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A filtration associated to an abelian inner ideal and the speciality of the subquotient of a Lie algebra

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Abstract

For any abelian inner ideal B of a Lie algebra L such that $[B, \mathrm{Ker}_L B]^n \subseteq B$ for some $n \in \mathbb{N}$ we build a bounded filtration $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ whose first nonzero term $\mathcal{F}_{-n}$ is B, $\mathcal{F}_{n-1} = \mathrm{Ker}_L B$ and $\mathcal{F}_n = L$. The extremes of the induced $\mathbb{Z}$-graded Lie algebra $\hat{L} = \mathcal{F}_{-n} \oplus \mathcal{F}_{-n+1}/\mathcal{F}_{-n} \oplus ... \oplus \mathcal{F}_n/\mathcal{F}_{n-1}$ are the Jordan pair $V = (\mathcal{F}_{-n},\ \mathcal{F}_n/\mathcal{F}_{n-1})$ and coincide with the subquotient $(B, L/\mathrm{Ker}_L B)$.

Thanks to this filtration, we can prove that when a Lie algebra L is strongly prime and $\mathrm{Ker}_L B$ is not a subalgebra of L, then subquotient $(B, L/\mathrm{Ker}_L B)$ is a special strongly prime Jordan pair.

1 Preliminaries

Throughout this chapter we are going to introduce definitions and results which are necessary for the development of subsequent sections.

By a ring of scalars Φ we understand that Φ is an associative, commutative and unitary ring. We will be deal with Lie algebras L, associative algebras R and linear Jordan pairs V over a ring of scalars Φ containing $\frac{1}{2}$ and $\frac{1}{3}$. As usual, $[x, y]$ will denote the Lie bracket of two elements x, y of L, and the product of elements of R will be written by juxtaposition. Any associative algebra R gives rise to a Lie algebra $R^{(-)}$ with Lie bracket $[x, y] := xy - yx$,

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for all $x, y \in R$. If R has an involution $*$ we will consider the Lie subalgebra $\mathrm{Skew}(R,*) = \{x \in R \mid x^* = -x\}$ of $R^{(-)}$. Jordan triple products of a Jordan pair $V = (V^+, V^-)$ will be written by $\{x, y, z\}$ for any $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$. The reader is referred to [1], [2] and [4] for basic results, notation and terminology on Lie algebras and Jordan pairs.

Definition 1. A Φ-module B of a Lie algebra L is called an abelian inner ideal if $[B, B] = 0$ and $[B, [B, L]] \subseteq B$. The kernel of an abelian inner ideal is

$$\mathrm{Ker}_L B = \{x \in L \mid [B, [B, x]] = 0\}$$

Associated to an abelian inner ideal B of L we can consider the subquotient $(B, L/\mathrm{Ker}_L B)$, which is a linear Jordan pair with products

$$\{b_1, \bar{x}, b_2\} = [[b_1, x], b_2] \qquad \{\bar{x}, b_1, \bar{y}\} = \overline{[[x, b_1], y]}$$

for every $b_1, b_2 \in B$ and every $\bar{x}, \bar{y} \in L/\mathrm{Ker}_L B$, see [7, 3.2].

Definition 2. Let Φ be a ring of scalars and let L be a Lie algebra over Φ. For each $x \in L$, we define the linear map $\mathrm{ad}_x : L \longrightarrow L$ as $\mathrm{ad}_x(y) := [x, y]$ for every $y \in L$. We will say that L is nondegenerate Lie algebra if, for every $x \in L$ such that $\mathrm{ad}_x^2(L) = 0$, then $x = 0$. An element x in a Lie algebra L is called a Jordan element of L if $\mathrm{ad}_x^3(L) = 0$.

Definition 3. Let L be a Lie algebra. A finite $\mathbb{Z}$-grading is a non-trivial $\mathbb{Z}$-grading of L such that the support $\mathrm{supp}\, L = \{m \in \mathbb{Z} \mid L_m \neq 0\}$ is finite. In this case $L = L_{-n} \oplus L_{-(n-1)} \oplus \ldots \oplus L_0 \oplus \ldots \oplus L_{n-1} \oplus L_n$ for some positive integer n. If $L_{-n} + L_n \neq 0$, we will call such a grading a $(2n+1)$-grading. Note that if L is nondegenerate then both L_{-n} and L_n are non-zero. If $L = L_{-n} \oplus \ldots \oplus L_n$ is a $(2n+1)$-graded Lie algebra, then $V = (L_{-n}, L_n)$ is a Jordan pair with products $\{x, y, z\} = [[x, y], z]$ and $\{y, x, t\} = [[y, x], t]$ for every $x, z \in L_{-n}$ and every $y, t \in L_n$, which is called the associated Jordan pair of L.

Definition 4. Let L be a Lie algebra over Φ. A $\mathbb{Z}$-filtration $\{\mathcal{F}_i\}_{i \in \mathbb{Z}}$ is a chain of submodules of L $\ldots \subset \mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ such that $[\mathcal{F}_i, \mathcal{F}_j] \subset \mathcal{F}_{i+j}$ for every $i, j \in \mathbb{Z}$. A $\mathbb{Z}$-filtration $\{\mathcal{F}_i\}_{i \in \mathbb{Z}}$ is bounded if there exist $n, m \in \mathbb{Z}$, with $n < m$, such that $\mathcal{F}_i = 0$ for every $i \leq n$ and $\mathcal{F}_j = L$ for every $j \geq m$. If $\{\mathcal{F}_i\}_{i \in \mathbb{Z}}$ is a $\mathbb{Z}$-filtration of a Lie algebra L over Φ, we can consider the Φ-module

$$\hat{L} = \ldots \oplus \underbrace{\mathcal{F}_{i-1}/\mathcal{F}_{i-2}}_{\hat{L}_{i-1}} \oplus \underbrace{\mathcal{F}_i/\mathcal{F}_{i-1}}_{\hat{L}_i} \oplus \underbrace{\mathcal{F}_{i+1}/\mathcal{F}_i}_{\hat{L}_{i+1}} \oplus \ldots \qquad (\star)$$

with product $[\bar{x}, \bar{y}] = \overline{[x, y]} \in \mathcal{F}_{i+j}/\mathcal{F}_{i+j-1}$ for every $\bar{x} = x + \mathcal{F}_{i-1} \in \mathcal{F}_i/\mathcal{F}_{i-1}$ and every $\bar{y} = y + \mathcal{F}_{j-1} \in \mathcal{F}_j/\mathcal{F}_{j-1}$. Thereby $\hat{L}$ has structure of $\mathbb{Z}$-graded Lie algebra and it is called the induced graded Lie algebra (see [5, p. 351]).

Definition 5. An associative algebra R is semiprime if, for every nonzero ideal I of R, $I^2 \neq 0$, and it is prime if $IJ \neq 0$ for every pair of nonzero ideals I, J of R. If R is an associative algebra with involution $*$, we say that and ideal I of R is an $*$-ideal if $y^* \in I$ for every $y \in I$, and we say that R is $*$-prime if $IJ \neq 0$ for every nonzero $*$-ideals I, I of R.

The extended centroid of R will be denoted by $C(R)$ (see [3, §2.3] for its definition and main properties). When R is semiprime, $C(R)$ is von Neumann regular, and when R is prime, $C(R)$ is a field. The central closure of R is $\hat{R} = C(R) + C(R)R$ and R is centrally closed if it coincides with its central closure. When R has an involution $*$, this involution extends to $C(R)$ and to $\hat{R}$. If R is a prime associative algebra with involution $*$, the involution is of the first kind when every element in $C(R)$ is symmetric with respect to $*$, and it is of the second kind if there are nonzero skew-symmetric elements in $C(R)$.

A Lie algebra L is said to be prime if $[I, J] \neq 0$ for every nonzero ideals I, J of L. If L is prime and nondegenerate, we say that L is a strongly prime Lie algebra. By [8, Theorem 1.6], we know that L is a strongly prime Lie algebra if and only if for every $x, y \in L$ such that $[x, [y, L]] = 0$, we have that $x = 0$ or $y = 0$.

Definition 6. An associative pair over a ring of scalars Φ is a pair $A = (A^+, A^-)$ of Φ-modules with a triple product such that $uv(xyz) = u(vxy)z = (uvx)yz$ for every $x, z, u \in A^\sigma$ and every $y, v \in A^{-\sigma}$, where $\sigma = \pm$.

Example 1. If X and Y are two Φ-modules over a ring of scalars Φ, then the pair $V = (\mathrm{Hom}_\Phi(X, Y), \mathrm{Hom}_\Phi(Y, X))$ with triple product $f_1 g_1 f_2$ and $g_1 f_1 g_2$ for every $f_1, f_2 \in \mathrm{Hom}_\Phi(X, Y)$ and for every $g_1, g_2 \in \mathrm{Hom}_\Phi(Y, X)$, is an associative pair.

If $A = (A^+, A^-)$ is an associative pair over a ring of scalars Φ, then the pair of Φ-modules (A^+, A^-) with products $\{x, y, z\} = xyz + zyx$ and $\{y, x, t\} = yxt + txy$ for every $x, z \in A^\sigma$ and for every $y, t \in A^{-\sigma}$, $\sigma = \pm$, is a Jordan pair denoted by $(A^+, A^-)^{(+)}$.

Definition 7. Let Φ be a ring of scalars an let $V = (V^+, V^-)$ be a Jordan pair over Φ. We say that V is special if is a subpair of the Jordan pair $A = (A^+, A^-)^{(+)}$ for some associative pair $A = (A^+, A^-)$ over Φ.

Example 2. Let X and Y be two modules over a ring of scalars Φ. Then the Jordan pair $V = (\mathrm{Hom}_\Phi(X, Y), \mathrm{Hom}_\Phi(Y, X))^{(+)}$ is special.

2 Filtration associated to an abelian inner ideal

In this section we will construct a filtration associated to an abelian inner ideal. If B is an abelian inner ideal of a Lie algebra L such that $[B, \mathrm{Ker}_L B]^n \subset$

B for some $n \in \mathbb{N}$, we will show that B induces a bounded filtration of L starting on B and whose second last submodule coincides with $\mathrm{Ker}_L B$.

In [9, Theorem 1.2] a bounded filtration from -2 to 2 associated to a Jordan element was built. Generalizing this idea, given an abelian inner ideal B of a Lie algebra L, we are going to construct a bounded filtration of L associated to B.

Theorem 1. *[6, Theorem 3.1] Let L be a Lie algebra and let B be an abelian inner ideal of L. Let us suppose that there exists $n \in \mathbb{N}$ such that $[B, Ker_L B]^n \subseteq B$. Then the chain*

$$\cdots \subset \mathcal{F}_{-n} \subset \mathcal{F}_{-n+1} \subset \cdots \subset \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n \subset \ldots,$$

where $\mathcal{F}_{-m} := \{0\}$ and $\mathcal{F}_m := L$ for every $m \geq n$, and

$$\begin{aligned} \mathcal{F}_{-n} &:= B,\ \mathcal{F}_{-k} := [B, \mathrm{Ker}_L B]^k + B \text{ for } k = 1, \ldots, n-1, \\ \mathcal{F}_0 &:= \{x \in L \mid [x, B] \subseteq B\} \\ \mathcal{F}_s &:= \mathrm{ad}^{n-s-1}_{[B,\mathrm{Ker}_L B]}(Ker_L B) + \mathcal{F}_0 \text{ for } s = 1, \ldots, n-1,\ \mathcal{F}_n := L \end{aligned}$$

is a bounded filtration of L.

Remark 1. The graded Lie algebra induced by this filtration $\hat{L} = \mathcal{F}_{-n} \oplus \mathcal{F}_{-n+1}/\mathcal{F}_{-n} \oplus \cdots \oplus \mathcal{F}_n/\mathcal{F}_{n-1}$ has associated Jordan pair $V = (\mathcal{F}_{-n}, \mathcal{F}_n/\mathcal{F}_{n-1})$ equal to the subquotient $(B, L/\mathrm{Ker}_L B)$.

In the following results we will show that the hypothesis $[B, \mathrm{Ker}_L B]^n = 0$ for some $n \in \mathbb{N}$ is quite natural for large families of Lie algebras.

Remark 2. In [10, Proposition 3.5(d)] it was shown that for any abelian inner ideal B of a centrally closed prime associative algebra R, $[B, \mathrm{Ker}_{R^{(-)}} B]$ is nilpotent of index k with $k \leq 3$. This result easily extends to semiprime associative algebras. Indeed, given an abelian inner ideal of a semiprime associative algebra R, since R is a subdirect product of prime associative algebras R_i, B decomposes into a subdirect product of abelian inner ideals B_i of R_i. For each i, let us consider the central closure $\hat{R}_i$, and let us extend B_i to an abelian inner ideal $\hat{B}_i = C(R_i)B_i$ of $\hat{R}_i$. Then $[\hat{B}_i, \mathrm{Ker}_{\hat{R}_i} \hat{B}_i]$ is nilpotent of index ≤ 3 for each i, and therefore $[B, \mathrm{Ker}_{R^{(-)}} B]$ is nilpotent of index ≤ 3. In particular, every abelian inner ideal B of a semiprime associative algebra R satisfies $[B, \mathrm{Ker}_{R^{(-)}} B]^k = 0 \subseteq B$ for some $k \leq 3$.

Proposition 1. *[6, Proposition 4.2] Let R be a centrally closed associative algebra with involution $*$. Suppose that R is $*$-prime and that $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \in \Phi$. Let $L = Skew(R, *)$ and let B be an abelian inner ideal of L. Then $[B, Ker_L B]$ is a nilpotent subalgebra of L of index n, where we have the following possibilities for n:*

(a) If R is $*$-prime not prime or if it is prime and the involution is of the second kind, $n \leq 3$.

(b) If R is prime and the involution is of the first kind, then $n \leq 4$. In particular, if $[L, L] = 0$, $n = 1$; otherwise $b^3 = 0$ for every $b \in B$ and either

- *there exists $b \in B$ such that $b^3 = 0$, $b^2 \neq 0$. In this case $n = 2$ and L admits a 3-grading $L = L_{-1} \oplus L_0 \oplus L_1$ with $B = L_{-1}$ and $KerB = L_{-1} \oplus L_0$, or*
- *$B^2 = 0$ and $n \leq 4$. If, moreover, $B(KerB)B = 0$, then $n \leq 3$.*

Since every semiprime associative algebra R with involution $*$ is a subdirect product of $*$-prime associative algebras R_i, given an abelian inner ideal B of $L = \text{Skew}(R, *)$ we can consider the projections B_i of B onto $L_i = \text{Skew}(R_i, *)$. Let $\hat{R}_i$ be the central closure of each R_i and let $\hat{B}_i = H(C(R_i), *)B_i$ be the abelian inner ideal generated by B_i in $\hat{L}_i = \text{Skew}(\hat{R}_i, *)$. By 1, $[\hat{B}_i, \text{Ker}_{\hat{L}_i} \hat{B}_i]$ is nilpotent of index ≤ 4, so $[B_i, \text{Ker}_{L_i} B_i]$ is also nilpotent of index ≤ 4 for each i. Thus $[B, \text{Ker}_L B]$ is nilpotent of index ≤ 4.

Remark 3. Let L be nondegenerate. Then for every nonzero abelian inner ideal B of finite length of L there exists a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ such that $B = L_n$ (this is always the case when L is nondegenerate finite dimensional), see [7, Corollary 6.2]. With respect to this grading, $\text{Ker}_L B = L_{-(n-1)} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$, so $[B, \text{Ker}_L B] \subset L_1 \oplus \cdots \oplus L_n$ implies that $[B, \text{Ker}_L B]^n \subseteq B$.

3 The speciality of the subquotient

In this last section we will apply the filtration associated to an abelian inner ideal to give a sufficient condition for the speciality of the subquotient associated to such an abelian inner ideal.

We will use the following lemma, which has appeared several times in the literature and can be found for example in [1, Theorem 11.34].

Lemma 1. *Let $L = L_{-n} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_n$ be a $(2n+1)-\mathbb{Z}$-graded Lie algebra. Then the pair $V := (L_{-n}, L_n)$ with product $\{x, y, z\} := [x, [y, z]]$ for $x, z \in L_{\sigma n}$ and $y \in L_{-\sigma n}$, $\sigma = \pm$ is a Jordan pair. Moreover, for any $i \in 1, 2, \ldots, n-1$ the pair of linear maps (Ψ_i, Ψ_{-i})*

$$\Psi_i : L_{-n} \to Hom(L_i, L_{i-n}) \qquad \Psi_{-i} : L_n \to Hom(L_{i-n}, L_i)$$

defined by $\Psi_{\sigma i}(x)(y) = ad_x y$ for any $x \in L_{-\sigma n}$ and any $y \in L_i$ if $\sigma = +$ or $y \in L_{i-n}$ if $\sigma = -$ is a homomorphism of Jordan pairs between V and the special Jordan pair $(Hom(L_i, L_{i-n}), Hom(L_{i-n}, L_i))^{(+)}$.

In the following theorem we will give sufficient conditions to assure that the pair of homomorphisms (Ψ_{n-1}, Ψ_{1-n}) is a monomorphism and therefore the subquotient $(B, L/\text{Ker}_L B)$ is a special Jordan pair.

Theorem 2. *[10, Corollary 3.4] Let L be a strongly prime Lie algebra over a ring of scalars Φ. Let B be an abelian inner ideal of L and consider $\mathrm{Ker}_L B$. Suppose that there exists $n \in \mathbb{N}$ such that $[B, \mathrm{Ker}_L B]^n \subseteq B$. If $\mathrm{Ker}_L B$ is not a subalgebra of L, then the subquotient $(B, L/\mathrm{Ker}_L B)$ is a special Jordan pair.*

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