Largest Eigenvalue Distribution of Noncircularly Symmetric Wishart-Type Matrices With Application to Hoyt-Faded MIMO Communications

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***Abstract*—This paper is concerned with the largest eigenvalue of the Wishart-type random matrix $\mathbf{W} = \mathbf{X}\mathbf{X}^\dagger$ (or $\mathbf{W} = \mathbf{X}^\dagger\mathbf{X}$), where $\mathbf{X}$ is a complex Gaussian matrix with unequal variances in the real and imaginary parts of its entries, i.e., $\mathbf{X}$ belongs to the noncircularly symmetric Gaussian subclass. By establishing a novel connection with the well-known complex Wishart ensemble, we here derive exact and asymptotic expressions for the largest eigenvalue distribution of $\mathbf{W}$, which provide new insights on the effect of the real-imaginary variance imbalance of the entries of $\mathbf{X}$. These new results are then leveraged to analyze the outage performance of multiantenna systems with maximal ratio combining subject to Nakagami-q (Hoyt) fading.**

***Index Terms*—MIMO, nakagami distribution, performance analysis, rayleigh channels, rician channels.**

I. Introduction

The largest eigenvalue distribution of the so-called Wishart-type random matrices plays an important role in a wide range of applications, including signal detection [1], fading modeling [2] and principal component analysis [3]. Despite the rich characterization of the well-known complex and real Wishart matrices [2]–[5], results for Wishart-type models generated from noncircularly-symmetric complex Gaussians are far more scarce. In particular, we are interested in Hermitian matrices of the form:

$$\mathbf{W} = \begin{cases} \mathbf{X}\mathbf{X}^\dagger, & p \leq n \\ \mathbf{X}^\dagger\mathbf{X}, & p > n \end{cases} \tag{1}$$

where $\mathbf{X} \in \mathbb{C}^{p\times n}$ has independent and identically distributed (i.i.d.) complex Gaussian entries with $\mathrm{Re}([\mathbf{X}]_{ij}) \sim \mathcal{N}(0, \sigma^2_{\mathrm{Re}})$ and $\mathrm{Im}([\mathbf{X}]_{ij}) \sim \mathcal{N}(0, \sigma^2_{\mathrm{Im}})$, i.e., with arbitrary variances in the real and imaginary parts, and where $\mathrm{Re}([\mathbf{X}]_{ij})$ and $\mathrm{Im}([\mathbf{X}]_{ij})$ are mutually independent. The model in (1) has been referred to as the cross-over ensemble between the Laguerre unitary (LUE) and orthogonal (LOE) ensembles [6]. When both variances are equal, i.e. $\sigma^2_{\mathrm{Re}} = \sigma^2_{\mathrm{Im}}$, $\mathbf{W}$ is a central complex Wishart matrix (LUE); when one of the variances is zero, $\mathbf{W}$ is a central real Wishart matrix (LOE). Similar intermediate ensembles have been a subject of interest in the context of (non Wishart-type) Gaussian ensembles, i.e., GUE and GOE, with applications to the study of energy-level spacings in nuclear physics [7].

With applications to fading modeling, the works in [8] and [6] made initial progress to characterize the intermediate Wishart-type ensemble of (1). In [8], the analysis is restricted to a 2×2 matrix $\mathbf{X}$ and the joint probability density function (jpdf) of the eigenvalues of $\mathbf{W}$ is given in a very complicated form involving six integrals. In [6], the eigenvalue jpdf is derived for arbitrary dimensions by using Brownian motion properties with a fictitious time variable which is related to the ratio between the variances σ^2_{Re} and σ^2_{Im}. However, this expression is still complicated and does not allow for any further insightful analysis. The complexity and scarcity of results for the model in (1) are mainly due to the challenge posed by the power imbalance between σ^2_{Re} and σ^2_{Im}, which renders classical random matrix properties no longer applicable. In particular, despite its cross-disciplinary interest, results for the largest eigenvalue are not available thus far, even for a 2×2 matrix $\mathbf{X}$, and the implications of this real-imaginary variance imbalance remain largely unknown.

In this paper, we propose a new approach to characterize the largest eigenvalue of $\mathbf{W}$ for arbitrary p, n. By exploiting a novel statistical connection between $\mathbf{W}$ and the well-known non-central complex Wishart matrix, we derive an exact expression and an asymptotic expansion (in the tail) for the distribution of the largest eigenvalue, which provides new insights on the effects of the real-imaginary variance imbalance of the entries of $\mathbf{X}$. We then use these expressions to study the performance of multiple-input multiple-output (MIMO) communication systems subject to Nakagami-q (Hoyt) fading, in a similar context as [6], [8]. In particular, our analytical results explain the impact of the fading parameter q on the outage performance of MIMO systems with maximal ratio combining (MIMO-MRC).

II. Connection With the Non-Central Complex Wishart Ensemble

In the model (1), it is convenient to rewrite $\mathbf{X}$ as

$$\mathbf{X} = \mathbf{X}_C + \mathbf{X}_R \tag{2}$$

where the matrices $\mathbf{X}_C \in \mathbb{C}^{p\times n}$ and $\mathbf{X}_R \in \mathbb{R}^{p\times n}$ are mutually independent with zero-mean i.i.d. entries. The entries $[\mathbf{X}_C]_{ij}$ are circularly-symmetric $\mathcal{C}N(0, \sigma_C^2)$, while $[\mathbf{X}_R]_{ij} \sim \mathcal{N}(0, \sigma_R^2)$ with the correspondence

$$\sigma_C^2 = 2\sigma_{\text{Im}}^2 \, , \qquad \sigma_R^2 = \sigma_{\text{Re}}^2 - \sigma_{\text{Im}}^2$$

where $\sigma_{\text{Re}} \geq \sigma_{\text{Im}}$. Note that this condition on σ_{Re} does not imply any loss of generality, since replacing σ_{Re} with σ_{Im} and vice versa does not affect the eigenvalue statistics of $\mathbf{W}$.

For $\sigma_C^2 = 0$ or $\sigma_R^2 = 0$, $\mathbf{W}$ collapses to a central (real or complex) Wishart matrix, the properties of which have been studied extensively. However, with both $\sigma_C, \sigma_R > 0$, one deviates from such classical matrix models, and the statistical characterization becomes more challenging.

The redefinition in (2) is instrumental to establish a novel connection between $\mathbf{W}$ and the non-central Wishart ensemble, which will in turn facilitate the subsequent analysis of the largest eigenvalue distribution. Key to our approach is to adopt a "condition and average" method which allows us to connect the statistical properties of $\mathbf{W}$ with those of non-central Wishart matrices, and to leverage existing results for such matrices. Specifically, let $\mathbf{W}_R = \mathbf{X}_R \mathbf{X}_R^\dagger$ if $p \leq n$ or $\mathbf{W}_R = \mathbf{X}_R^\dagger \mathbf{X}_R$ otherwise. Also, let $s = \min(p, n)$, $t = \max(p, n)$. When conditioned on $\mathbf{W}_R \in \mathbb{R}^{s \times s}$, $\mathbf{W}$ follows a non-central complex Wishart distribution with t degrees of freedom and real non-centrality parameter $\mathbf{W}_R$, i.e., $\mathbf{W}|\mathbf{W}_R \sim \mathcal{CW}(t, \sigma_C^2 \mathbf{I}_s, \mathbf{W}_R)$. The marginal eigenvalue distributions of $\mathbf{W}$, such as the largest eigenvalue distribution (but similarly for others, e.g., the smallest), can be obtained by averaging those of $\mathbf{W}|\mathbf{W}_R$ over $\mathbf{W}_R$, which belongs to the space of real positive semi-definite matrices.

III. Largest Eigenvalue Distribution of $\mathbf{W}$

A key advantage of the above connection is that it circumvents the need to directly integrate over the joint eigenvalue density of $\mathbf{W}$. Such joint density is known for arbitrary s, t [6], however the expression is complicated, involving Pfaffians of matrices whose entries contain double infinite series with terms involving products of generalized Laguerre polynomials. As such, it appears difficult to compute the distribution of the largest eigenvalue by marginalizing the joint eigenvalue density in [6]. In the following, by exploiting the non-central Wishart connection and adopting a "condition and average" approach, we provide for the first time exact and asymptotic results for the largest eigenvalue distribution of $\mathbf{W}$.

A. Exact Expression

Proposition 1: Consider $\mathbf{W}$ in (1), with $\sigma_C, \sigma_R > 0$. The cumulative distribution function (cdf) of the largest eigenvalue of $\mathbf{W}$ admits

$$F_{\phi_{\max}}(x) = \frac{\text{Pf}(\mathbf{\Xi}_1(x))}{\text{Pf}(\mathbf{\Xi}_2)} \tag{3}$$

where $\text{Pf}(\cdot)$ denotes the matrix Pfaffian operation (see Appendix A), $\mathbf{\Xi}_1(x)$ is $s \times s$ with entries

$$[\mathbf{\Xi}_1(x)]_{ij} = \int_0^\infty \int_0^\infty f_i(x, u) f_j(x, z) \text{sgn}(z - u) du dz \tag{4}$$

with $\text{sgn}(\cdot)$ the signum function, and

$$f_k(x, y) = \sqrt{\frac{e^{-y/\sigma_R^2}}{(\sigma_C^2/2)^{2k} y}} \Bigg[Q_{s+t-2k+1, t-s}\left(\sqrt{2y/\sigma_C^2}, 0\right) - Q_{s+t-2k+1, t-s}\left(\sqrt{2y/\sigma_C^2}, \sqrt{2x/\sigma_C^2}\right) \Bigg] \tag{5}$$

where $Q_{\cdot,\cdot}(\cdot, \cdot)$ is the Nuttall Q-function (see Appendix B) [9]. The matrix $\mathbf{\Xi}_2$ is $s \times s$ with entries

$$[\mathbf{\Xi}_2]_{ij} = \int_0^{+\infty} \int_0^{+\infty} g_i(u) g_j(z) \text{sgn}(z - u) du dz \tag{6}$$

where

$$g_k(y) \triangleq \lim_{x \to \infty} f_k(x, y) = \sqrt{\frac{e^{-y/\sigma_R^2}}{(\sigma_C^2/2)^{2k} y}} Q_{s+t-2k+1, t-s}\left(\sqrt{2y/\sigma_C^2}, 0\right).$$

Proof: Let $\tau = t - s$. Conditioned on $\mathbf{W}_R$, the cdf of the largest eigenvalue $\phi_{\max}$ of $\mathbf{W}$ is given by [2]

$$F_{\phi_{\max}|\mathbf{W}_R}(x) = \Pr(\phi_{\max} \leq x | \mathbf{W}_R) = \frac{\sigma_C^{2s(s-1)-2st} \prod_{i=1}^s e^{-\sigma_C^{-2} w_i}}{(\tau!)^s \prod_{i<j}^s (w_i - w_j)} |\mathbf{\Lambda}(x)| \tag{7}$$

where $w_1 > \ldots > w_s \geq 0$ are the eigenvalues of $\mathbf{W}_R$. The entries of $\mathbf{\Lambda}(x)$ are given in [2, eq. (50)] as a difference between a confluent hypergeometric function and a Nuttall Q-function. Here, for computational reasons, we use the alternative form [10, eq. (20)]

$$[\mathbf{\Lambda}(x)]_{ij} = \frac{\tau! \sigma_C^{3t-s+2i+2} 2^{i-s}}{(2w_j)^{\tau/2} e^{-w_j/\sigma_C^2}} \Bigg[Q_{s+t-2i+1, \tau}\left(\sqrt{2w_j/\sigma_C^2}, 0\right) - Q_{s+t-2i+1, \tau}\left(\sqrt{2w_j/\sigma_C^2}, \sqrt{2x/\sigma_C^2}\right) \Bigg].$$

Since (7) only depends on the eigenvalues of $\mathbf{W}_R$, we remove the conditioning by averaging w.r.t. $f_{w_1,\ldots,w_s}(w_1, \ldots, w_s)$, the jpdf of $w_1 > \ldots > w_s$, which corresponds to the eigenvalue jpdf of a central real Wishart matrix [11, eq. (58)]. This gives

$$F_{\phi_{\max}}(x) = \int_{\mathcal{L}} F_{\phi_{\max}|w_1,\ldots,w_s}(x) \times f_{w_1,\ldots,w_s}(w_1, \ldots, w_s) dw_1 \ldots dw_s = c^{-1} \int_{\mathcal{L}} |\mathbf{\Lambda}(x)| \prod_{i=1}^s w_i^{\frac{\tau-1}{2}} e^{-\sigma_{eq}^2 w_i} dw_1 \ldots dw_s$$

where $\mathcal{L} = \{0 \leq w_s < \ldots < w_1 < \infty\}$, $\sigma_{eq}^2 = \sigma_C^{-2} + \sigma_R^{-2}/2$ and c is a normalization constant. Using [12, eq. (4.5)], the multiple integral is expressed as

$$F_{\phi_{\max}}(x) = c^{-1} \text{Pf}(\mathbf{\Xi}_1(x)) \tag{8}$$

where the elements of $\mathbf{\Xi}_1(x)$ are given by (4). Moreover, from (8), it is clear that $c = \lim_{x \to \infty} \text{Pf}(\mathbf{\Xi}_1(x))$, which can be also expressed as $c = \text{Pf}(\mathbf{\Xi}_2)$ after interchanging the limit and the double integral thanks to Lebesgue's dominated convergence theorem. The elements of Ξ_2 are given in (6), since the second Nuttall Q-function in (5) vanishes when $x \to \infty$. ■

The specific computation of the Pfaffians in (3) depends on whether the matrix dimension is even or odd. In either case, however, they may be evaluated as the square root of a matrix determinant, as detailed in Appendix A.

Although this expression is exact, it does not give insight into how the largest eigenvalue distribution behaves when the variance imbalance between the real and imaginary parts of $\mathbf{X}$ changes. With this in mind, our next objective is to establish a simplified tail expansion for the distribution, which is again obtained by exploiting the non-central Wishart connection, allowing us to leverage known results for such Wishart model.

B. Asymptotic Expansion in the Tail

Proposition 2: As $x \to 0$,

$$F_{\phi_{\max}}(x) = h_{s,t}\, a_{s,t}^{\mathrm{CW}} x^{st} + o(x^{st}) \tag{9}$$

where

$$a_{s,t}^{\mathrm{CW}} = \prod_{i=1}^{s} \frac{(s-i)!}{(s+t-i)!}$$

and

$$h_{s,t} = \frac{1}{\left(\sigma_C \sqrt{2\sigma_R^2 + \sigma_C^2}\right)^{st}}\ . \tag{10}$$

Proof: When conditioned on $\mathbf{W}_R$, as $x \to 0$ we have [10, eq. (28)]

$$F_{\phi_{\max}|\mathbf{W}_R}(x) = \frac{\sigma_C^{-2st} \prod_{i=1}^{s}(s-i)!}{\prod_{i=1}^{s}(s+t-i)!} \prod_{i=1}^{s} e^{-\sigma_C^{-2} w_i}\, x^{st} + o(x^{st}).$$

Removing the conditioning as in the proof of Proposition 1, we get

$$\begin{aligned} F_{\phi_{\max}}(x) = {} & \frac{2^{-\frac{st}{2}} \pi^{\frac{s^2}{2}} \sigma_R^{-st} \sigma_C^{-2st} \prod_{i=1}^{s}(s-i)! x^{st}}{\prod_{i=1}^{s} \Gamma(\frac{t}{2}-i+1)\Gamma(\frac{s}{2}-i+1)(s+t-i)!} \\ & \times \int_{\mathcal{L}} \prod_{m=1}^{s} e^{-\sigma_{eq}^2 w_m} w_m^{\frac{\tau-1}{2}} \prod_{i<j}^{s} (w_i - w_j) \prod_{l=1}^{s} dw_l \\ & + o(x^{st}). \end{aligned}$$

Making the multiple change of variables $\lambda_m = \sigma_{eq}^2 w_m$, $m = 1, \ldots, s$, the remaining multiple integral can be identified with the Selberg integral for the central real Wishart matrix [11, eq. (58)]. This leads to the result. ■

In (9), $a_{s,t}^{\mathrm{CW}}$ corresponds to the expansion coefficient for the largest eigenvalue distribution of a central complex Wishart matrix [10], [13]. Hence, in the left-hand tail, the effect of the unbalanced variances of the real and imaginary components of $\mathbf{X}$ is clearly revealed, and is decoupled into the function $h_{s,t}$ in (10). This simplicity is quite remarkable, particularly when considering the complexity of the exact largest eigenvalue distribution (3), as well as that of the joint eigenvalue distribution derived in [6].

To better interpret the result in (9), we express the function $h_{s,t}$ in terms of the original variances σ_{Re}^2 and σ_{Im}^2, while we fix the total variance as $\sigma_{\mathrm{Re}}^2 + \sigma_{\mathrm{Im}}^2 = 1$. Then, for $\sigma_{\mathrm{Re}} \in (0,1)$

$$h_{s,t} = \frac{1}{\left(4\sigma_{\mathrm{Re}}^2(1-\sigma_{\mathrm{Re}}^2)\right)^{st/2}}\ . \tag{11}$$

From (11), $h_{s,t}$ is clearly minimized at the "balanced" case, $\sigma_{\mathrm{Re}}^2 = 1/2$, for which $h_{s,t} = 1$, as it should. One observes a significant deviation, however, as the variances of the real and imaginary components become more unbalanced. That is, the higher the imbalance, the faster the decay of the left-hand tail of $F_{\phi_{\max}}(x)$.

IV. Outage Performance of MIMO-MRC Systems in Nakagami-q (Hoyt) Environments

We here provide an application example for the derived results. Consider a communication link between a transmitter, equipped with N_t antennas, and a receiver with N_r antennas. The multi-antenna link is subject to Nakagami-q (Hoyt) fading, typically assumed in satellite-based communications [14] or, in general, when the fading conditions are more severe than those of a Rayleigh-faded environment. The channel is modeled by $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$ with zero-mean i.i.d. entries $[\mathbf{H}]_{ij}$ representing the complex gain between the j-th transmit and the i-th receive antennas, where $\mathrm{Re}([\mathbf{H}]_{ij}) \sim \mathcal{N}(0, \sigma_{\mathrm{Re}}^2)$ and $\mathrm{Im}([\mathbf{H}]_{ij}) \sim \mathcal{N}(0, \sigma_{\mathrm{Im}}^2)$ are mutually independent, $\sigma_{\mathrm{Re}}^2 = 1/(1+q^2)$, $\sigma_{\mathrm{Im}}^2 = q^2/(1+q^2)$, and $q \in (0,1)$ denotes the Hoyt fading parameter, which defines the power imbalance between the real and imaginary channel components. Invoking the redefinition in (2),

$$\mathbf{H} = \sqrt{\frac{2q^2}{1+q^2}}\mathbf{H}_C + \sqrt{\frac{1-q^2}{1+q^2}}\mathbf{H}_R$$

where the entries of $\mathbf{H}_C$ are circularly-symmetric $\mathcal{CN}(0,1)$, while those of $\mathbf{H}_R$ are $\mathcal{N}(0,1)$. Define $\mathbf{W}_{\mathrm{H}} = \mathbf{H}\mathbf{H}^\dagger$, if $N_r \le N_t$, or $\mathbf{W}_{\mathrm{H}} = \mathbf{H}^\dagger\mathbf{H}$ otherwise.

We further assume that the receiver has perfect knowledge of $\mathbf{H}$ while the transmitter, with only partial knowledge, uses the well-known beamforming (BF) principle [15] to send data with a total fixed power P. The noise at each receive antenna is assumed independent $\mathcal{CN}(0,1)$ and we define the transmit signal to noise ratio (SNR) as $\bar{\gamma} \triangleq P$. The received signal vector $\mathbf{r} \in \mathbb{C}^{N_r}$ can then be expressed as

$$\mathbf{r} = \sqrt{\bar{\gamma}}\mathbf{H}\mathbf{w}x + \mathbf{n}$$

where x is the transmitted symbol with $\mathbb{E}[|x|^2] = 1$, $\mathbf{n}$ is the noise vector, and $\mathbf{w}$ is the BF vector with $||\mathbf{w}|| = 1$. The detection of x is optimal when $\mathbf{w}$ equals the eigenvector corresponding to the largest eigenvalue of $\mathbf{W}_{\mathrm{H}}$ and when the MRC principle is applied to the received signal $\mathbf{r}$, which yields a post-processing SNR [15]

$$\gamma = \bar{\gamma}\lambda_{\max}$$

where $\lambda_{\max}$ denotes the largest eigenvalue of $\mathbf{W}_{\mathrm{H}}$.

Defining γ_{th} as the minimum required SNR for a reliable communication (i.e., with x reliably detected), the outage probability is exactly obtained from Proposition 1 as

$$P_{\mathrm{out}} = \Pr(\gamma \le \gamma_{\mathrm{th}}) = F_{\phi_{\max}}\left(\frac{\gamma_{\mathrm{th}}}{\bar{\gamma}}\right) \tag{12}$$

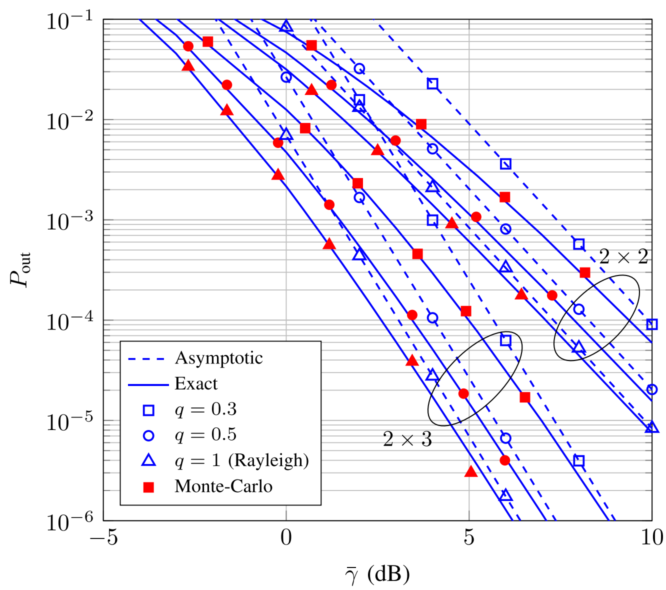

Fig. 1. Outage probabilities of 2×2 and 2×3 MIMO-MRC systems under Nakagami-q (Hoyt) fading for different values of q; $\gamma_{\text{th}} = 0$ dB.

with $\sigma_C^2 = 2q^2/(1+q^2), \sigma_R^2 = (1-q^2)/(1+q^2), s = \min(N_t, N_r)$ and $t = \max(N_t, N_r)$.

To give further insight, we approximate P_{out} by leveraging the asymptotic characterization of Proposition 2. In practice, we are interested in small outage probabilities, i.e., implying small values of $\gamma_{\text{th}}/\bar{\gamma}$. From Proposition 2, as $\gamma_{\text{th}}/\bar{\gamma} \to 0$,

$$P_{\text{out}} \approx \hat{P}_{\text{out}} = a_{s,t}^{\text{CW}} \left(\frac{\gamma_{\text{th}}}{\theta(q)\bar{\gamma}} \right)^{st} \tag{13}$$

for $q > 0$, where $\theta(q) = 2q/(1+q^2)$ and st is the diversity order. This explicitly reveals that the power gain one gets by steering the signal along the top channel eigenmode decreases as the real and imaginary channel components get more imbalanced. Specifically, the effect of such imbalance can be seen as a reduction in the average SNR—w.r.t. the perfectly balanced case (Rayleigh fading, $q = 1$)—by a factor of $\theta(q) < 1$. The outage probability is then degraded (increased) by a factor of $\theta(q)^{-st}$, which can be approximated by $(2q)^{-st}$ for small q. We remark that this degradation is exponentially accentuated as the number of antennas increases.

Fig. 1 depicts the outage probability as a function of the SNR $\bar{\gamma}$ for 2×2 and 2×3 MIMO systems, operating in three different fading conditions ($q = 1$, $q = 0.5$ and $q = 0.3$). For each q, we plot: (i) the empirical probability—obtained from Monte-Carlo simulations (10^6 realizations), (ii) the exact P_{out} in (12)—where we compute $F_{\phi_{\max}}(x)$ from (3) numerically,[1] and (iii) the asymptotic outage probability $\hat{P}_{\text{out}}$ in (13). We observe a good agreement between (exact) analytical and simulated results in all cases and, as expected, the asymptotic result converges to the exact one as P_{out} gets smaller. As anticipated, P_{out} degrades significantly as q decreases, with such degradation being more pronounced in the 2×3 setting. Even for a dual-antenna (2×2) system, this degradation at $\bar{\gamma} = 10$ dB is approximately one order of magnitude for the case $q = 0.3$ (90% of the channel gain in the real part and 10% in the imaginary part, or vice versa) w.r.t. the case $q = 1$ (Rayleigh). This is consistent with the analytical prediction above, where the P_{out} degradation was given by the factor $\theta(q)^{-st}$ as $\gamma_{\text{th}}/\bar{\gamma} \to 0$. For $q = 0.3$ and $st = 4$, this factor yields ~ 10.9.

[1]To compute (4) and (6), we use the Gauss-Laguerre integration method, where the Nuttall Q-functions are computed as finite sums of Marcum Q-functions, since the difference of their orders is odd (see Appendix B).

TABLE I
Data Rate Loss of a 2×2 MIMO System at $\bar{\gamma} = 10$ dB, $\epsilon = 10^{-3}$

q	0.5	0.4	0.3	0.2
RL	9%	17%	26%	37%
$\hat{RL}$	9%	18%	27%	43%

We are also interested in the outage data rate, defined as the largest transmission rate (in bits/s/Hz) that can be reliably guaranteed at least $(1-\epsilon) \times 100\%$ of the time, i.e.

$$R_{\text{out}}(\epsilon) = \sup_{R \geq 0}(R : P_{\text{out}}(R) < \epsilon)$$

where ϵ is the maximum outage level and $P_{\text{out}}(R)$ is the outage probability for a given data rate R, i.e.

$$P_{\text{out}}(R) \triangleq \Pr\left(\log_2(1+\gamma) \leq R\right) = F_{\phi_{\max}}((2^R - 1)/\bar{\gamma}).$$

Thus,

$$R_{\text{out}}(\epsilon) = \log_2(1 + \bar{\gamma} F_{\phi_{\max}}^{-1}(\epsilon)) \tag{14}$$

where $F_{\phi_{\max}}^{-1}(\cdot)$ denotes the inverse function of $F_{\phi_{\max}}(\cdot)$. Again, Proposition 2 allows us to approximate $R_{\text{out}}(\epsilon)$ for small values of ϵ by a compact and insightful expression; as $\epsilon \to 0$,

$$R_{\text{out}}(\epsilon) \approx \hat{R}_{\text{out}}(\epsilon, q) = \log_2\left(1 + \theta(q)\bar{\gamma}\left(\frac{\epsilon}{a_{s,t}^{\text{CW}}}\right)^{1/st}\right)$$

where, once more, we see the effect of the Nakagami-q fading through the isolated factor $\theta(q)$ which, for $q < 1$, causes a reduction in the "effective" average SNR. To better illustrate the proportional outage rate degradation w.r.t. the Rayleigh case ($q = 1$), we define the (approximated) fractional rate loss

$$\hat{RL}(\%) \approx \frac{\hat{R}_{\text{out}}(\epsilon, 1) - \hat{R}_{\text{out}}(\epsilon, q)}{\hat{R}_{\text{out}}(\epsilon, 1)} \times 100. \tag{15}$$

Table I shows the approximated $\hat{RL}$ along with the exact RL, computed with (14) by numerically inverting $F_{\phi_{\max}}(\cdot)$, for a 2×2 setting with a maximum outage level of $\epsilon = 10^{-3}$ and $\bar{\gamma} = 10$ dB. The numbers reveal a substantial rate degradation as q gets smaller, up to $\sim 37\%$ loss ($q = 0.2$).

V. Conclusion

We have simplified the study of noncircularly-symmetric Wishart-type matrices, for which results are generally scarce thus far due to the significant analytical challenge posed by the underlying real-imaginary variance imbalance of the entries of the model. In particular, we have introduced a new analytical approach, based on a novel connection between the statistics of these matrices and the well-known non-central complex Wishart ensemble, which allowed us to derive, for the first time, exact and asymptotic expressions for the distribution of the largest eigenvalue. Particularly noteworthy is the insight brought by such asymptotic expansion, which clearly explains the effect of the real-imaginary variance imbalance of the model. The usefulness of this result has been exemplified through the analysis of MIMO-MRC systems under Nakagami-q (Hoyt) fading, where we have characterized the outage performance loss associated with the level of fading severity (value of q). Specifically, we have observed that a greater imbalance between the real and imaginary channel components (i.e., smaller q) significantly degrades the outage performance, with the outage probability being increased (w.r.t. the Rayleigh case, $q = 1$) by a factor $\theta(q)^{-N_t N_r}$, where $\theta(q) = 2q/(1+q^2)$.

As a closing remark, one could use the same analytical approach, based on the non-central Wishart connection here unveiled, to derive other statistics of the noncircularly-symmetric Wishart-type model that are not yet available, e.g., the distribution of the smallest eigenvalue or, more generally, that of the kth largest eigenvalue, useful in the characterization of multichannel beamforming wireless systems.

Appendix

A. Pfaffian Computation

The Pfaffians in (3) can be computed as follows [12]. Let $\mathbf{A}$ be an $s \times s$ matrix with entries

$$[\mathbf{A}]_{ij} = \int_a^b \int_a^b \psi_i(x)\psi_j(y)\text{sgn}(y-x)dxdy$$

for some functions $\psi_i(\cdot)$, $i = 1, \ldots, s$. Then,

$$\text{Pf}(\mathbf{A}) = \begin{cases} \sqrt{|\mathbf{A}|}\,, & s \text{ even} \\ \sqrt{|\mathbf{A}^+|}\,, & s \text{ odd} \end{cases}$$

where $\mathbf{A}^+$ is a $(s+1) \times (s+1)$ matrix with entries

$$[\mathbf{A}^+]_{ij} = \begin{cases} [\mathbf{A}]_{ij} & i,j = 1, \ldots, s \\ \int_a^b \psi_i(x)dx & i = 1, \ldots, s+1, j = s+1 \\ -\int_a^b \psi_j(x)dx & i = s+1, j = 1, \ldots, s+1 \\ 0 & i = j = s+1. \end{cases}$$

B. Nuttall Q-Function

The Nuttall Q-function is defined as [9]

$$Q_{m,n}(a,b) = \int_b^\infty x^m e^{-\frac{x^2+a^2}{2}} I_n(ax)dx$$

where $a \in \mathbb{R}^+$, $b \in \mathbb{R} < \infty$, $m, n \in \mathbb{Z}^+$ and $I_\nu(x)$ is the νth order modified Bessel function of the first kind with $\nu \in \{\mathbb{R} > -\frac{1}{2}\}$, i.e. [16, eq. (9.6.18)]

$$I_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} e^{\pm zt} dt.$$

When $m + n$ is odd, we can express the Nuttall Q-function in terms of finite sum of generalized Marcum Q- and modified Bessel functions of the first kind, i.e. [17, eq. (8)]

$$Q_{n+2k+1,n}(a,b) = \sum_{l=1}^{k+1} c_{k,l} a^{n+2(l-1)} Q_{n+l}(a,b) + e^{-\frac{a^2+b^2}{2}} \times \sum_{r=1}^{k} P_{k,r}(b^2) a^{r-1} b^{n+r+1} I_{n+r-1}(ab)$$

where $n, k \in \mathbb{N}$,

$$c_{k,l} = 2^{k-l+1} \frac{k!}{(l-1)!} \binom{k+n}{k-l+1}$$

and $P_{k,r}(b^2) = \sum_{j=0}^{k-r} d_{j,k,r} b^{2j}$ with

$$d_{j,k,r} = 2^{k-r-j} \frac{(k-1-j)!}{(r-1)!} \binom{k+n}{k-r-j}.$$

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