BASIC IDEALS IN EVOLUTION ALGEBRAS

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Abstract. With the aim of finding useful tools and invariants to classify finite dimensional evolution algebras, we introduce and study the notion of a basic ideal. Every n-dimensional perfect evolution algebra has a maximal basic ideal I which is unique except when the dimension of I is n-1. An application of our results leads to the description of the four dimensional perfect non-simple evolution algebras over a field with mild restrictions.

1. Introduction and preliminary results

The motivation for this paper has been the search of algebraic tools which allow to have a deeper understanding of finite dimensional evolution algebras, and also to find invariant properties under change of basis.

Evolution algebras, which are algebras modelling non-Mendelian genetics, appeared for the first time in the 2006 paper [11] by Tian and Vojtechovsky. Two years later Tian publishes [10], a pioneering monograph where many connections with other mathematical branches (such as graph theory, stochastic processes, group theory, dynamical systems, mathematical physics, and others) are established. These publications were followed by a flurry of activity on this new area. In [3], the authors accomplish a systematic study of evolution algebras of arbitrary dimension. This paper also analyzes the notions of evolution subalgebra, ideal, non-degeneracy and simplicity. In particular, the authors characterize the simple evolution algebras using the description of the ideal generated by one element. Of special interest are the results related to the decomposition of an evolution algebra into its irreducible components.

The classification of a particular type of algebras is always a major problem, and the case of evolution algebras is not an exception. In the paper [6] the two dimensional evolution algebras over the complex field are classified. The classification over a general field with some restrictions on the characteristic is achieved in [2], while the general case has been analyzed in [8, 5]. There are 10 mutually non-isomorphic types of two dimensional evolution algebras.

The three dimensional evolution algebras over a field having characteristic different from 2 and in which there are roots of orders 2, 3 and 7 were classified in [2, 4]: there exist 116 non-isomorphic families. The difficulty increases exponentially while the dimension increases linearly. There are several reasons. First, the computations become more complicated because of the number of cases, and second, the properties considered in each dimension are not sufficient or are not affordable for bigger dimensions. To visualize this difficulty, note that, in the classification of three dimensional evolution algebras, the matrices that can act as change

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of basis matrices (when the algebra is perfect) are the permutation matrices, only six, while in the four dimensional case there are twenty four.

We center our attention in the perfect evolution algebras (an algebra A is said to be perfect if $A^2 = A$). The notion we found fruitful has been that of basic ideal, i.e., an ideal having a natural basis which can be extended to a natural basis of the whole algebra. We prove that there exist maximal basic ideals, which are unique except when their dimension is $n-1$ (where n is the dimension of the evolution algebra). Thanks to this, we find that the number of zeros in the structure matrix is an invariant (except when there are maximal (n-1)-dimensional basic ideals).

While the first part of this paper is devoted to finite dimensional perfect evolution algebras, in the second one we specialize to four dimensional evolution algebras. We start the classification of the four dimensional evolution algebras and determine the perfect non-simple ones over a field having characteristic different from 2 containing roots of orders 2, 3 and 7.

The possible isomorphisms among evolution algebras in the same family are determined in [1].

The article is divided into five sections, being the first one expended to the introduction and the preliminaries. In the second one we introduce the notion of basic ideal. An ideal is said to be basic if it has the extension property, i.e., it contains a natural basis which can be extended to a natural basis of the whole algebra (see [3] for the extension property). The notion of basic ideal does not depends on the natural basis it contains (Lemma 2.3. We prove that there exist maximal i-basic ideals (maximal basic ideals of dimension i) for some $i < \dim(A)$ (Lemma 2.5), which are not necessarily unique (Example 2.6) and characterize the basic simplicity of perfect evolution algebras (Proposition 2.7).

The maximal basic ideals will play a fundamental role in the classification of the four dimensional perfect non simple evolution algebras. On the one hand, the classification relays on the dimension of the maximal basic ideals; on the other hand, if I is a maximal basic ideal of a perfect evolution algebra A having dimension n, then A/I is a basic simple evolution algebra (that is, an algebra having no basic ideals) whenever the dimension of I is i, for $i \in \{0, 1, \ldots, n-1\}$ (Proposition 2.10).

Important key points for the general classification are: Firstly, showing that the number of zero entries in the structure matrix is invariant (Proposition 2.13). Secondly, writing the structure matrix of a perfect non-simple evolution algebra in the form $\begin{pmatrix} W & U \\ 0 & Y \end{pmatrix}$, where W is the structure matrix of some maximal i-basic ideal I and Y is the structure matrix of the evolution algebra A/I (Proposition 2.9). And thirdly, proving that for $0 \neq s \leq \left[\frac{n}{2}\right]$ the possible permutations producing isomorphic evolution algebras having structure matrices of this form act as permutations in each block (Theorem 2.15); a consequence is that, in these cases, the number of zeros in W, U and Y is invariant.

Under a certain restriction, a perfect n-dimensional evolution algebra has an $n-1$-basic ideal producing a “nice” block decomposition of the structure matrix, i.e., a decomposition such that the number of zeros is invariant in each block (see Proposition 2.19). This restriction is not to satisfy Condition $(n-1, n-2, n-1)$ (see Definition 2.24).

The rest of the paper is devoted to the classification of perfect four dimensional evolution algebras. To this end we consider whether or not the evolution algebra is irreducible. When the algebra can not be decomposed we distinguish cases depending on the dimension of a maximal basic ideal (which always exists), which can be one, two or three. Each of these cases is considered in Sections 3, 4 and 5, respectively.

Sections 3 and 4 follow the same philosophy: we fix W and classify depending on the number of zeros in U. In Section 3 the matrix W is always the 1×1 matrix (1), and the number of zeros in U can be 2, 1 or 0. The matrix W in Section 4 corresponds to the structure matrix of a 2-dimensional perfect evolution algebra. Looking at the classification of these evolution algebras (see [2]) we distinguish five cases (4.1 through 4.5), each of which is classified depending on the number of zeros in U (3, 2, 1 or 0). Once we have fixed W and U, we discuss which are the matrices Y associated to simple three, respectively two dimensional, evolution algebras that can be considered: three dimensional in Section 3 and two dimensional in Section 4.

The last section deals with the case in which W is the structure matrix of a 3-basic ideal, say I. We will consider only the cases in which A is an irreducible evolution algebra (this is explained in Figure 1). Subsections 5.1 and 5.2 distinguish the cases for which I does or does not have, respectively, a 2-basic ideal. Finally, Subsection 5.1 is divided into two cases depending on A having another 3-basic ideal J such that $I \cap J$ is a 2-basic ideal of I and J (in this case A is said to satisfy Condition (3,2,3)). We use the associated graph to determine all the algebras satisfying this condition.

From now on A will be a four dimensional evolution algebra having characteristic different from 2 and in which there are roots of orders 2, 3 and 7. Recall that an *evolution algebra* over a field $\mathbb{K}$ is a $\mathbb{K}$-algebra A provided with a basis $B = \{e_i \mid i \in \Lambda\}$ such that $e_i e_j = 0$ whenever $i \neq j$ (where Λ is a non-empty arbitrary set). Such a basis B is called a *natural basis*. Fixed a natural basis B in A, the scalars $\omega_{ki} \in \mathbb{K}$ such that $e_i^2 := e_i e_i = \sum_{k \in \Lambda} \omega_{ki} e_k$ will be called the *structure constants* of A *relative to* B, and the matrix $M_B := (w_{ki})$ is said to be the *structure matrix of* A *relative to* B. We will write $M_B(A)$ to emphasize the evolution algebra we refer to and simply will speak about the structure matrix. Every evolution algebra is uniquely determined by its structure matrix.

An algebra A is said to be *simple* if $A^2 \neq 0$ and A has no nonzero proper ideals.

To reach the classification of four dimensional perfect non-simple evolution algebras, one of the key points will be the graph associated to the evolution algebra. We recall here some of the essential notions for a graph. A *directed graph* is a 4-tuple $E = (E^0, E^1, r_E, s_E)$ consisting of two disjoint sets E^0, E^1 and two maps $r_E, s_E : E^1 \to E^0$. The elements of E^0 are called the *vertices* of E and the elements of E^1 the *edges* of E, while for $f \in E^1$ the vertices $r_E(f)$ and $s_E(f)$ are called the *range* and the *source* of f, respectively. If there is no confusion with respect to the graph we are considering, we simply write $r(f)$ and $s(f)$. If E^0 and E^1 are finite we will say that E is *finite*.

A vertex which emits no edges is called a *sink*. A vertex which does not receive any vertex is called a *source*. A *path* μ in a graph E is a finite sequence of edges $\mu = f_1 \dots f_n$ such that $r(f_i) = s(f_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) := s(f_1)$ and $r(\mu) := r(f_n)$ are the *source* and *range* of μ, respectively, and n is the *length* of μ. This fact will be denoted by $|\mu| = n$. We also say that μ is *a path from* $s(f_1)$ *to* $r(f_n)$ and denote by μ^0 the set of its vertices, i.e., $\mu^0 := \{s(f_1), r(f_1), \dots, r(f_n)\}$. On the other hand, by μ^1 we denote the set of edges appearing in μ, i.e., $\mu^1 := \{f_1, \dots, f_n\}$. We view the elements of E^0 as paths of length 0. The set of all paths of a graph E is denoted by $\text{Path}(E)$. Let $\mu = f_1 \dots f_n \in \text{Path}(E)$. If $n = |\mu| \geq 1$, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at* v. If $\mu = f_1 \dots f_n$ is a closed path based at v and $s(f_i) \neq s(f_j)$ for every $i \neq j$, then μ is called a *cycle based at* v or simply a *cycle*. A cycle of length 1 will be said to be a *loop*.

Given a finite graph E, the *adjacency matrix* is the matrix $Ad_E = (a_{ij}) \in \mathbb{Z}^{(E^0 \times E^0)}$ given by $a_{ij} = |\{\text{edges from } i \text{ to } j\}|$.

A graph E is said to satisfy *Condition* (Sing) if among two vertices of E^0 there is at most one edge. There are different ways in which we can associate a graph to an evolution algebra, but for our classification purposes, it will be enough to consider only graphs satisfying Condition (Sing), that is, the graph we will consider will satisfy that $Ad_E = (a_{ij})$ has all its entries in $\{0, 1\}$. A graph E is said to be *strongly connected* if given two different vertices u, v, there exists a path μ such that $s(\mu) = u$ and $r(\mu) = v$.

Given a natural basis $B = \{e_i \mid i \in \Lambda\}$ of an evolution algebra A and given its structure matrix $M_B = (\omega_{ji}) \in \mathrm{M}_\Lambda(\mathbb{K})$, consider the matrix $P^t = (p_{ji}) \in \mathrm{M}_\Lambda(\mathbb{K})$ such that $p_{ji} = 0$ if $\omega_{ji} = 0$ and $p_{ji} = 1$ if $\omega_{ji} \neq 0$. The *graph associated to the evolution algebra* A (relative to the basis B), denoted by E_A^B (or simply by E if the algebra A and the basis B are understood) is the graph whose adjacency matrix is $P = (p_{ij})$ (see, for example, [3]).

Remark 1.1. In general, the graph associated to an evolution algebra A depends on the selected natural basis and non-isomorphic graphs can give rise to the same evolution algebra (see [3, Example 2.34]). However, when the algebra A is finite dimensional and satisfies $A = A^2$, the graph is uniquely determined, as follows by [7, Corollary 4.5]. We will denote it by E_A. Moreover, when the evolution algebra is non-degenerate (which is the case, as being perfect implies being non-degenerate), the associated graph is connected if and only if the evolution algebra is irreducible (see, for example [3, Corollary 5.8]).

2. Basic ideals

A notion that will be very useful in order to classify the evolution algebras we are studying is that of basic ideal, which we introduce here.

Definitions 2.1. Let A be an evolution algebra, I be a proper ideal of A and $i \in \{0, 1, 2 \dots\}$. We will say that I is an *i-basic ideal relative to* a natural basis $B = \{e_j \mid j \in \Lambda\}$ (where Λ is a nonempty set) if I is generated by i vectors from B. A basic i-ideal I (relative to a basis B) will be called a *maximal i-basic ideal relative to* B if there are no basic ideals relative to B having dimension $j > i$.

If A has no basic ideals relative to any basis, then we will say that A is *basic simple*.

Remark 2.2. Note that every nonzero basic ideal is an evolution ideal and has the extension property (see [3, Definitions 2.4] for the definitions).

Lemma 2.3. *Let A be a perfect finite dimensional evolution algebra and let I be an i-basic ideal relative to a natural basis B. Then I is an i-basic ideal relative to any natural basis of A.*

Proof. Since $A^2 = A$ then the only natural basis that A has consists of a permutation of the elements of B (by [7, Theorem 4.4]). □

Lemma 2.3 gives rise to the following definitions.

Definitions 2.4. Let A be a perfect finite dimensional evolution algebra and let I be an ideal of A.

(i) I is an *i-basic ideal* if it is an i-basic ideal relative to any natural basis.
(ii) I is a *maximal i-basic ideal* if it is a maximal i-basic ideal relative to any natural basis.
(iii) A is *basic simple* if the ideal generated by every element in any natural basis is A.

Lemma 2.5. *Let A be a perfect finite dimensional evolution algebra. Then A contains a maximal i-basic ideal for some $i \in \{0, 1, \dots, n-1\}$.*

Proof. As follows by Lemma 2.3, the existence of maximal basic ideals does not depend on the natural basis. Let B be any natural basis of A. Since A is perfect, $e^2 \neq 0$ for any $e \in B$. If the ideal generated by every element of B is A, then 0 is a maximal basic ideal. Otherwise, let $i \neq \dim(A)$ be the maximum of the dimensions of the ideals generated by elements in the basis B. Then A has a maximal i-basic ideal. □

The example that follows shows that maximal basic ideals are not, necessarily unique. However, there are some cases in which the unicity is true.

Example 2.6. A maximal basic ideal is not necessarily unique. For an example, consider the evolution algebra A having a natural basis $B = \{e_1, e_2, e_3, e_4\}$ and with product given by $e_1^2 = e_2$, $e_2^2 = e_1$, $e_3^2 = e_1 + e_2 + e_3$, $e_4^2 = e_2 + e_4$. and take I and J the ideals generated by $\{e_1, e_2, e_3\}$ and $\{e_1, e_2, e_4\}$, respectively. Then I and J are maximal 3-basic ideals.

Moreover, the uniqueness does not depend on the irreducibility of the algebra, as the evolution algebra A we have considered in this example is irreducible.

Now we relate the notions of simple and basic simple evolution algebra.

Proposition 2.7. *Let A be a finite dimensional evolution algebra A. The following are equivalent conditions:*

(i) *A is simple.*
(ii) *$A = A^2$ and E_A is strongly connected.*
(iii) *$A = A^2$ and A is basic simple.*

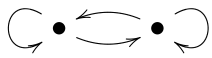
Proof. (i) ⇒ (ii). If A is simple, then $A = A^2 \neq 0$ and the graph E_A is unique (see [7, Corollary 4.5]). Moreover, E_A has to be strongly connected because if there are two vertices which are not connected, then there exists a proper nonzero ideal.

(i) ⇔ (iii) follows by [4, Corollary 4.6].

(ii) ⇒ (iii). Since E_A is unique (because A is perfect) and a nonzero proper basic ideal would give rise to a non strongly connected graph, the result follows. □

We cannot eliminate the hypothesis $A = A^2$ in Proposition 2.7, as shown in the example that follows.

Example 2.8. Consider the evolution algebra A having a natural basis $\{e_1, e_2\}$ such that $e_i^2 = e_1 + e_2$, for $i = 1, 2$. This evolution algebra is neither simple nor basic simple as $e_1 + e_2$ generates a one dimensional ideal (which is a basic ideal). However, one associated graph for A is



which is strongly connected.

In terms of the structure matrix, the simplicity of an evolution algebra can be characterized as in [3, Corollary 4.6]. For the sake of completeness, we include the statement of that result.

Proposition 2.9. *Let A be a finite-dimensional evolution algebra of dimension n and $B = \{e_i \mid i \in \Lambda\}$ a natural basis of A. Then A is simple if and only if the determinant of the structure matrix $M_B(A)$ is nonzero and B cannot be reordered in such a way that the corresponding structure matrix is as follows:*

$$\begin{pmatrix} W_{m\times m} & U_{m\times(n-m)} \\ 0_{(n-m)\times m} & Y_{(n-m)\times(n-m)} \end{pmatrix},$$

for some $m \in \mathbb{N}$ with $m < n$ and matrices $W_{m\times m}$, $U_{m\times(n-m)}$ and $Y_{(n-m)\times(n-m)}$.

For a subset S of an algebra A we will denote by $\langle S\rangle$ the ideal of A generated by S.

Proposition 2.10. *Let A be a finite n-dimensional perfect evolution algebra (for $n \geq 2$) and let I be a maximal i-basic ideal, where $i \in \{0, \ldots, n-1\}$. Then I is a perfect evolution algebra and A/I is a basic simple perfect evolution algebra.*

Proof. The case $i = 0$ follows immediately. Assume $i \neq 0$. Let $B = \{e_j \mid j \in \{1, \ldots, n\}\}$ be a natural basis of A such that $I = \langle\{e_1, \ldots, e_i\}\rangle$. Write $M_B = \begin{pmatrix} W & U \\ 0_{n-i} & Y \end{pmatrix}$, where $W \in M_i(\mathbb{K})$, $U \in M_{i\times n-i}(\mathbb{K})$, $Y \in M_{n-i}(\mathbb{K})$.

It is clear that I is an evolution algebra having natural basis $B_I := \{e_1, \ldots, e_i\}$. Also, A/I is an evolution algebra with $\overline{B} := \{\overline{e_{i+1}}, \ldots, \overline{e_n}\}$ being a natural basis. Note that $M_{B_I} = W$ and $M_{\overline{B}} = Y$. We claim that I and A/I are perfect evolution algebra.

By [9, Theorem 2] we have $|M_B| = |W||Y|$. Since A is perfect, $|M_B| \neq 0$; therefore $|W| \neq 0$ and $|Y| \neq 0$. This shows our claim.

Now we prove that A/I is basic simple. For $i = n-1$ the result follows because A/I is a one-dimensional evolution algebra. Assume $0 < i < n-1$. If A/I is not basic simple, there exists a j-basic ideal $\overline{J}$ of A/I, for some $j \neq 0$. Since A/I is perfect (as we have proved before), by Lemma 2.3 the basic ideals of A/I do not depend on the basis. This means that we may assume that there exists $\overline{B_J} = \{\overline{e_k} \mid k \in \Lambda_j\ \}$, where $\Lambda_j \subsetneq \{i+1, \ldots, n\}$, such that $\overline{J}$ is generated by $\overline{B_J}$. Then $\{e_1, \ldots, e_i\} \cup \{e_k\}_{k\in\Lambda_j}$ generates an $(i+j)$-basic ideal, a contradiction to the maximality of I. □

Definition 2.11. Let A be an evolution algebra and assume that there exists a natural basis B such that $M_B = \begin{pmatrix} W & U \\ 0 & Y \end{pmatrix}$. We will say that the *number of zeros* of W *is invariant* if for any basis B' such that $M_{B'} = \begin{pmatrix} W' & U' \\ 0 & Y' \end{pmatrix}$, the number of zeros in W' coincides with the number of zeros in W. Analogous definitions can be given for U and Y.

Examples 2.12. Let A be an evolution algebra and assume that there exists a natural basis B such that $M_B = \begin{pmatrix} W & U \\ 0 & Y \end{pmatrix}$.

(i) The number of zeros of U and of W is not necessarily invariant. Take A as the evolution algebra with natural basis $B = \{e_1, e_2, e_3, e_4\}$ and product given by $M_B := \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$.

Now, take $B' := \{e_4, e_1, e_2, e_3\}$. Then $M_{B'} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$, and the reader can see that the statement is true.

(ii) The number of zeros in Y is not necessarily invariant. Consider the evolution algebra with natural basis $B = \{e_1, e_2, e_3, e_4\}$, and let $B' := \{e_3, e_1, e_2, e_4\}$. Then $M_B := \left(\begin{array}{c|ccc} 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right)$ and $M_{B'} = \left(\begin{array}{c|ccc} 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$, which proves the statement.

(iii) Even if the number of zeros of U is invariant, the number of zeros of W and Y need not be invariant. Consider the evolution algebra with natural basis $B = \{e_1, e_2, e_3, e_4\}$ and product given by $M_B := \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$. Then, for every permutation $\sigma \in S_4$ such that

$M_{B^\sigma} = \begin{pmatrix} W^\sigma & U^\sigma \\ 0 & Y^\sigma \end{pmatrix}$, where $B^\sigma := \{e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, e_{\sigma(4)}\}$, we have that the number of zeros of U^σ is three. This means that the number of zeros of U is invariant. Now, take $B' := \{e_3, e_1, e_4, e_2\}$. Then $M'_B = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$, what shows that the number of zeros of W and Y is not invariant.

(iv) Even if the number of zeros of W is invariant, the number of zeros of U and Y need not be. Take B and B' as in (ii).

(v) Even if the number of zeros of Y is invariant, the number of zeros of W and U need not be. Take B and B' as in (i).

Following the same notation as in [4, Subsection 3.1], for any natural number n, we define the semidirect product $S_n \rtimes (\mathbb{K}^\times)^n$. It is not difficult to see that any matrix P in $S_n \rtimes (\mathbb{K}^\times)^n$ is a change of basis matrix from a natural basis B into another natural basis B' and the relationship among the structure matrices M_B and $M_{B'}$ and the matrix P is as given in [4, Condition (5)], that is, $P^{-1}M_B P^{(2)} = M_{B'}$. This is the reason because we defined in [4, Subsection 3.1] the action of P on M_B by:

$$P \cdot M_B = P^{-1}M_B P^{(2)}.$$

For any $\sigma \in S_n$, we denote by I_σ the matrix obtained from the identity changing the columns as determined by σ (i.e. the i-column in I_σ is $C_{\sigma(i)}$, where C_j denotes the j-column in the identity matrix).

The result that follows is a generalization to n-dimensional evolution algebras of [4, Proposition 3.2]. We do not include its proof as it is similar.

Proposition 2.13. *For any natural number n, any $P \in S_n \rtimes (\mathbb{K}^\times)^n$ and any $M \in \mathcal{M}_n(\mathbb{K})$ we have:*

(i) *The number of zero entries in M coincides with the number of zero entries in $P \cdot M$.*

(ii) *The number of zero entries in the main diagonal of M coincides with the number of zero entries in the main diagonal of $P \cdot M$.*

(iii) *The rank of M and the rank of $P \cdot M$ coincide.*

(iv) *Assume that M is the structure matrix of an evolution algebra A relative to a natural basis B. Assume that $A^2 = A$. If N is the structure matrix of A relative to a natural basis B' then there exists $Q \in S_n \rtimes (\mathbb{K}^\times)^n$ such that $N = Q \cdot M$.*

In order to study the change of basis matrices producing isomorphic evolution algebras whose structure matrix is the same as the initial one, we establish Theorem 2.15. First we need some definitions. Recall that a *reducible evolution algebra* is an evolution algebra A which can be decomposed as the direct sum (in the sense of [3, Definition 5.3]) of two nonzero evolution algebras, equivalently, of two nonzero evolution ideals, equivalently, of two nonzero ideals, as shown in [3, Lemma 5.2]. An evolution algebra which is not reducible will be called *irreducible*.

Notation 2.14. For $n \in \mathbb{N}$, we let $\left[\frac{n}{2}\right]$ denote the integer part of $\frac{n}{2}$.

In the theorem that follows we assume that there exists a maximal s-basic ideal I such that $s \leq \left[\frac{n}{2}\right]$. Note that Lemma 2.3 says that there exist maximal basic ideals, but the dimension of these ideals is not, necessarily, smaller or equal than $\left[\frac{n}{2}\right]$.

Theorem 2.15. *Let A be an n-dimensional perfect evolution algebra (for $n \geq 3$), and assume that there exists a maximal s-basic ideal I, for $0 \neq s \leq \left[\frac{n}{2}\right]$. This implies that there exists a natural basis $B = \{e_1, \ldots, e_n\}$ such that $I = \langle\{e_1, \ldots, e_s\}\rangle$ and $M_B = \begin{pmatrix} W & U \\ 0 & Y \end{pmatrix}$, where $W \in M_s(\mathbb{K})$ (in fact, W is the structure matrix of I relative to the basis $\{e_1, \ldots, e_s\}$ of I), $U \in M_{s\times(n-s)}(\mathbb{K})$, $Y \in M_{n-s}(\mathbb{K})$.*

(i) *For $s = 1$, the algebra A is irreducible, the only possible permutations σ producing isomorphic evolution algebras whose structure matrix has the same form as M_B must satisfy $\sigma(1) = 1$.*
(ii) *For $s = 1$, the number of zeros in W, U and Y is invariant.*
(iii) *Assume $s \leq \left[\frac{n}{2}\right]$ and A irreducible. Then, the only possible permutations σ producing isomorphic evolution algebras whose structure matrix has the same form as M_B must satisfy $\sigma(i) \in \{1, \ldots, s\}$ for all $i \in \{1, \ldots, s\}$ (and, consequently, $\sigma(j) \in \{s+1, \ldots, n\}$ for all $j \in \{s+1, \ldots, n\}$).*
(iv) *For $s \leq \left[\frac{n}{2}\right]$ and A irreducible, the number of zeros in W, U and Y is invariant. If A is not irreducible, then the result is not true. If $s > \left[\frac{n}{2}\right]$, then the result is not true.*

Proof. Write $M_B = (\omega_{ji})$, where $\omega_{ji} \in \mathbb{K}$.

(i) and (ii). Assume first $s = 1$. It is trivial that A has to be irreducible because otherwise there will be a maximal t-basic ideal of dimension $1 < t < n-1$, which is not the case. It is immediate to see that the number of zeros of W is invariant. Now we see that the number of zeros in Y is invariant, which consequently implies that the number of zeros in U is also invariant by (i) in Proposition 2.13.

Assume that we may change e_1 to e_i, for some $i \in \{2, \ldots, n\}$. Then $e_i^2 = \omega_{ii} e_i$ and this would imply that $\{e_1, e_i\}$ generates a 2-basic ideal, which is not possible as I is a maximal 1-basic ideal. This means that any other change is given by a permutation $\sigma \in S_n$ such that $\sigma(e_1) = e_1$. Therefore, the first row of the matrix $I_\sigma M_B$ is just $\begin{pmatrix}\omega_{11} & \omega_{1\sigma(2)} & \ldots & \omega_{1\sigma(n)}\end{pmatrix}$, which has the same number of zeros that the first row of the matrix M_B, which coincides with the number of zeros of U.

(iii) and (iv). Assume $s \leq \left[\frac{n}{2}\right]$ and A irreducible. Note that Y is the structure matrix of the evolution algebra A/I relative to the natural basis $\{\overline{e_{s+1}}, \ldots, \overline{e_n}\}$. By Proposition 2.10, the evolution algebra A/I is perfect and basic simple. Decompose $\{1, \ldots, n\} = \Lambda_1 \sqcup \Lambda_2$, where $\Lambda_1 := \{1, \ldots, s\}$ and $\Lambda_2 := \{s+1, \ldots, n\}$. We are going to prove that the only bases $B' = \{e_{\sigma(1)}, \ldots, e_{\sigma(n)}\}$, for $\sigma \in S_n$, such that $M_{B'} = \begin{pmatrix} W' & U' \\ 0 & Y' \end{pmatrix}$, where $W' \in M_s(\mathbb{K})$, are those such that $\sigma(\Lambda_i) = \Lambda_i$, for $i = 1, 2$.

Indeed, let $\sigma \in S_n$ and define $\Gamma_k^j := \{i \in \Lambda_k \mid \sigma(i) \in \Lambda_j\}$, for $j, k \in \{1, 2\}$. Note that $|\Gamma_2^1| = |\Gamma_1^2|$ and $\Lambda_1 = \Gamma_1^1 \sqcup \Gamma_1^2$. Define

$$J = \langle \{e_k \mid k \in \Gamma_1^1 \sqcup \Gamma_1^2 \sqcup \Gamma_2^1)\}\rangle,$$

which is an ideal of A. If $J = A$, then $\Gamma_2^2 = \emptyset$, which implies $n - s = |\Gamma_2^1| \leq |\Lambda_1| = s$, that is, $s \geq \frac{n}{2}$. Since $s \leq \left[\frac{n}{2}\right]$, necessarily n is even and $s = \frac{n}{2}$.This implies

$$A = \langle\{e_k \mid k \in \Gamma_2^1\}\rangle \oplus \langle\{e_k \mid k \in \Gamma_1^2\}\rangle,$$

being each direct summand nonzero, which is a contradiction because we are assuming that A is irreducible. Therefore, J is a proper ideal of A having dimension strictly bigger that the dimension of I (because $\Gamma_2^1 \neq \emptyset$), which is a contradiction to the maximality of I. This shows our claim and (iii). Item (iv) follows immediately from (iii).

Now we see that the result in (ii) is not true when the algebra is not irreducible. For an example, consider the evolution algebra A having a natural basis $B = \{e_1, e_2, e_3, e_4\}$ such that $M_B = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$. Then for $B' = \{e_3, e_4, e_1, e_2\}$ we have $M_{B'} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right)$, and we observe that the number of zeros in the pieces is not invariant.

Finally we show that the result is not true when $s > \left[\frac{n}{2}\right]$. To this end, take A as the evolution five dimensional evolution algebra having structure matrix $M_B = \left(\begin{array}{ccc|cc} 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$, relative to a natural basis $B = \{e_1, \dots, e_5\}$. In this case $\Lambda_1 = \{1, 2, 3\}$ and $\Lambda_2 = \{4, 5\}$. Take the natural basis $B' = \{e_4, e_5, e_3, e_1, e_2\}$. We have $M_{B'} = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right)$. Observe that the number of zeros in the blocks of M_B and $M_{B'}$ is not the same. □

Corollary 2.16. *Let A be an n-dimensional irreducible perfect evolution algebra (for $n \geq 3$). Then, there exists a unique maximal basic ideal I whose dimension is $s \leq \left[\frac{n}{2}\right]$.*

Proof. The existence of maximal basic ideals follows by 2.3. For $s = 1$, if A has two maximal 1-basic ideals, then the sum of both gives a 2-basic ideal of A, contradicting the maximality of each one.

Now, assume $s \geq \left[\frac{n}{2}\right]$ and use the same notation as in the proof of Theorem 2.15 (iii). Let $I = \{e_1, \dots, e_s\}$ be a maximal s-basic ideal. Assume that J is another maximal s-basic ideal of A. This means that there exists $\sigma \in S_n$ such that $\{e_{\sigma(1)}, \dots, e_{\sigma(s)}\}$ is a basis for J. By the proof of (iii) in Theorem 2.15, $\{\sigma(1), \dots, \sigma(s)\} = \{1, \dots, s\}$. This shows $I = J$. □

Remark 2.17. (i) If $s > \left[\frac{n}{2}\right]$, then there is no necessarily a unique maximal basic ideal. See the last example in the proof of Theorem 2.15.

(ii) The integer s in Theorem 2.15 cannot be $n - 1$. For an example, consider $n = 4$, $s = 3$ and take A as in Examples 2.12 (i).

(iii) The integer s in Theorem 2.15 cannot be 0 if A is not perfect (i.e. if A is not perfect and 0 is a maximal ideal, then the number of zeros is not an invariant). For an example, consider the evolution algebra A having a natural basis $B = \{e_1, e_2, e_3\}$ and product given by $M_B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Then for $B' = \{e_1, e_2 + e_3, -e_2 + e_3\}$ we have $M_{B'} = \begin{pmatrix} 0 & 2 & 2 \\ 1/2 & 1 & 1 \\ -1/2 & 1 & 1 \end{pmatrix}$. Note that while M_B has four zeros, $M_{B'}$ has only one.

Another case in which the number of zeros in W, U and Y is invariant appears in Proposition 2.19. First we need the definition that follows (see [3, Definitions 3.1]).

Definition 2.18. Let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of an evolution algebra A and let $i_0 \in \Lambda$. The *first-generation descendents* of i_0 are the elements of the subset $D^1(i_0)$ given by:

$$D^1(i_0) := \left\{ k \in \Lambda \mid e_{i_0}^2 = \sum_k \omega_{k i_0} e_k \text{ with } \omega_{k i_0} \neq 0 \right\}.$$

In an abbreviated form, $D^1(i_0) := \{j \in \Lambda \mid \omega_{j i_0} \neq 0\}$. Note that $j \in D^1(i_0)$ if and only if, $\pi_j(e_{i_0}^2) \neq 0$ (where π_j is the canonical projection of A over $\mathbb{K}e_j$).

Proposition 2.19. *Let A be a perfect n-dimensional evolution algebra having an $(n-1)$-basic ideal I and let $B = \{e_1, \ldots, e_n\}$ be a natural basis. Assume (there is no loss in generality in doing this) that $I = \langle\{e_1, \ldots, e_{n-1}\}\rangle$. Let*

$$M_B = \left(\begin{array}{ccc|c} & W & & U \\ \hline 0 & \cdots & 0 & \omega_{nn} \end{array}\right), \tag{1}$$

where $W \in M_{n-1}(\mathbb{K})$, $U \in M_{(n-1)\times 1}(\mathbb{K})$ and $\omega_{nn} \in \mathbb{K}^{\times}$. Then, there exists a natural basis B' of A with

$$M_{B'} = \left(\begin{array}{ccc|c} & W' & & U' \\ \hline 0 & \cdots & 0 & \omega'_{nn} \end{array}\right),$$

where $W' \in M_{n-1}(\mathbb{K})$, $U' \in M_{(n-1)\times 1}(\mathbb{K})$ and $\omega'_{nn} \in \mathbb{K}^{\times}$, such that the number of zeros in W and in W' does not coincide (and, consequently the number of zeros in U and in U' is different) if and only if there exists $i \in \{1, \ldots, n-1\}$ such that $i \notin D^1(j)$ for any $j \in \{1, \ldots, n\} \setminus \{i\}$ and $|D^1(i)| \neq |D^1(n)|$.

Proof. Assume first that the number of zeros is not invariant for some basis B' such that $M_{B'}$ is as in the statement. Since (up to scalars) the only changes of basis are the permutation matrices (by [7, Theorem 4.4]), there exists $i \in \{1, \ldots, n-1\}$ and $\sigma \in S_n$ such that $\sigma(i) = n$, and $B' = \{e_{\sigma(i)} \mid i \in \{1, \ldots, n\}\}$. Taking into account the last row in $M_{B'}$ we obtain $i \notin D^1(j)$ for any $j \in \{1, \ldots, n\} \setminus \{i\}$. Since we are assuming that the number of zeros in W and in W' is different, necessarily $|D^1(i)| \neq |D^1(n)|$.

Now we prove the reverse implication. Assume that there exists i satisfying the conditions in the statement. Take $\sigma \in S_n$ such that $\sigma(i) = n$ and $\sigma(n) = i$ and define $B' = \{e_{\sigma(i)} \mid i \in \{1, \ldots, n\}\}$. Then $M_{B'}$ is as required; concretely, $\omega'_{nn} \in \mathbb{K}^{\times}$ and the number of zeros in W' is not the same as the number of zeros in W. □

Remark 2.20. In the proof of Proposition 2.19, it is shown that the condition $i \notin D^1(j)$ for any $j \in \{1, \ldots, n\} \setminus \{i\}$ means that the change of basis obtained from B by putting e_i instead of e_n produces an structure matrix having the same form as (1). This is the reason to classify taking into account this condition.

Using Proposition 2.19, and Remark 2.20 we get the following.

Corollary 2.21. *Let A be a perfect n-dimensional evolution algebra having an $(n-1)$-basic ideal I and let $B = \{e_1, \ldots, e_n\}$ be a natural basis. Assume that $I = \langle\{e_1, \ldots, e_{n-1}\}\rangle$. Let*

$$M_B = \left(\begin{array}{ccc|c} & W & & U \\ \hline 0 & \cdots & 0 & \omega_{nn} \end{array}\right),$$

where $W \in M_{n-1}(\mathbb{K})$, $U \in M_{n-1\times 1}(\mathbb{K})$ and $\omega_{nn} \in \mathbb{K}^{\times}$. Then, there exists $i \in \{1, \ldots, n-1\}$ such that $i \notin D^1(j)$ for any $j \in \{1, \ldots, n\} \setminus \{i\}$ if and only if there exist $k_1, \ldots, k_{n-2} \in \{1, \ldots, n-1\} \setminus \{i\}$, with $k \neq l$, such that $\operatorname{lin}\{e_{k_1}, \ldots, e_{k_{n-2}}\}$ is an $(n-2)$-dimensional evolution ideal of A and $\operatorname{lin}\{e_{k_1}, \ldots, e_{k_{n-2}}, e_n\}$ is an $(n-1)$-dimensional evolution ideal.

Remark 2.22. When the conditions in Corollary 2.21 are satisfied, then the evolution algebra A, which contains an $(n-1)$-basic ideal I, also contains another $(n-1)$-basic ideal, say J, such that $I \cap J$ is an $(n-2)$-basic ideal of both, I and J. We will use this fact for the classification.

Remark 2.23. Note that for perfect evolution algebras, the intersection of basic ideals is again a basic ideal.

Definition 2.24. An evolution algebra A having dimension n will be said to satisfy *Condition* (n-1,n-2,n-1) if A has two different $(n-1)$-basic ideals I and J such that $I \cap J$ is an $(n-2)$-basic ideal of I and also of J.

Note that in Sections 3 and 4, the maximal basic ideal is unique by Corollary 2.16. This is not the case in Section 5, where we can see that the ideal I in Proposition 2.19, which is basic and maximal, is not, in general the unique basic and maximal ideal.

In the sections that follow we apply the results we have proved in order to classify the four dimensional perfect non-simple evolution algebras.

3. Classification of four dimensional perfect non-simple evolution algebras. Case: The maximal basic ideal is one-dimensional

In this section we will classify the four dimensional perfect non-simple evolution algebras.

In the general classification of four dimensional evolution algebras, we can distinguish the reducible and the irreducible cases. Assume first that the evolution algebra is reducible. This means that $A = I \oplus J$, where I and J are evolution ideals having dimensions 1 and 3, or having both of them dimension 2. When this happens, the classification is achieved by considering the classification of two dimensional evolution algebras [6, 2] and the classification of three dimensional evolution algebras (see [4]).

Thus, from now on, we will consider only irreducible evolution algebras. We classify these evolution algebras A taking into account the dimension of the maximal basic ideals (they exist by Lemma 2.5 and are unique when their dimension is smaller or equal than 2, by Theorem 2.15). For I a maximal basic ideal of A, since A is non-simple, the dimension I, denote it by i, is at least one.

Let $B = \{e_1, e_2, e_3, e_4\}$ be a natural basis of A, write $\{1, 2, 3, 4\} = \Lambda_1 \sqcup \Lambda_2$, where I is the vector space generated by $\{e_j\}_{j\in\Lambda_1}$. There is no loss in generality if we assume that $\Lambda_1 = \{1, \dots, i\}$. Then,

$$M_B = \begin{pmatrix} W & U \\ 0 & Y \end{pmatrix}, \tag{2}$$

where W is the structure matrix of I relative to the natural basis $\{e_1, \dots, e_i\}$. By Proposition 2.10, A/I is a basic simple evolution algebra and Y can be seen as the structure matrix of A/I relative to the natural basis $\{\overline{e_{i+1}}, \dots, \overline{e_4}\}$.

We start considering that the maximal basic ideal I of A is one-dimensional. Write

$$M_B = \left(\begin{array}{c|ccc} 1 & \omega_{12} & \omega_{13} & \omega_{14} \\ \hline 0 & & & \\ 0 & & Y & \\ 0 & & & \end{array}\right). \tag{3}$$

Note that, in this case U$= (\omega_{12}\ \ \omega_{13}\ \ \omega_{14})$. By Theorem 2.15, the number of zeros in U and Y is invariant. We will use this fact to classify. Concretely, the cases we are going to consider depends on the number of zeros in U. Notice that ω_{12}, ω_{13} and ω_{14} cannot be zero at the same time because otherwise the algebra will be reducible.

3.1. The matrix U has two zero entries.

The possible matrices are of the form:

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & \omega_{14} \\ \hline 0 & & & \\ 0 & & Y & \\ 0 & & & \end{array}\right), \left(\begin{array}{c|ccc} 1 & 0 & \omega_{13} & 0 \\ \hline 0 & & & \\ 0 & & Y & \\ 0 & & & \end{array}\right), \left(\begin{array}{c|ccc} 1 & \omega_{12} & 0 & 0 \\ \hline 0 & & & \\ 0 & & Y & \\ 0 & & & \end{array}\right),$$

where the entries in the first row are nonzero. It happens that the three of them produce isomorphic evolution algebras and the change of basis matrices are given by $I_{(3,4)}$ (producing the isomorphism between the first and the second) and by $I_{(2,4)}$ (producing the isomorphism between the first and the third). Moreover, by multiplying conveniently one of the vectors in the basis by an scalar, we may assume that the matrix is:

$$\left(\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ \hline 0 & & & \\ 0 & & Y & \\ 0 & & & \end{array}\right). \tag{4}$$

In order to get an irredundant classification we may keep the first row and the first columns as in (4) (use what has been explained above and use Theorem 2.15). This implies that the only possible change of basis which is allowed is $I_{(3,4)}$. Moreover, the matrix Y corresponds to a basic simple three-dimensional evolution algebra. Since $|M_B| \neq 0$, $|Y| \neq 0$ and by Proposition 2.7, this is equivalent to saying that Y is the structure matrix of a simple three-dimensional evolution algebra. Therefore, in the matrices we consider, we have inserted, as matrix Y, all the different matrices associated to simple three-dimensional evolution algebras, which appear in the classification in [2, 4].

Now, we explain how do we select the matrices using the classification of the three-dimensional evolution algebras. Take as a reference, for example, Table 19 in [2] and look at the fifth row (corresponding to the first simple three-dimensional evolution algebras in that table), which is the following:

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & \mu & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & \mu \end{pmatrix}$	$\begin{pmatrix} \mu & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & \mu & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & \mu \end{pmatrix}$

Since the only change of basis matrix producing non-isomorphic four-dimensional evolution algebras is $I_{(3,4)}$, the matrix Y can be the first one in the table before (which is isomorphic to the fourth one in the table), but because of the restriction in the permutations which are allowed, we have to take also the second matrix in the table (which is isomorphic to the last one) and the third one (which is isomorphic to the fifth one). Taking into account this remark, the families of non-isomorphic evolution algebras under these conditions are of the form $\left(\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ \hline 0 & & & \\ 0 & & Y & \\ 0 & & & \end{array}\right)$, where Y is one of the matrices that follow.

$$\begin{pmatrix} 0 & 0 & \omega_{24} \\ \omega_{32} & 0 & 0 \\ 0 & \omega_{43} & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{22} & 0 & \omega_{24} \\ \omega_{32} & 0 & 0 \\ 0 & \omega_{43} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & 0 \\ 0 & \omega_{33} & \omega_{34} \\ \omega_{42} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{34} \\ \omega_{42} & 0 & \omega_{44} \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & \omega_{24} \\ \omega_{32} & 0 & 0 \\ 0 & \omega_{43} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & 0 \\ \omega_{32} & 0 & \omega_{34} \\ \omega_{42} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{34} \\ \omega_{42} & \omega_{43} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \omega_{23} & 0 \\ \omega_{32} & 0 & \omega_{34} \\ \omega_{42} & 0 & \omega_{44} \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & \omega_{24} \\ \omega_{32} & 0 & 0 \\ 0 & \omega_{43} & \omega_{44} \end{pmatrix} \quad \begin{pmatrix} \omega_{22} & 0 & \omega_{24} \\ \omega_{32} & 0 & \omega_{34} \\ 0 & \omega_{43} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & 0 \\ \omega_{32} & 0 & \omega_{34} \\ 0 & \omega_{43} & \omega_{44} \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & \omega_{24} \\ \omega_{32} & 0 & 0 \\ \omega_{42} & 0 & \omega_{44} \end{pmatrix} \quad \begin{pmatrix} \omega_{22} & \omega_{23} & 0 \\ \omega_{32} & 0 & \omega_{34} \\ 0 & \omega_{43} & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{22} & \omega_{23} & \omega_{24} \\ \omega_{32} & 0 & 0 \\ 0 & \omega_{43} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \omega_{23} & 0 \\ \omega_{32} & \omega_{33} & \omega_{34} \\ \omega_{42} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{34} \\ \omega_{42} & \omega_{43} & \omega_{44} \end{pmatrix} \quad \begin{pmatrix} \omega_{22} & 0 & \omega_{24} \\ \omega_{32} & \omega_{33} & 0 \\ 0 & \omega_{43} & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{22} & \omega_{23} & 0 \\ 0 & \omega_{33} & \omega_{34} \\ \omega_{42} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & 0 \\ 0 & \omega_{33} & \omega_{34} \\ \omega_{42} & 0 & \omega_{44} \end{pmatrix} \quad \begin{pmatrix} \omega_{22} & 0 & \omega_{24} \\ \omega_{32} & 0 & 0 \\ \omega_{42} & \omega_{43} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & 0 \\ 0 & \omega_{33} & \omega_{34} \\ \omega_{42} & \omega_{43} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \omega_{23} & \omega_{24} \\ 0 & 0 & \omega_{34} \\ \omega_{42} & 0 & \omega_{44} \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & \omega_{24} \\ \omega_{32} & 0 & \omega_{34} \\ 0 & \omega_{43} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & \omega_{24} \\ \omega_{32} & 0 & \omega_{34} \\ \omega_{42} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & 0 \\ \omega_{32} & 0 & \omega_{34} \\ \omega_{42} & \omega_{43} & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{22} & \omega_{23} & 0 \\ 0 & \omega_{33} & \omega_{34} \\ \omega_{42} & 0 & \omega_{44} \end{pmatrix} \quad \begin{pmatrix} \omega_{22} & \omega_{23} & \omega_{24} \\ \omega_{32} & 0 & 0 \\ \omega_{42} & 0 & \omega_{44} \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{23} & 0 \\ \omega_{32} & \omega_{33} & \omega_{34} \\ 0 & \omega_{43} & \omega_{44} \end{pmatrix}$$

$$\begin{pmatrix}\omega_{22}&0&\omega_{24}\\0&0&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&0&0\\0&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&0\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&0\\0&0&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&0\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&0\\\omega_{32}&0&\omega_{34}\\0&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&0\\0&\omega_{43}&\omega_{44}\end{pmatrix}$$

$$\begin{pmatrix}\omega_{22}&\omega_{23}&0\\\omega_{32}&0&\omega_{34}\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&0\\0&\omega_{33}&\omega_{34}\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&0&\omega_{34}\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&0&\omega_{34}\\0&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&0&\omega_{24}\\\omega_{32}&0&\omega_{34}\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&0&0\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&0\\\omega_{32}&0&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}$$

$$\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\0&0&\omega_{34}\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&0\\0&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&0\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&0&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&0\\0&\omega_{33}&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&0&0\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&0\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\0&0&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}$$

$$\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&0&\omega_{34}\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&0\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&0\\\omega_{32}&\omega_{33}&\omega_{34}\\0&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&0&\omega_{24}\\0&\omega_{33}&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&0\\0&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&0\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&0\\0&\omega_{33}&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}$$

$$\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&0&0\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&0\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\0&0&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&0&\omega_{34}\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&\omega_{34}\\0&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&0&\omega_{24}\\\omega_{32}&0&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&0&\omega_{44}\end{pmatrix}$$

$$\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&0&\omega_{34}\\0&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&0&\omega_{24}\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&0&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&0&\omega_{34}\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&\omega_{34}\\0&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&0&\omega_{24}\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}$$

$$\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&\omega_{34}\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}$$

Denote the set of all these matrices by Ω.

3.2. **The matrix U has one zero entry.** We are reasoning in a similar way as in Case 3.1. Here, the possible matrices are of the form:

$$\left(\begin{array}{c|ccc}1&0&\omega_{13}&\omega_{14}\\\hline 0&&&\\0&&Y&\\0&&&\end{array}\right),\quad\left(\begin{array}{c|ccc}1&\omega_{12}&0&\omega_{14}\\\hline 0&&&\\0&&Y&\\0&&&\end{array}\right),\quad\left(\begin{array}{c|ccc}1&\omega_{12}&\omega_{13}&0\\\hline 0&&&\\0&&Y&\\0&&&\end{array}\right).$$

All of them produce isomorphic evolution algebras and the change of basis matrices are given by $I_{(2,3)}$ (producing the isomorphism between the first and the second) and by $I_{(2,4)}$ (producing the isomorphism between the first and the third). Moreover, by multiplying conveniently one of the vectors in the basis by an scalar, we may assume that the matrix is $\left(\begin{array}{c|ccc}1&0&1&1\\\hline 0&&&\\0&&Y&\\0&&&\end{array}\right)$. Reasoning as in Case 3.1, we get that $Y \in \Omega$.

3.3. **The matrix U has no nonzero entries.** Once we have analyzed which are the change of basis matrices that are allowed we get that $I_\sigma M$ is of the same form as M whenever σ belongs to the group $\{\mathrm{id}, (3,4), (2,3), (2,3,4), (2,4,3), (2,4)\}$.

In order to classify we proceed by inserting as Y in (2) all the different matrices associated to simple three-dimensional evolution algebras which appear in the classification in [2, 4]. They are

$$\begin{pmatrix}0&0&\omega_{24}\\\omega_{32}&0&0\\0&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&0&\omega_{24}\\\omega_{32}&0&0\\0&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&0&0\\0&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&0\\\omega_{32}&0&\omega_{34}\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&0\\\omega_{32}&0&\omega_{34}\\0&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&0&0\\0&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&0&\omega_{24}\\\omega_{32}&\omega_{33}&0\\0&\omega_{43}&0\end{pmatrix}$$

$$\begin{pmatrix}\omega_{22}&0&\omega_{24}\\\omega_{32}&0&0\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&0&\omega_{34}\\0&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&0\\0&\omega_{33}&\omega_{34}\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&0&0\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&0&0\\0&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&0\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&0\\0&\omega_{43}&\omega_{44}\end{pmatrix}$$

$$\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&0&\omega_{34}\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&0&0\\\omega_{42}&\omega_{43}&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&0\\0&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&0&0\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{23}&\omega_{24}\\\omega_{32}&0&\omega_{34}\\\omega_{42}&\omega_{43}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&0\\\omega_{42}&0&\omega_{44}\end{pmatrix}\quad\begin{pmatrix}\omega_{22}&\omega_{23}&\omega_{24}\\\omega_{32}&\omega_{33}&0\\0&\omega_{43}&\omega_{44}\end{pmatrix}$$

$$\begin{pmatrix}\omega_{22} & \omega_{23} & \omega_{24}\\ \omega_{32} & 0 & 0\\ \omega_{42} & \omega_{43} & \omega_{44}\end{pmatrix} \quad \begin{pmatrix}\omega_{22} & \omega_{23} & \omega_{24}\\ \omega_{32} & 0 & \omega_{34}\\ \omega_{42} & 0 & \omega_{44}\end{pmatrix} \quad \begin{pmatrix}0 & \omega_{23} & \omega_{24}\\ \omega_{32} & \omega_{33} & \omega_{34}\\ \omega_{42} & 0 & \omega_{44}\end{pmatrix} \quad \begin{pmatrix}0 & \omega_{23} & \omega_{24}\\ \omega_{32} & 0 & \omega_{34}\\ \omega_{42} & \omega_{43} & \omega_{44}\end{pmatrix} \quad \begin{pmatrix}\omega_{22} & \omega_{23} & \omega_{24}\\ \omega_{32} & \omega_{33} & \omega_{34}\\ \omega_{42} & 0 & \omega_{44}\end{pmatrix} \quad \begin{pmatrix}\omega_{22} & \omega_{23} & \omega_{24}\\ \omega_{32} & \omega_{33} & \omega_{34}\\ \omega_{42} & \omega_{43} & 0\end{pmatrix} \quad \begin{pmatrix}\omega_{22} & \omega_{23} & \omega_{24}\\ \omega_{32} & \omega_{33} & \omega_{34}\\ \omega_{42} & \omega_{43} & \omega_{44}\end{pmatrix}$$

4. Classification of four dimensional perfect non-simple evolution algebras. Case: The maximal basic ideal is two-dimensional

In this case, the structure matrix M_B is as follows:

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{W} & U \\ \hline 0 & 0 & \\ 0 & 0 & Y \end{array}\right). \tag{5}$$

Notice that the matrix U cannot be the zero matrix because otherwise the algebra will be reducible.

In order to classify, we need to find the possible matrices W, Y and U. The W must corresponds to a perfect two dimensional evolution algebra because $|M_B| \neq 0$ and by [9, Theorem 2] this implies $|W|, |Y| \neq 0$. The matrix Y corresponds to a basic simple two dimensional evolution algebra because it is the structure matrix of A/I, where I is the maximal 2-basic ideal of A, and it is basic simple by Proposition 2.10. Once we have the matrices W and Y, we classify taking into account the number of zeros in U. This is because Theorem 2.15 says that the number of zeros in U, (also in W and in Y) is invariant

We start fixing W, which, by the classification of perfect two-dimensional evolution algebras ([2, Theorem 3.3. (III)]), must be one matrix in the following set:

$$\Gamma := \left\{ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ \omega_{21} & 1\end{pmatrix}, \begin{pmatrix}1 & \omega_{12}\\ \omega_{21} & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & \omega_{22}\end{pmatrix} \right\}$$

with $\omega_{ij} \neq 0$ and $|W| \neq 0$.

For each $W \in \Gamma$ we consider the different possibilities for U depending on the number on nonzero entries it has. In each of these cases we consider as matrix Y one in the set $\Gamma' := \Gamma'_1 \cup \Gamma'_2$, where

$$\Gamma'_1 := \left\{ \begin{pmatrix}1 & \omega_{12}\\ \omega_{21} & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & \omega_{22}\end{pmatrix} \right\} \quad \text{and} \quad \Gamma'_2 := \left\{ \begin{pmatrix}\omega_{11} & 1\\ 1 & 0\end{pmatrix} \right\}.$$

Although the evolution algebras having structure matrices $\begin{pmatrix}0 & 1\\ 1 & \omega_{22}\end{pmatrix}$ and $\begin{pmatrix}\omega_{11} & 1\\ 1 & 0\end{pmatrix}$ are isomorphic, as parts of the matrix M_B both will appear in some cases, not in all of them. This is the reason to write Γ' as that union.

4.1. Take $W = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

4.1.1. *There are three zeros in U.* The four cases that appear

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & \omega_{24}\\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}}\\ 0 & 0 & & \end{array}\right), \quad \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0\\ 0 & 1 & \omega_{23} & 0\\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}}\\ 0 & 0 & & \end{array}\right), \quad \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14}\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}}\\ 0 & 0 & & \end{array}\right), \quad \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}}\\ 0 & 0 & & \end{array}\right)$$

are isomorphic to the first one via the permutations: $(3,4)$, $(1,2)$ and $(1,2)(3,4)$, respectively. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & \omega_{24}\\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}}\\ 0 & 0 & & \end{array}\right)$$

where $Y \in \Gamma'$. We have four families of mutually non-isomorphic evolution algebras.

4.1.2. *There are two zeros in U.* The six cases appearing are:

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & \omega_{14} \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ 0 & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & 0 \\ 0 & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ 0 & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & 0 \\ 0 & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right).$$

The first and the second ones are isomorphic, as well as the third and the fourth ones, and the fifth and the last ones. The permutations giving these isomorphisms are $(1,2)$ in the first case and $(3,4)$ in the other cases. In order to classify we choose the matrices in the odd position. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y_1}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ 0 & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y_2}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ 0 & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y_3}} \\ 0 & 0 & & \end{array}\right),$$

where $Y_1, Y_3 \in \Gamma_1'$ and $Y_2 \in \Gamma'$. We have ten families of mutually non-isomorphic evolution algebras.

4.1.3. *There is one zero in U.* The four cases that appear

$$\left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & \omega_{14} \\ 0 & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & \omega_{14} \\ 0 & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & 0 \\ 0 & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ 0 & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right) \tag{6}$$

are isomorphic to the first one via the permutations $(3,4)$, $(1,2)$ and $(1,2)(3,4)$, respectively. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & \omega_{14} \\ 0 & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right),$$

where $Y \in \Gamma'$. We have four families of mutually non-isomorphic evolution algebras.

4.1.4. *There are no zeros in U.* There is only one possibility for U and, therefore, the matrix in (5) takes the form:

$$\left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & \omega_{14} \\ 0 & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right),$$

where $Y \in \Gamma_1'$. We have three families of mutually non-isomorphic evolution algebras.

4.2. Take $W = \begin{pmatrix} 1 & 0 \\ \omega_{21} & 1 \end{pmatrix}$.

4.2.1. *There are three zeros in U.* As in Case 4.1.1, there are four possibilities for U and the structure matrix has the form:

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ \omega_{21} & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & 0 \\ \omega_{21} & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right).$$

The first matrix is isomorphic to the second one and the third one is isomorphic to the last one, in both cases, via the permutation $(3,4)$. For the classification we select the first and the third matrices. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ \omega_{21} & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right).$$

For any election of Y in Γ' we have mutually non-isomorphic evolution algebras. The number of families is eight.

4.2.2. *There are two zeros in U.* As in Case 4.1.2 there are six possibilities for U. The matrix in (5) becomes:

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \omega_{21} & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & \omega_{14} \\ \omega_{21} & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & 0 \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & 0 \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right).$$

The third and the fourth are isomorphic, as well as the fifth and the last one. The permutation giving these isomorphisms is $(3,4)$. We fix the first, the second, the third and the fifth matrices. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \omega_{21} & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y_1} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & \omega_{14} \\ \omega_{21} & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{Y_2} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y_3} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{Y_4} \\ 0 & 0 & & \end{array}\right),$$

where $Y_1, Y_2 \in \Gamma'_1$ and $Y_3, Y_4 \in \Gamma'$. We have fourteen families of mutually non-isomorphic evolution algebras.

4.2.3. *There is one zero in U.* As in Case 4.1.3, there are four possibilities for U and the structure matrix can have the form:

$$\left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & \omega_{14} \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & 0 \\ \omega_{21} & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right).$$

The first one is isomorphic to the second one and the third one is isomorphic to the last one, via the permutation $(3,4)$. For the classification we fix the first matrix and the third one. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & 0 \\ \omega_{21} & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right).$$

For any election of Y in Γ' we have mutually non-isomorphic evolution algebras. The number of families is eight.

4.2.4. *There are no zeros in U.* There is only one possibility for U and, therefore, the matrix in (5) takes the form:

$$\left(\begin{array}{cc|cc} 1 & 0 & \omega_{13} & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right),$$

where $Y \in \Gamma'_1$. We have three families of mutually non-isomorphic evolution algebras.

4.3. Take $W = \begin{pmatrix} 1 & \omega_{12} \\ \omega_{21} & 1 \end{pmatrix}$.

4.3.1. *There are three zeros in U.* There are four possibilities for U providing four different matrices as in (5):

$$\left(\begin{array}{cc|cc} 1 & \omega_{12} & 0 & 0 \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & \omega_{12} & 0 & 0 \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & \omega_{12} & 0 & \omega_{14} \\ \omega_{21} & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 1 & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right),$$

They are isomorphic to the first one via the permutations $(3,4)$, $(1,2)$ and $(1,2)(3,4)$, respectively. For the classification we fix the first matrix. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 1 & \omega_{12} & 0 & 0 \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{Y} \\ 0 & 0 & & \end{array}\right),$$

where $Y \in \Gamma'$. We have four families of mutually non-isomorphic evolution algebras.

4.3.2. *There are two zeros in U.* Again there are six possibilities for U. The matrix in (5) can be:

$$\left(\begin{array}{cc|cc} 1 & \omega_{12} & 0 & 0 \\ \omega_{21} & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 1 & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 1 & \omega_{12} & 0 & \omega_{14} \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 1 & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 1 & \omega_{12} & 0 & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 1 & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right).$$

The first one and the second one are isomorphic, as well as the third and the fourth and the fifth and the last one. The permutations giving these isomorphisms are $(1,2)$ in the first case and $(3,4)$ in the other cases. We fix the first, the third and the fifth matrices. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 1 & \omega_{12} & 0 & 0 \\ \omega_{21} & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y_1} \end{array}\right), \left(\begin{array}{cc|cc} 1 & \omega_{12} & 0 & \omega_{14} \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y_2} \end{array}\right), \left(\begin{array}{cc|cc} 1 & \omega_{12} & 0 & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y_3} \end{array}\right),$$

where $Y_1, Y_3 \in \Gamma'_1$ and $Y_2 \in \Gamma'$. We have ten families of mutually non-isomorphic evolution algebras.

4.3.3. *There is one zero in U.* Again, we have four possibilities for U and there are four cases appearing:

$$\left(\begin{array}{cc|cc} 1 & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right), \quad \left(\begin{array}{cc|cc} 1 & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & 1 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right), \quad \left(\begin{array}{cc|cc} 1 & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right), \quad \left(\begin{array}{cc|cc} 1 & \omega_{12} & 0 & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right),$$

They are isomorphic to the first one via the permutations $(3,4)$, $(1,2)$ and $(1,2)(3,4)$, respectively. For the classification we fix the first matrix. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 1 & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right),$$

where $Y \in \Gamma'$. We have four families of mutually non-isomorphic evolution algebras.

4.3.4. *There are no zeros in U.* There is only one possibility for U and, therefore, the matrix in (5) takes the form:

$$\left(\begin{array}{cc|cc} 1 & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & 1 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right),$$

where $Y \in \Gamma'_1$. We have three families of mutually non-isomorphic evolution algebras.

4.4. Take $W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

4.4.1. *There are three zeros in U.* We have four possibilities for U and the structure matrices are

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right), \quad \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right), \quad \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right), \quad \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right).$$

All of them are isomorphic to the first one via the permutations $(3,4)$, $(1,2)$ and $(1,2)(3,4)$, respectively. For the classification we fix the first matrix. The possible evolution algebras are:

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{|c}{} \\ 0 & 0 & \multicolumn{2}{|c}{Y} \end{array}\right),$$

where $Y \in \Gamma'$. We have four families of mutually non-isomorphic evolution algebras.

4.4.2. *There are two zeros in U.* There are six possibilities for U and the matrix in (5) can be

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & \omega_{14} \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & 0 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & 0 \\ 1 & 0 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & 0 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & 0 \\ 1 & 0 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right).$$

The first and the second are isomorphic, as well as the third and the fourth one, and the fifth and the last one. The permutations giving these isomorphisms are $(1,2)$ in the first case and $(3,4)$ in the other cases. For the classification we choose the matrices appearing in the odd positions. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y_1} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & 0 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y_2} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & 0 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y_3} \end{array}\right),$$

where $Y_1, Y_3 \in \Gamma_1'$ and $Y_2 \in \Gamma'$. We have ten families of mutually non-isomorphic evolution algebras.

4.4.3. *There is one zero in U.* Again we have four possibilities for U and the possible structre matrices are

$$\left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & \omega_{14} \\ 1 & 0 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & \omega_{14} \\ 1 & 0 & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & 0 \\ 1 & 0 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & 0 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right).$$

All of them are isomorphic to the first one via the permutations $(3,4)$, $(1,2)$ and $(1,2)(3,4)$, respectively. For the classification we fix the first matrix. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & \omega_{14} \\ 1 & 0 & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right),$$

where $Y \in \Gamma'$. We have four families of mutually non-isomorphic evolution algebras.

4.4.4. *There are no zeros in U.* There is only one possibility for U and, therefore, the matrix in (5) takes the form:

$$\left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & \omega_{14} \\ 1 & 0 & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right),$$

where $Y \in \Gamma_1'$. We have three families of mutually non-isomorphic evolution algebras.

4.5. Take $W = \begin{pmatrix} 0 & 1 \\ 1 & \omega_{22} \end{pmatrix}$.

4.5.1. *There are three zeros in U.* In this case there are four possibilities for U and the matrix in (5) can take the form:

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & \omega_{22} & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & \omega_{22} & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & \omega_{22} & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & 0 \\ 1 & \omega_{22} & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right).$$

The first one is isomorphic to the second one and the third one is isomorphic to the last one, via, in both cases, the permutation $(3,4)$. We fix the first matrix and the third one. The possible evolution algebras are:

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & \omega_{22} & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & \omega_{22} & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{} \\ 0 & 0 & \multicolumn{2}{c}{Y} \end{array}\right).$$

For any election of Y in Γ' we have mutually non-isomorphic evolution algebras. The number of families is eight.

4.5.2. *There are two zeros in U.* As before, U has six possibilities. The matrix in (5) becomes:

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & \omega_{22} & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & \omega_{14} \\ 1 & \omega_{22} & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & \omega_{22} & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & 0 \\ 1 & \omega_{22} & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & \omega_{22} & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & 0 \\ 1 & \omega_{22} & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right).$$

The third and the fourth are isomorphic, as well as the fifth and the last one. The permutation giving these isomorphisms is $(3, 4)$. We fix the first, the second, the third and the fifth matrices. The possible evolution algebras are

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & \omega_{22} & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y_1}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & \omega_{14} \\ 1 & \omega_{22} & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y_2}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & \omega_{22} & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y_3}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & \omega_{22} & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y_4}} \\ 0 & 0 & & \end{array}\right),$$

where $Y_1, Y_2 \in \Gamma'_1$ and $Y_3, Y_4 \in \Gamma'$. We have fourteen families of mutually non-isomorphic evolution algebras.

4.5.3. *There is one zero in U.* There are four possibilities for U. In this case the matrix in (5) can take the form:

$$\left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & \omega_{14} \\ 1 & \omega_{22} & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & \omega_{14} \\ 1 & \omega_{22} & 0 & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & 0 \\ 1 & \omega_{22} & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & 0 & \omega_{14} \\ 1 & \omega_{22} & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right).$$

The first one is isomorphic to the second one and the third one is isomorphic to the last one, via, in both cases, the permutation $(3, 4)$. For the classification we fix the first one and the third one:

$$\left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & \omega_{14} \\ 1 & \omega_{22} & \omega_{23} & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & 0 \\ 1 & \omega_{22} & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right).$$

For any election of Y in Γ' we have mutually non-isomorphic evolution algebras. The number of families is eight.

4.5.4. *There are no zeros in U.* There is only one possibility for U and, therefore, the matrix in (5) takes the form:

$$\left(\begin{array}{cc|cc} 0 & 1 & \omega_{13} & \omega_{14} \\ 1 & \omega_{22} & \omega_{23} & \omega_{24} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & 0 & & \end{array}\right),$$

where $Y \in \Gamma'_1$. We have three families of mutually non-isomorphic evolution algebras.

5. Classification of four dimensional perfect non-simple evolution algebras. Case: The evolution algebra has a maximal 3-basic ideal

Assume that I is a maximal 3-basic ideal of A. Write: $B = \{e_1, e_2, e_3, e_4\}$, $I = \text{lin}\{e_1, e_2, e_3\}$ and

$$M_B = \left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\text{W}} & \text{U} \\ \hline 0 & 0 & 0 & 1 \end{array}\right). \tag{7}$$

As Example 2.12 (i) shows, the number of zeros of the matrices U and W is not necessarily invariant. This happens only when the conditions in Proposition 2.19 are satisfied. Therefore, in order to classify the evolution algebras which appear, we need a different approach. For this reason we pay attention to the change of basis matrices (we know that they are permutations of the basis up to scalars, as explained before) producing structure matrices having the same form as in (7). Looking at the matrix M_B it seems natural to think that these are those fixing e_4. This will not be true in general. It will only happen under certain extra hypothesis, as shown in Corollary 2.21. This is the reason because several cases appear. We consider whether or not I has a 2-basic ideal. In the affirmative, we distinguish whether or not this 2-basic ideal is contained in another 3-basic ideal.

5.1. A **has a 3-basic ideal which has a 2-basic ideal.** We may assume that $\{e_1, e_2\}$ is a 2-basic ideal of I, where I is a 3-basic ideal having basis $\{e_1, e_2, e_3\}$.

In the tables that follow we analyze which types of U and W produce a structure matrix such that A is irreducible, where W is the matrix of I. These matrices appear in the classification of three-dimensional evolution algebras (done in [2, 4]). Whenever we obtain an irreducible evolution algebra A, we add $*$ when A does not satisfy Condition (3,2,3). We remark again that this means that e_4 cannot be changed to any other element in the natural basis in order to produce another matrix having the same form.

W	$U=\begin{pmatrix}1\\0\\0\end{pmatrix}$	$U=\begin{pmatrix}0\\1\\0\end{pmatrix}$	$U=\begin{pmatrix}0\\0\\1\end{pmatrix}$	$U=\begin{pmatrix}1\\1\\0\end{pmatrix}$	$U=\begin{pmatrix}1\\0\\1\end{pmatrix}$	$U=\begin{pmatrix}0\\1\\1\end{pmatrix}$	$U=\begin{pmatrix}1\\1\\1\end{pmatrix}$
$\begin{pmatrix}\omega_{11}&0&0\\0&\omega_{22}&0\\0&0&\omega_{33}\end{pmatrix}$	Reducible	Reducible	Reducible	Reducible	Reducible	Reducible	Irreducible*
$\begin{pmatrix}0&\omega_{12}&0\\\omega_{21}&0&0\\0&0&\omega_{33}\end{pmatrix}$	Reducible	Reducible	Reducible	Reducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&\omega_{12}&0\\0&\omega_{22}&0\\0&0&\omega_{33}\end{pmatrix}$	Reducible	Reducible	Reducible	Reducible	Irreducible	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&\omega_{12}&0\\\omega_{21}&0&0\\0&0&\omega_{33}\end{pmatrix}$	Reducible	Reducible	Reducible	Reducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&0&0\\0&0&\omega_{33}\end{pmatrix}$	Irreducible	Irreducible	Irreducible*	Irreducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\0&\omega_{22}&0\\0&0&\omega_{33}\end{pmatrix}$	Irreducible	Irreducible	Irreducible	Irreducible	Irreducible	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&\omega_{12}&0\\\omega_{21}&\omega_{22}&0\\0&0&\omega_{33}\end{pmatrix}$	Reducible	Reducible	Reducible	Reducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&\omega_{12}&0\\0&\omega_{22}&\omega_{23}\\0&0&\omega_{33}\end{pmatrix}$	Irreducible	Irreducible	Irreducible*	Irreducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&0&\omega_{13}\\0&\omega_{22}&\omega_{23}\\0&0&\omega_{33}\end{pmatrix}$	Irreducible	Irreducible	Irreducible*	Irreducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&0&0\\0&0&\omega_{33}\end{pmatrix}$	Irreducible	Irreducible	Irreducible*	Irreducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&\omega_{12}&0\\\omega_{21}&0&\omega_{23}\\0&0&\omega_{33}\end{pmatrix}$	Irreducible	Irreducible	Irreducible*	Irreducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&0&\omega_{23}\\0&0&\omega_{33}\end{pmatrix}$	Irreducible	Irreducible	Irreducible*	Irreducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&0\\0&0&\omega_{33}\end{pmatrix}$	Irreducible	Irreducible	Irreducible*	Irreducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\0&\omega_{22}&\omega_{23}\\0&0&\omega_{33}\end{pmatrix}$	Irreducible	Irreducible	Irreducible*	Irreducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&0&\omega_{23}\\0&0&\omega_{33}\end{pmatrix}$	Irreducible	Irreducible	Irreducible*	Irreducible	Irreducible*	Irreducible*	Irreducible*
$\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\0&0&\omega_{33}\end{pmatrix}$	Irreducible	Irreducible	Irreducible*	Irreducible	Irreducible*	Irreducible*	Irreducible*

FIGURE 1. A has a 3-basic ideal which has a 2-basic ideal

Now, we analyze whether or not A satisfies Condition (3,2,3).

5.1.1. *A satisfies Condition (3,2,3).* The following tables describe the different types of mutually non-isomorphic evolution algebras. In order to differentiate the non-isomorphic algebras we use as invariants: the number of zeros in the matrix; the number of zeros in the main diagonal (these are invariant as shown in Proposition 2.13 (i) and (ii)); the indegree and the outdegree of each vertex in the associated graph (which is an invariant as, up to scalars, the only change of basis matrices are the permutations and any permutation preserves these degrees); and the graph (since the algebra is perfect, by [7, Corollary 4.5], the graph is an invariant).

There are 23 mutually non-isomorphic evolution algebras. In the table below two matrices having the same type correspond to isomorphic evolution algebras.

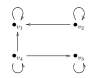
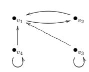
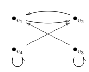
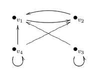
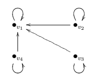
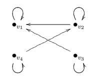
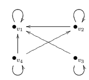
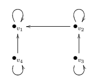
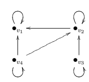
	Number of zeros	Number of zeros in the main diagonal	$(deg^+(v), deg^-(v))$	Graph	Type
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ 0 & \omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	9	0	(1,3)(2,1)(1,2)(3,1)		1
$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	10	2	(1,3)(1,1)(2,1)(2,1)		2
$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	10	2	(1,2)(1,2)(2,1)(2,1)		3
$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	9	2	(1,3)(1,2)(2,1)(3,1)		4
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ 0 & \omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	9	0	(1,4)(2,1)(2,1)(2,1)		5
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	9	0	(1,3)(2,2)(2,1)(2,1)		6
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	0	(1,4)(2,2)(2,1)(3,1)		7
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	9	0	(1,3)(2,2)(2,1)(2,1)		6
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	0	(1,3)(2,3)(2,1)(3,1)		9

FIGURE 2. *A* satisfies Condition (3,2,3)

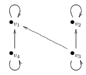
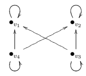
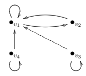
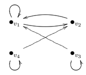
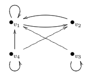
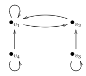
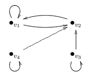
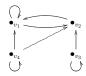
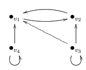
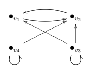
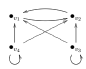
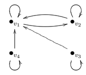
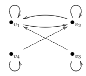
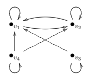
	Number of zeros	Number of zeros in the main diagonal	$(deg^+(v), deg^-(v))$	Graph	Type
$\left(\begin{array}{ccc\|c} \omega_{11} & 0 & \omega_{13} & 1 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	9	0	(1,3)(1,2)(3,1)(2,1)		1
$\left(\begin{array}{ccc\|c} \omega_{11} & 0 & \omega_{13} & 1 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	0	(1,3)(1,3)(3,1)(3,1)		10
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	9	1	(2,4)(1,1)(2,1)(2,1)		11
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	9	1	(2,3)(1,2)(2,1)(2,1)		12
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	1	(2,4)(1,2)(2,1)(3,1)		13
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	9	1	(2,3)(1,2)(2,1)(2,1)		12
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 0 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	9	1	(2,2)(1,3)(2,1)(2,1)		14
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	1	(2,3)(1,3)(2,1)(3,1)		15
$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	9	2	(1,3)(1,2)(3,1)(2,1)		4
$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	2	(1,2)(1,3)(3,1)(2,1)		16
$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	2	(1,3)(1,3)(3,1)(3,1)		17
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & \omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	0	(2,4)(2,2)(2,1)(2,1)		18
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	0	(2,3)(2,3)(2,1)(2,1)		19
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	7	0	(2,4)(2,3)(2,1)(3,1)		20

FIGURE 3. A satisfies Condition (3,2,3)

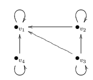
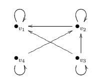
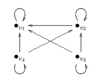
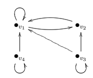
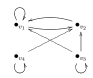
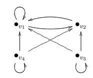
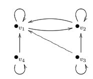
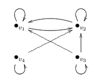
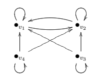
	Number of zeros	Number of zeros in the main diagonal	$(deg^+(v), deg^-(v))$	Graph	Type
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	0	(1,4)(2,2)(3,1)(2,1)		7
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	0	(1,3)(2,3)(3,1)(2,1)		9
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	7	0	(1,4)(2,3)(3,1)(3,1)		21
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	1	(2,4)(1,2)(3,1)(2,1)		13
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	8	1	(2,3)(1,3)(3,1)(2,1)		15
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	7	1	(2,4)(1,3)(3,1)(3,1)		22
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	7	0	(2,4)(2,3)(3,1)(2,1)		20
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	7	0	(2,3)(2,4)(3,1)(2,1)		20
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	6	0	(2,4)(2,4)(3,1)(3,1)		23

FIGURE 4. A satisfies Condition (3,2,3)

5.1.2. *The algebra A does not satisfy Condition (3,2,3).* We start by explaining how we proceed when the algebra A does not satisfy Condition (3,2,3) in order to obtain all the mutually non isomorphic evolution algebras.

We are classifying matrices of the form (7), where W and U are as in Figures 4 and 1 and the matrix (7) corresponds to an "Irreducible*" case, so, we start with 54 cases (which can be seen in Figure 1). Observe that two matrices as in (7) are associated to isomorphic evolution algebras if and only if the corresponding W's are the matrices of three-dimensional isomorphic evolution algebras. The reason is that the possible change of basis matrices are those permutation matrices keeping the last column invariant.

Choose one matrix as in (7) and locate W in [2, Tables 18-22] (which is the classification of three dimensional perfect evolution algebras having a 2-basic ideal generated by the first and the second vectors in the given basis). If there is another W' in the mentioned tables such that the three-dimensional evolution algebras with associated matrices W and W' are isomorphic and such that both algebras have a 2-basic ideal generated by the first and the second vectors in the given basis, then the four-dimensional evolution algebras having these W and W', with the corresponding U and U', are isomorphic. After doing this, we group by isomorphisms the evolution algebras provided in Figure 1.

The tables below show which matrices included among the 54 correspond to isomorphic evolution algebras. Two matrices in the same row correspond to isomorphic evolution algebras and matrices in different rows are not isomorphic.

$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 0 \\ \omega_{12} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 0 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & 0 & 1 \\ \omega_{12} & \omega_{11} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 0 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{11} & 0 & \omega_{12} & 0 \\ 0 & \omega_{33} & 0 & 1 \\ 0 & 0 & \omega_{22} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 0 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & 0 & 1 \\ 0 & \omega_{33} & \omega_{32} & 0 \\ 0 & 0 & \omega_{11} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & 0 & 1 \\ \omega_{12} & \omega_{11} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{11} & 0 & \omega_{12} & 1 \\ 0 & \omega_{33} & 0 & 1 \\ 0 & 0 & \omega_{22} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & 0 & 1 \\ 0 & \omega_{33} & \omega_{31} & 1 \\ 0 & 0 & \omega_{11} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 0 \\ \omega_{12} & \omega_{11} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 0 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 1 \\ \omega_{12} & \omega_{11} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 1 \\ \omega_{12} & \omega_{11} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$

FIGURE 5. Isomorphic evolution algebras

	(1,2)
$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 0 \\ \omega_{12} & 0 & \omega_{13} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 0 \\ \omega_{12} & 0 & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 1 \\ \omega_{12} & 0 & \omega_{13} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 1 \\ \omega_{12} & 0 & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & 0 & 1 \\ \omega_{12} & \omega_{11} & \omega_{13} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & 0 & 1 \\ \omega_{12} & \omega_{11} & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & \omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & \omega_{21} & 0 & 0 \\ \omega_{12} & \omega_{11} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 0 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & \omega_{23} & 0 \\ \omega_{12} & \omega_{11} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & \omega_{23} & 0 \\ \omega_{12} & \omega_{11} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 0 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & \omega_{23} & 1 \\ \omega_{12} & \omega_{11} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$

FIGURE 6. Isomorphic evolution algebras

	(1,2)
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & \omega_{23} & 1 \\ \omega_{12} & \omega_{11} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & 0 & \omega_{13} & 1 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & \omega_{23} & 0 \\ 0 & \omega_{11} & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 0 \\ \omega_{12} & \omega_{11} & \omega_{13} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 0 \\ \omega_{12} & \omega_{11} & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 1 \\ \omega_{12} & \omega_{11} & \omega_{13} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & 0 & 1 \\ \omega_{12} & \omega_{11} & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 0 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & \omega_{23} & 0 \\ \omega_{12} & \omega_{11} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & \omega_{23} & 0 \\ \omega_{12} & \omega_{11} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 0 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & \omega_{23} & 1 \\ \omega_{12} & \omega_{11} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & \omega_{23} & 1 \\ \omega_{12} & \omega_{11} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$

FIGURE 7. Isomorphic evolution algebras

	(1,2)
$\left(\begin{array}{ccc\|c} 0 & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & \omega_{23} & 0 \\ \omega_{12} & 0 & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & \omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & \omega_{21} & 0 & 0 \\ \omega_{12} & \omega_{11} & \omega_{13} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & \omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & \omega_{21} & 0 & 0 \\ \omega_{12} & \omega_{11} & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & \omega_{21} & 0 & 1 \\ \omega_{12} & \omega_{11} & \omega_{13} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & \omega_{21} & 0 & 1 \\ \omega_{12} & \omega_{11} & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & \omega_{23} & 0 \\ \omega_{12} & \omega_{11} & \omega_{13} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & \omega_{23} & 0 \\ \omega_{12} & \omega_{11} & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & \omega_{23} & 1 \\ \omega_{12} & \omega_{11} & \omega_{13} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & 0 & \omega_{23} & 1 \\ \omega_{12} & \omega_{11} & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & \omega_{23} & 0 \\ \omega_{12} & \omega_{11} & \omega_{13} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$

FIGURE 8. Isomorphic evolution algebras

	(1,2)
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & \omega_{23} & 0 \\ \omega_{12} & \omega_{11} & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & \omega_{23} & 1 \\ \omega_{12} & \omega_{11} & \omega_{13} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 0 & \omega_{21} & \omega_{23} & 1 \\ \omega_{12} & \omega_{11} & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$
$\left(\begin{array}{ccc\|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|c} \omega_{22} & \omega_{21} & \omega_{23} & 0 \\ \omega_{12} & \omega_{11} & \omega_{13} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$

FIGURE 9. Isomorphic evolution algebras

After removing in the set of the 54 matrices the ones corresponding to isomorphic evolution algebras, the remaining are:

$$\left(\begin{array}{ccc|c} \omega_{11} & 0 & 0 & 1 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & \omega_{12} & 0 & 0 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 0 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 1 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 0 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & \omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 0 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 1 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 0 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 1 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & 0 & \omega_{13} & 0 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & 0 & \omega_{13} & 1 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & 0 & \omega_{13} & 1 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ 0 & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 0 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 0 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & 0 & 1 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & \omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & \omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & \omega_{22} & 0 & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ 0 & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & 0 & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & \omega_{22} & \omega_{23} & 0 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} \omega_{11} & \omega_{12} & \omega_{13} & 1 \\ \omega_{21} & \omega_{22} & \omega_{23} & 1 \\ 0 & 0 & \omega_{33} & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

5.2. No 3-basic ideal of A has a 2-basic ideal.

5.2.1. *The matrix U has one nonzero entry.* The possible matrices are of the form:

$$\begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & \omega_{22} & \omega_{23} & 0 \\ \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & 0 \\ \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} \\ \omega_{31} & \omega_{32} & \omega_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & \omega_{22} & \omega_{23} & 0 \\ \omega_{31} & \omega_{32} & \omega_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the entries in the last column are nonzero. It happens that the three of them produce isomorphic evolution algebras and the change of basis matrices are given by $I_{(2,3)}$ (producing the isomorphism between the first and the second) and by $I_{(1,3)}$ (producing the isomorphism between the first and the third). Moreover, by multiplying conveniently one of the vectors in the basis by an scalar, we may assume that the matrix is:

$$\left(\begin{array}{ccc|c} & & & 0 \\ & W & & 0 \\ & & & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right). \tag{8}$$

In order to get an irredundant classification we may keep the last row and the last column as in (8) (use what has been explained above and use Theorem 2.15). This implies that the only possible change of basis which is allowed is $I_{(1,2)}$. Moreover, the matrix W corresponds to a three-dimensional evolution algebra not having a 2-basic ideal. (These appear in the classification in [2, 4]). Concretely, they are one of the following:

$$\begin{pmatrix} 0 & 0 & \omega_{13} \\ \omega_{21} & 0 & 0 \\ 0 & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & 0 \\ \omega_{21} & 0 & 0 \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ 0 & 0 & \omega_{23} \\ 0 & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ \omega_{21} & 0 & 0 \\ 0 & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & 0 \\ 0 & 0 & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ 0 & 0 & \omega_{23} \\ \omega_{31} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & 0 \\ 0 & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & 0 \\ 0 & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ 0 & 0 & \omega_{23} \\ \omega_{31} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ \omega_{21} & 0 & 0 \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ 0 & 0 & \omega_{23} \\ 0 & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ \omega_{21} & 0 & 0 \\ 0 & \omega_{32} & \omega_{33} \end{pmatrix}$$

$$\begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ 0 & 0 & \omega_{23} \\ 0 & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ 0 & \omega_{22} & \omega_{23} \\ \omega_{31} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & 0 \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ 0 & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & 0 \\ \omega_{21} & 0 & \omega_{23} \\ 0 & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ \omega_{21} & 0 & \omega_{23} \\ 0 & \omega_{32} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \omega_{13} \\ 0 & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & 0 \\ \omega_{21} & 0 & 0 \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ 0 & 0 & \omega_{23} \\ 0 & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & 0 \\ 0 & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & 0 \\ 0 & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ 0 & 0 & \omega_{23} \\ \omega_{31} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ 0 & \omega_{32} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \omega_{12} & 0 \\ 0 & \omega_{22} & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ 0 & 0 & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ \omega_{21} & 0 & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ 0 & 0 & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ 0 & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & 0 \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ 0 & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ 0 & \omega_{22} & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ 0 & \omega_{22} & \omega_{23} \\ 0 & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & 0 \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ 0 & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix}$$

$$\begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & 0 \\ 0 & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ 0 & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ 0 & \omega_{22} & \omega_{23} \\ \omega_{31} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ 0 & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ 0 & \omega_{32} & \omega_{33} \end{pmatrix}$$

$$\begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ 0 & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ 0 & \omega_{22} & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & 0 \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ 0 & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & 0 \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ 0 & \omega_{22} & \omega_{23} \\ 0 & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ 0 & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & 0 \\ 0 & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ 0 & 0 & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix}$$

$$\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&0&0\\\omega_{31}&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\0&0&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&0&\omega_{23}\\\omega_{31}&0&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&0&\omega_{23}\\\omega_{31}&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&0\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&0&\omega_{13}\\0&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&0\\0&\omega_{32}&\omega_{33}\end{pmatrix}$$

$$\begin{pmatrix}\omega_{11}&\omega_{12}&0\\0&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\0&\omega_{22}&\omega_{23}\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&0&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&0\\\omega_{21}&\omega_{22}&0\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&0&0\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\0&0&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&0&0\end{pmatrix}$$

$$\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&0&\omega_{23}\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&0&\omega_{13}\\\omega_{21}&0&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&0\\\omega_{31}&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&0&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&0\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&0&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}$$

$$\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&0&\omega_{23}\\\omega_{31}&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&0&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&0\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}$$

Denote by Ω' the set whose elements are the matrices above.

5.2.2. *The matrix U has two nonzero entries.* We reason in a similar way as in Case 5.2.1. Here, the possible matrices are of the form:

$$\left(\begin{array}{ccc|c} & & & \omega_{14}\\ & W & & \omega_{24}\\ & & & 0\\ \hline 0&0&0&1\end{array}\right),\quad \left(\begin{array}{ccc|c} & & & \omega_{14}\\ & W & & 0\\ & & & \omega_{34}\\ \hline 0&0&0&1\end{array}\right),\quad \left(\begin{array}{ccc|c} & & & 0\\ & W & & \omega_{24}\\ & & & \omega_{34}\\ \hline 0&0&0&1\end{array}\right).$$

All of them produce isomorphic evolution algebras and the change of basis matrices are given by $I_{(2,3)}$ (producing the isomorphism between the first and the second) and by $I_{(1,3)}$ (producing the isomorphism between the first and the third). Moreover, by multiplying conveniently one of the vectors in the basis by an scalar, we may assume that the matrix is:

$$\left(\begin{array}{ccc|c} & & & 1\\ & W & & 1\\ & & & 0\\ \hline 0&0&0&1\end{array}\right), \tag{9}$$

where $W \in \Omega'$.

5.2.3. *The matrix U has no nonzero entries.* In this case the structure matrix M_B is as follows:

$$\left(\begin{array}{ccc|c} & & & 1\\ & W & & 1\\ & & & 1\\ \hline 0&0&0&1\end{array}\right). \tag{10}$$

In order to determine the possible values for W, we analyze which are the allowed changes of basis. We get that $I_\sigma M_B$ is, up to scalars, of the same form as the matrix in (10) if σ belongs to the subgroup of S_4 whose elements are $\{\mathrm{id}, (2,3), (1,2), (1,2,3), (1,3,2), (1,3)\}$, which consists of all permutations in S_4 which leave 4 invariant.

With this information in hand we proceed by choosing the possible matrices W. It happens that they are the ones associated to three-dimensional evolution algebras not having a 2-basic ideal which appear in the classification in [2, 4]. Concretely, W can be one of the following:

$$\begin{pmatrix}0&0&\omega_{13}\\\omega_{21}&0&0\\0&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&0\\\omega_{21}&0&0\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&0&\omega_{13}\\\omega_{21}&0&0\\0&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&0&0\\0&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&0\\\omega_{21}&0&0\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&0\\\omega_{21}&0&0\\0&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&0\\\omega_{21}&0&\omega_{23}\\\omega_{31}&0&\omega_{33}\end{pmatrix}$$

$$\begin{pmatrix}0&\omega_{12}&0\\\omega_{21}&0&\omega_{23}\\0&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&0\\\omega_{21}&0&0\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&0&0\\0&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&0&\omega_{13}\\\omega_{21}&\omega_{22}&0\\0&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&0&\omega_{13}\\\omega_{21}&0&0\\\omega_{31}&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&0&\omega_{23}\\0&\omega_{32}&0\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&0\\0&\omega_{22}&\omega_{23}\\\omega_{31}&0&\omega_{33}\end{pmatrix}$$

$$\begin{pmatrix}\omega_{11}&\omega_{12}&0\\\omega_{21}&\omega_{22}&0\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&0&0\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&0&0\\0&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&0\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&0\\0&\omega_{32}&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&0&\omega_{23}\\\omega_{31}&0&\omega_{33}\end{pmatrix}\quad\begin{pmatrix}0&\omega_{12}&\omega_{13}\\\omega_{21}&0&0\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}$$

$$\begin{pmatrix} 0 & \omega_{12} & 0 \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ 0 & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & 0 \\ 0 & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix}$$

$$\begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & 0 \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & 0 & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & 0 & \omega_{32} \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix}$$

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