasing weights v. In order to formulate the results we need some more definitions. As in [6], we say that a continuous function $\rho : \mathbb{D} \to \mathbb{R}$ belongs to the class $\mathcal{L}_0$, if $\lim_{|z|\to 1} \rho(z) = 0$,

$$\sup_{z,\zeta\in\mathbb{D}, z\neq\zeta} \frac{|\rho(z) - \rho(\zeta)|}{|z-\zeta|} < \infty \tag{5.1}$$

and for every $\varepsilon > 0$ there exists a compact $E \subset \mathbb{D}$ such that $|\rho(z) - \rho(\zeta)| \leqslant \varepsilon|z-\zeta|$ whenever $z, \zeta \in \mathbb{D}\setminus E$. Furthermore, according to [6], a twice continuously differentiable, real valued subharmonic function φ on $\mathbb{D}$ is said to belong to the class $\mathcal{W}_0$, if $\Delta\varphi > 0$ and there exists $\rho \in \mathcal{L}_0$ such that

$$\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho. \tag{5.2}$$

on $\mathbb{D}$.

We say that a weight v on $\mathbb{D}$ belongs to the class $\mathcal{E}$, if it is of the form $v = e^{-\varphi}$ for some *radial* function $\varphi \in \mathcal{W}_0$. As examples of weights in this class, we mention

$$v(z) = \exp\big(-\alpha/(1-|z|^\ell)^\beta\big) \tag{5.3}$$

where $\alpha, \beta, \ell > 0$ are constants. For these weights one can take $\rho(z) = (1-|z|)^{1+\beta/2}$.

For the weight v as in (5.3) with $\ell, \beta = 1$ and $\omega = v^2$, the boundedness of P_ω on L^∞_v was proved in [2]. In the setting of exponentially decreasing weights satisfying the general condition (B) of [9, 10], examples of bounded projections from L^∞_v onto H^∞_v were constructed in [10]. (These projections are not necessarily of the Bergman type. All weights

(5.3) satisfy condition (B), but there are many others, see the mentioned references.) We also mention the following result of [6].

Theorem E *If* $v = e^{-\varphi}$ *with* $\varphi \in \mathcal{W}_0$ *and* $\omega = v^2$*, then the projection* P_ω *is bounded from* L^∞_v *onto* H^∞_v*.*

We will prove in Theorem 20 a generalization of this result for the class $\mathcal{E}$ but it is first worthwhile to recall known, related negative results. Indeed, in [3] it was shown that given a weight ω as in (5.3) with $\ell = 2$, the projection P_ω is bounded in L^p_ω, $1 \leqslant p \leqslant \infty$, if and only if $p = 2$. The result was extended in [20]. Furthermore, in [1, Theorems 1 and 2] it was shown that given v as in (5.3) and another weight $\omega = \exp\big(-\tilde{\alpha}/(1-|z|^\ell)^{\tilde{\beta}}\big)$, then P_ω is unbounded in L^∞_v, if $\tilde{\alpha} \neq 2\alpha$ or $\tilde{\beta} \neq \beta$.

5.2 Study of the boundedness $P_\omega : L^\infty_v \to H^\infty_v$ for perturbed exponentially decreasing weights

In view of Theorem E and the above mentioned negative results, it is of interest to know if the projection P_ω is bounded in L^∞_v for some other, "smaller" perturbations v of the weight $\omega^{1/2}$. We will prove the following generalization, which is our main result concerning exponentially decreasing weights.

Theorem 20 *Assume the weight* $w = e^{-\varphi}$ *belongs to* $\mathcal{E}$ *with* φ, ρ *satisfying* (5.2)*. Moreover, let* $\nu \in \widehat{\mathcal{D}}$ *and define the weights* $\omega = w^2$ *and* $v = w\widehat{\nu}^t \rho^\sigma$*, where* $t \in \{-1, 0, 1\}$ *and* σ *is any real number. Then, the projection* P_ω *is bounded from* L^∞_v *onto* H^∞_v*.*

Here, the factor $\widehat{\nu}^t \rho^\sigma$ can be considered as a modest multiplicative perturbation of the exponential weight w. For example, for a weight w as in (5.3), the perturbation could be any negative or positive power of the boundary distance.

Proof We present the proof for $t = 1$; similar proofs work for $t \in \{-1, 0\}$. We will need some results of [6]. The Bergman reproducing kernel B^ω_z in

$$P_\omega f(z) = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\omega(\zeta)dA(\zeta) \tag{5.4}$$

has by [6, Theorem 3.2] the upper bound

$$|B^\omega_z(\zeta)| \leqslant C\frac{e^{\varphi(z)+\varphi(\zeta)}}{\rho(z)\rho(\zeta)}e^{-\alpha d_\rho(z,\zeta)} \tag{5.5}$$

for some $\alpha > 0$. Moreover the last factor also has for any $M > 0$ the upper bound

$$e^{-\alpha d_\rho(z,\zeta)} \leqslant C(M)\Big(\frac{\min(\rho(z),\rho(\zeta))}{|z-\zeta|}\Big)^M, \tag{5.6}$$

see formula (23) of [6]. We will choose a large enough M later. In (5.6), d_ρ is defined according to [6] as the distance function

$$d_\rho(z,\zeta) = \inf_\gamma \int_0^1 |\gamma'(t)|\frac{dt}{\rho(\gamma(t))}, \tag{5.7}$$

where $z, \zeta \in \mathbb{D}$ and the infimum is taken over all piecewise C^1 curves $\gamma : [0,1] \to \mathbb{D}$ with $\gamma(0) = z, \gamma(1) = \zeta$.

Let $\nu \in \widehat{\mathcal{D}}$ and $v = w\widehat{\nu}\rho^\sigma = e^{-\varphi}\widehat{\nu}\rho^\sigma$ be as in the assumptions of the theorem, and let $f \in L_v^\infty$ be such that $\|f\|_{\infty,v} \leqslant 1$. We can estimate for every $z \in \mathbb{D}$, by using (5.5) and $|f(\zeta)| \leqslant \rho(\zeta)^{-\sigma} e^{\varphi(\zeta)}/\widehat{\nu}(\zeta)$,

$$
\begin{aligned}
v(z)|P_\omega f(z)| &\leqslant w(z)\rho(z)^\sigma \widehat{\nu}(z) \int_{\mathbb{D}} |B_z^\omega(\zeta)||f(\zeta)|e^{-2\varphi(\zeta)} dA(\zeta) \\
&\leqslant w(z)\widehat{\nu}(z)\rho(z)^\sigma e^{\varphi(z)} \int_{\mathbb{D}} \frac{e^{\varphi(\zeta)}}{\rho(z)\rho(\zeta)} e^{-\alpha d_\rho(z,\zeta)} \frac{1}{e^{\varphi(\zeta)}\widehat{\nu}(\zeta)\rho(\zeta)^\sigma} dA(\zeta) \\
&\leqslant \widehat{\nu}(z)\rho(z)^{\sigma-1} \int_{\mathbb{D}} \frac{e^{-\alpha d_\rho(z,\zeta)}}{\widehat{\nu}(\zeta)\rho(\zeta)^{\sigma+1}} dA(\zeta).
\end{aligned} \tag{5.8}
$$

We next divide the integration domain $\mathbb{D}$ into two subsets,

$$
\mathbb{A}_z = \left\{ \zeta \in \mathbb{D} : |\zeta| \geqslant \frac{1}{2}(1+|z|) \right\} \quad \text{and} \quad \mathbb{B}_z = \mathbb{D} \setminus \mathbb{A}_z. \tag{5.9}
$$

Then, there holds for $\zeta \in \mathbb{B}_z$

$$
\widehat{\nu}(\zeta) \geqslant \widehat{\nu}\Big(\frac{1}{2}(1+|z|)\Big) \geqslant C\widehat{\nu}(z), \tag{5.10}
$$

since $\widehat{\nu}$ is decreasing with respect to the radius and $\nu \in \widehat{\mathcal{D}}$. This yields

$$
\begin{aligned}
\widehat{\nu}(z)\rho(z)^{\sigma-1} \int_{\mathbb{B}_z} \frac{e^{-\alpha d_\rho(z,\zeta)}}{\widehat{\nu}(\zeta)\rho(\zeta)^{\sigma+1}} dA(\zeta) &\leqslant C\rho(z)^{\sigma-1} \int_{\mathbb{B}_z} \frac{e^{-\alpha d_\rho(z,\zeta)}}{\rho(\zeta)^{\sigma+1}} dA(\zeta) \\
&\leqslant C\rho(z)^{\sigma-1} \int_{\mathbb{D}} \frac{e^{-\alpha d_\rho(z,\zeta)}}{\rho(\zeta)^{\sigma+1}} dA(\zeta).
\end{aligned} \tag{5.11}
$$

By Corollary 3.1 of [6], this expression is bounded by a constant.

To estimate the integral over $\mathbb{A}_z$ we first note that since $\nu \in \widehat{\mathcal{D}}$, Lemma B(ii) shows that there are constants $C, \beta > 0$ such that

$$
\frac{\widehat{\nu}(z)}{\widehat{\nu}(\zeta)} \leqslant C \frac{(1-|z|)^\beta}{(1-|\zeta|)^\beta}
$$

for all z, ζ with $|z| \leqslant |\zeta|$. We use this and (5.6) to write

$$
\begin{aligned}
\widehat{\nu}(z)\rho(z)^{\sigma-1} \int_{\mathbb{A}_z} \frac{e^{-\alpha d_\rho(z,\zeta)}}{\widehat{\nu}(\zeta)\rho(\zeta)^{\sigma+1}} dA(\zeta) &\leqslant C(1-|z|)^\beta \rho(z)^{\sigma-1} \int_{\mathbb{A}_z} \frac{e^{-\alpha d_\rho(z,\zeta)}}{\rho(\zeta)^{\sigma+1}(1-|\zeta|)^\beta} dA(\zeta) \\
&\leqslant C(M)(1-|z|)^\beta \rho(z)^{\sigma-1} \int_{\mathbb{A}_z} \frac{1}{(1-|\zeta|)^\beta \rho(\zeta)^{\sigma+1}} \left(\frac{\min\big(\rho(z), \rho(\zeta)\big)}{|z-\zeta|} \right)^M dA(\zeta)
\end{aligned} \tag{5.12}
$$

We now choose the number $\gamma > 0$ such that $\sigma + \gamma - 1 > 0$ and then M such that $M > \sigma + \beta + \gamma + 3$. Then, we use

$$
\min\big(\rho(z), \rho(\zeta)\big)^M \leqslant \rho(z)^\gamma \rho(\zeta)^{M-\gamma}
$$

to bound (5.12) by

$$C_M(1-|z|)^{\beta}\rho(z)^{\sigma+\gamma-1}\int\limits_{\mathbb{A}_z}\frac{\rho(\zeta)^{M-(\gamma+\sigma+1)}}{(1-|\zeta|)^{\beta}|z-\zeta|^M}dA(\zeta). \tag{5.13}$$

Note that by (5.1), the function ρ has the bound $\rho(\zeta)\leqslant C(1-|\zeta|)$ for all $\zeta\in\mathbb{D}$. Thus, (5.13) is not larger than

$$C_M(1-|z|)^{\sigma+\beta+\gamma-1}\int\limits_{\mathbb{A}_z}\frac{(1-|\zeta|)^{M-(\sigma+\beta+\gamma+1)}}{|z-\zeta|^M}dA(\zeta). \tag{5.14}$$

Since $|z-\zeta|\geqslant C|1-z\bar{\zeta}|$ for $\zeta\in\mathbb{A}$, the Forelli-Rudin estimate [21, Lemma 3.10] and the choice of the number M show that (5.14) is bounded by a constant.

Combining (5.8) and (5.11)–(5.14) we get $\sup_{z\in\mathbb{D}} v(z)|P_\omega f(z)|\leqslant C$ for all $f\in L_v^\infty$ with $\|f\|_{\infty,v}\leqslant 1$, which proves the theorem. □

Finally, we will apply Theorem 20 in order to get the following result for the special exponential weights. Let us set $w=e^{-\varphi}$, $\varphi(z)=\alpha(1-|z|^2)^{-\beta}$, and

$$v(z)=(1-|z|^2)^{\gamma}w(z),\ \omega(z)=(1-|z|^2)^{2\sigma}w(z)^2, \tag{5.15}$$

where the parameters satisfy $\alpha,\beta>0$, $\sigma,\gamma\in\mathbb{R}$.

Proposition 21 *Let the weights v and ω be as in (5.15). Then, the projection P_ω is bounded from L_v^∞ onto H_v^∞.*

Proof We define

$$\psi(z)=\frac{\alpha}{(1-|z|^2)^{\beta}}-\sigma\log(1-|z|^2)$$

and observe that $\psi\in\mathcal{W}_0$, since there holds $1/\sqrt{\Delta\psi(z)}\asymp R(z)$ for the function $R(z)=(1-|z|^2)^{1+\beta/2}\in\mathcal{L}_0$. We define the weight $W=e^{-\psi}\in\mathcal{E}$. Then, obviously, $\omega=W^2$ and $v(z)=R(z)^{\frac{\gamma-\sigma}{1+\beta/2}}W(z)$ so that the result follows from Theorem 20 by choosing $W\to w$ and $R\to\rho$ and setting $t=0$. □

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