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ON THE SPECIALITY OF JORDAN ALGEBRAS AND SUBQUOTIENTS OF LIE ALGEBRAS

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Abstract. In [7] Jordan algebras were attached to ad-nilpotent elements of index less than or equal to three of Lie algebras. In this paper we study conditions on the own structure of the Lie algebras that imply the speciality of these Jordan algebras. Similar results are obtained when dealing with subquotients associated to abelian inner ideals.

Key words: Lie algebra, Jordan algebra, subquotient, speciality.
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1. Introduction

The possibility of embedding a Jordan system into an associative system has been considered from the very beginning of the Jordan Theory. Jordan algebras were invented in the 1930s in the search for an exceptional algebraic setting for quantum mechanics (exceptional in the sense that its structure was not determined by some unobservable associative algebra). In their seminal paper of 1933, Jordan, von Neumann, and Wigner classified the finite-dimensional formally real linear Jordan algebras [12]. Already in 1934 A. A. Albert showed that the only simple algebra in the list which was exceptional was a certain 27-dimensional algebra of 3×3 hermitian matrices with entries from an 8-dimensional Cayley algebra [1]. Afterwards, several authors dealt with this problem until 1979 when Zelmanov published his outstanding result that the only simple exceptional linear Jordan algebra was the 27-dimensional Albert algebra [16].

Some Jordan algebras come attached to ad-nilpotent elements in Lie algebras. In their paper [7] the first-named two authors of this work together with A. Fernández gave a construction that mimics the construction of local algebras in Jordan systems. Given any Jordan system J and any element $a \in J$, we can consider the Jordan algebra J_a which is a quotient of the a-homotope algebra $J^{(a)}$ by the set $\operatorname{Ker} a$. When this construction is carried to the Lie setting, an extra condition must be imposed on the element a – it must be ad-nilpotent of index less than or equal to three – and the structure L_a obtained after quotiening out the kernel of a turns out to be a Jordan algebra.

In this paper we investigate some conditions under which the Jordan algebras L_a attached to strongly prime Lie algebras are special. These conditions are related to the kernel of a. Notice that for any nondegenerate Lie algebra L and any nonzero ad-nilpotent element a of index three, $\operatorname{Ker} a$ is always nonzero (because it contains

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$[a, L]$). The key point for the speciality of L_a will be related to $\operatorname{Ker} a$ not being a Lie subalgebra. Our main result is the following (see Corollary 2.5):

For a strongly prime Lie algebra L and an ad-nilpotent element $0 \neq a \in L$ of index 3 with $\operatorname{Ker} a$ *not being a subalgebra of L, the Jordan algebra L_a is special.*

Our proof directly gives a specialization of L_a. We want to point out that our result gives just a sufficient condition for the speciality of L_a. There exist Lie algebras L and elements $a \in L$ with $\operatorname{ad}_a^3(L) = 0$ and $\operatorname{Ker} a$ being a subalgebra such that L_a is special: for example if $L = \mathfrak{sl}(2) :=< \{e, h, f\} >$ over a field $\mathbb{F}$, then $\operatorname{Ker} e =< \{e, h\} >$, which is a subalgebra of L and $L_e \cong \mathbb{F}$ which is special.

Our proof is based on the fact that any ad-nilpotent element $a \in L$ with $\operatorname{ad}_a^3 L = 0$ gives rise to a filtration $\mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$ with $\mathcal{F}_{-2} = \Phi a + [a, [a, L]]$, $\mathcal{F}_1 = \operatorname{Ker} a$ and $\mathcal{F}_2 = L$, inducing a 5-grading

$$\hat{L} = \mathcal{F}_{-2} \oplus \mathcal{F}_{-1}/\mathcal{F}_{-2} \oplus \mathcal{F}_0/\mathcal{F}_{-1} \oplus \mathcal{F}_1/\mathcal{F}_0 \oplus \mathcal{F}_2/\mathcal{F}_1.$$

For any 5-graded Lie algebra $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, one can define a pair of Jordan homomorphisms $\Psi_1 : L_{-2} \to \operatorname{Hom}(L_1, L_{-1})$, $\Psi_{-1} : L_2 \to \operatorname{Hom}(L_{-1}, L_1)$ by $\Psi_{\sigma 1}(x) = \operatorname{ad}_x$. We will impose conditions to assure the injectivity of (Ψ_1, Ψ_{-1}) for the particular case of the 5-grading induced by an ad-nilpotent element $a \in L$ with $\operatorname{ad}_a^3 L = 0$. In particular, if L is simple, the Jordan pair (L_{-2}, L_2) is simple (see [5, Theorem 11.32], [17, Lemma 1.5]) and it is special as soon as (Ψ_1, Ψ_{-1}) is nonzero. As an example of this situation, let us consider a strongly prime Lie algebra L with an ad-nilpotent element $a \in L$ of index 3 such that there exists $b \in L$ with $[a, [a, b]] = -2a$. We can define $h := [a, b]$; ad_h is a semisimple element with eigenvalues $0, \pm 1, \pm 2$, and L is a 5-graded Lie algebra, $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, where each L_i is the eigenspace of L associated to the eigenvalue i, $i = 0, \pm 1, \pm 2$, see [2, Lemma 2.1]. If L_{-1} or L_1 are nonzero, (L_{-1}, L_1) is a module for the Jordan pair (L_{-2}, L_2) via the maps (Ψ_1, Ψ_{-1}). In particular, if L is simple, the Jordan pair (L_{-2}, L_2) is simple, and (Ψ_1, Ψ_{-1}) is injective since it cannot be zero (the map $\Psi_1(a) = \operatorname{ad}_a$ is a bijection between L_1 and L_{-1}, see [2, Lemma 2.1(3)]).

The last section is devoted to studying conditions under which the subquotients associated to abelian inner ideals B of a strongly prime Lie algebra L are special. Since filtrations associated to abelian inner ideals are not known, we directly construct the Jordan homomorphisms between the subquotients and the pairs of homomorphisms. The conditions we require to get injectivity include $[B, \operatorname{Ker} B]$ being nilpotent (which holds directly in the case of ad-nilpotent elements) and $\operatorname{Ker} B$ not being a subalgebra. Our main result in this section is the following (see Corollary 3.4):

Given a strongly prime Lie algebra L and an abelian inner ideal B of L, if $[B, \operatorname{Ker} B]$ *is nilpotent and* $\operatorname{Ker} B$ *is not a subalgebra, the subquotient* $(B, L/\operatorname{Ker} B)$ *is a special Jordan pair.*

In the particular case of $L = R^-$ for a prime associative algebra R, we will show that $[B, \operatorname{Ker} B]$ is always nilpotent of index less than or equal to three. Moreover, any abelian inner ideal B can be enlarged to another abelian inner ideal B^* of $\hat{R}^-$ (the central closure of R) that gives rise to a filtration of $\hat{R}^-$: $\mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$, with $\mathcal{F}_{-2} = B^*$, $\mathcal{F}_1 = \operatorname{Ker} B$ and $\mathcal{F}_2 = \hat{R}$.

2. Speciality of the Jordan algebras of a Lie algebra

Throughout this paper and unless otherwise specified, we will be dealing with Lie algebras, rings and Jordan systems over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3} \in \Phi$. The reader is referred to [11, 15] for basic results, notation and terminology on Lie algebras and Jordan systems, respectively.

In this section we will give a sufficient condition for the speciality of the Jordan algebra L_a at a Jordan element a. We remark that although L_a is a Jordan algebra, the condition for its speciality will be a Lie condition: $[\operatorname{Ker} a, \operatorname{Ker} a] \not\subset \operatorname{Ker} a$.

Lemma 2.1. *Let $L = L_{-n} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_n$ be a $(2n+1) - \mathbb{Z}$-graded Lie algebra. Then the pair $V := (L_{-n}, L_n)$ with product $\{x, y, z\} := [x, [y, z]]$ for $x, z \in L_{\sigma n}$ and $y \in L_{-\sigma n}$, $\sigma = \pm$ is a Jordan pair. Moreover, for any $i \in 1, 2, \dots, n-1$ the pair of linear maps (Ψ_i, Ψ_{-i})*

$$\Psi_i : L_{-n} \to \operatorname{Hom}(L_i, L_{i-n}) \qquad \Psi_{-i} : L_n \to \operatorname{Hom}(L_{i-n}, L_i)$$

defined by $\Psi_{\sigma i}(x)(y) = \operatorname{ad}_x y$ for any $x \in L_{-\sigma n}$ and any $y \in L_i$ if $\sigma = +$ or $y \in L_{i-n}$ if $\sigma = -$ is a homomorphism of Jordan pairs between V and the special Jordan pair $(\operatorname{Hom}(L_i, L_{i-n}), \operatorname{Hom}(L_{i-n}, L_i))^{(+)}$.

Proof. V is a Jordan pair by [17, p. 351].

Moreover, as mentioned in the proof of [5, Theorem 11.34] it is routine to verify that, for any $i = 1, \dots, n-1$, the pair of maps (Ψ_i, Ψ_{-i}) is a Jordan pair homomorphism. □

2.2. Let L be a Lie algebra over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3} \in \Phi$. Ad-nilpotent elements of index less than or equal to 3 will be called Jordan elements. Given a Jordan element $a \in L$, define $\operatorname{Ker} a = \{x \in L \mid [a, [a, x]] = 0\}$. Then $L_a := L/\operatorname{Ker} a$ with product $\bar{x} \circ \bar{y} = \overline{[[x, a], y]}$ becomes a Jordan algebra, called the Jordan algebra of L at a, see [7].

Associated to a Jordan element a of L we can consider a filtration

$$\mathcal{F}_i = 0,\ i \leq -3, \qquad \mathcal{F}_{-2} = \Phi a + [a, [a, L]] \qquad \mathcal{F}_{-1} = \Phi a + [a, \operatorname{Ker} a]$$
$$\mathcal{F}_0 = \{x \in L \mid [x, a] \in [a, [a, L]]\} \qquad \mathcal{F}_1 = \operatorname{Ker} a \qquad \mathcal{F}_j = L,\ j \geq 2.$$

Then $\{\mathcal{F}_i\}_i$ is a bounded filtration of L, called the principal filtration of L defined by a, see [10, 1.2, 1.3]. Furthermore,

$$\hat{L} = \mathcal{F}_{-2} \oplus \mathcal{F}_{-1}/\mathcal{F}_{-2} \oplus \mathcal{F}_0/\mathcal{F}_{-1} \oplus \mathcal{F}_1/\mathcal{F}_0 \oplus \mathcal{F}_2/\mathcal{F}_1$$

is a 5-graded Lie algebra. In particular, $V = (\mathcal{F}_{-2}, \mathcal{F}_2/\mathcal{F}_1)$ is a Jordan pair associated to a.

When L is nondegenerate, the set $\operatorname{Ker} a$ coincides with

$$\operatorname{Ker}(a) = \{x \in L \mid [(a), [(a), x]] = 0\}$$

for $(a) = \Phi a + [a, [a, L]]$, see [13, 3.7]. Then V is the subquotient of L induced by the abelian inner ideal (a), [13, 3.2], and the Jordan algebra L_a can be regarded as the a-homotope of the Jordan pair V at a, see [13, 3.6].

In the following result we will use Lemma 2.1 to define a specialization homomorphism.

Lemma 2.3. *Let L be a Lie algebra over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3} \in \Phi$, let $a \in L$ be a Jordan element, and let $\{\mathcal{F}_i\}_i$ be the principal filtration of L defined by a. Then the pair of maps (Ψ_1, Ψ_{-1})*

$$\Psi_1 : \Phi a + [a,[a,L]] \to \operatorname{Hom}(\mathcal{F}_1/\mathcal{F}_0,\, \mathcal{F}_{-1}/\mathcal{F}_{-2})$$
$$\Psi_{-1} : L/\operatorname{Ker} a \to \operatorname{Hom}(\mathcal{F}_{-1}/\mathcal{F}_{-2},\, \mathcal{F}_1/\mathcal{F}_0)$$

given by $\Psi_1(b) = \mathrm{ad}_b$, $b \in \Phi a + [a,[a,L]]$, and $\Psi_{-1}(x + \operatorname{Ker} a) = \mathrm{ad}_{x+\operatorname{Ker} a}$, $x + \operatorname{Ker} a \in L/\operatorname{Ker} a$, is a homomorphism of Jordan pairs between

$$V = (\mathcal{F}_{-2},\, \mathcal{F}_2/\mathcal{F}_1) \text{ and } (\operatorname{Hom}(\mathcal{F}_1/\mathcal{F}_0,\, \mathcal{F}_{-1}/\mathcal{F}_{-2}), \operatorname{Hom}(\mathcal{F}_{-1}/\mathcal{F}_{-2},\, \mathcal{F}_1/\mathcal{F}_0))^{(+)}.$$

Proof. Consider the graded Lie algebra

$$\hat{L} = \mathcal{F}_{-2} \oplus \mathcal{F}_{-1}/\mathcal{F}_{-2} \oplus \mathcal{F}_0/\mathcal{F}_{-1} \oplus \mathcal{F}_1/\mathcal{F}_0 \oplus \mathcal{F}_2/\mathcal{F}_1.$$

The maps Ψ_1, Ψ_{-1} are well defined and (Ψ_1, Ψ_{-1}) is a Jordan pair homomorphism between $V = (\Phi a + [a,[a,L]],\, L/\operatorname{Ker} a)$ and the special Jordan pair

$$(\operatorname{Hom}(\mathcal{F}_1/\mathcal{F}_0,\, \mathcal{F}_{-1}/\mathcal{F}_{-2}), \operatorname{Hom}(\mathcal{F}_{-1}/\mathcal{F}_{-2},\, \mathcal{F}_1/\mathcal{F}_0))^{(+)}.$$

by Lemma 2.1. □

Proposition 2.4. *Let L be a Lie algebra over ring of scalars Φ with $\frac{1}{2}, \frac{1}{3} \in \Phi$ and let $a \in L$ be a Jordan element. The pair*

$$\mathcal{I} = (\mathcal{I}^+, \mathcal{I}^-) = ([a,[a,[\operatorname{Ker} a, \operatorname{Ker} a]]],\, ([\operatorname{Ker} a, \operatorname{Ker} a] + \operatorname{Ker} a)/\operatorname{Ker} a)$$

is an ideal of the Jordan pair $V = (V^+, V^-) = (\Phi a + [a,[a,L]],\, L/\operatorname{Ker} a)$.

Moreover, if L is nondegenerate,

$$(\operatorname{Ker}\Psi_1, \operatorname{Ker}\Psi_{-1}) \subset \operatorname{Ann}_V(\mathcal{I}).$$

Proof. Let us check that $\mathcal{I} = (\mathcal{I}^+, \mathcal{I}^-)$ is an ideal of V: For any $k_1, k_2 \in \operatorname{Ker} a$, any $x, y \in L$ and any $\lambda \in \Phi$, if we use the Jacobi identity and take into account the multiplication properties of the filtration,

$$\begin{aligned}
&[[a,[a,[k_1,k_2]]],[x,([a,[a,y]] + \lambda a)]] \in [[a,[a,[k_1,k_2]]],[\mathcal{F}_2,\mathcal{F}_{-2}]] \subset \\
&\quad \subset [[a,[a,[\mathcal{F}_1,\mathcal{F}_1]]],\mathcal{F}_0] \subset \\
&\quad \subset [[a,\mathcal{F}_0],[a,[\mathcal{F}_1,\mathcal{F}_1]]] + [a,[[a,\mathcal{F}_0],[\mathcal{F}_1,\mathcal{F}_1]]] + [a,[a,[\mathcal{F}_1,\mathcal{F}_1]] \subset \\
&\quad \subset [\mathcal{F}_{-2},[a,[\mathcal{F}_1,\mathcal{F}_1]]] + [a,[\mathcal{F}_{-2},[\mathcal{F}_1,\mathcal{F}_1]] + [a,[a,[\mathcal{F}_1,\mathcal{F}_1]]] \subset \\
&\quad \subset [a,[\mathcal{F}_{-1},\mathcal{F}_1]] + [a,[a,[\mathcal{F}_1,\mathcal{F}_1]]] \subset [\mathcal{F}_{-1},\mathcal{F}_{-1}] = \\
&\quad = [\Phi a + [a,\operatorname{Ker} a], \Phi a + [a,\operatorname{Ker} a]] \subset [[a,\operatorname{Ker} a],[a,\operatorname{Ker} a]] \subset [a,[a,[\operatorname{Ker} a,\operatorname{Ker} a]]]
\end{aligned}$$

because $[[a,k],[a,l]] = \frac{1}{2}[a,[a,[k,l]]]$ for every $k, l \in \operatorname{Ker} a$. This proves that

$$\{\mathcal{I}^+, V^-, V^+\} \subset \mathcal{I}^+.$$

Moreover, using the Jacobi identity and the multiplication properties of the filtration,

$$\begin{aligned}
[x,&[[a,[a,[k_1,k_2]]],y]] = 2\,[x,[[[a,k_1],[a,k_2]],y]] \subset [x,[\operatorname{Ker} a,[a,\operatorname{Ker} a]]] \subset \\
&\subset [[x,\operatorname{Ker} a],[a,\operatorname{Ker} a]] + [\operatorname{Ker} a,[x,[a,\operatorname{Ker} a]] \subset \\
&\subset [[\mathcal{F}_2,\mathcal{F}_1],[\mathcal{F}_{-2},\mathcal{F}_1]] + [\mathcal{F}_1,[\mathcal{F}_2,[\mathcal{F}_{-2},\mathcal{F}_1]]] \subset \\
&\subset [\mathcal{F}_2,\mathcal{F}_{-1}] + [\mathcal{F}_1,\mathcal{F}_1] \subset \operatorname{Ker} a + [\operatorname{Ker} a,\operatorname{Ker} a], \text{ so}
\end{aligned}$$

$$\{V^-, \mathcal{I}^+, V^-\} \subset \mathcal{I}^-.$$

Finally

(1) $[[k_1,k_2],[[a,[a,x]],y]] \in [[\operatorname{Ker} a, \operatorname{Ker} a], \mathcal{F}_0] \subset [\operatorname{Ker} a, \operatorname{Ker} a]$,

(2) $[[k_1,k_2],[a,y]] \in [[\operatorname{Ker} a, \operatorname{Ker} a], \mathcal{F}_0] \subset [\operatorname{Ker} a, \operatorname{Ker} a]$,

(3) $[[a,[a,x]],[[k_1,k_2],[a,[a,y]]]] \in [\mathcal{F}_{-2},[[\mathcal{F}_1,\mathcal{F}_1],\mathcal{F}_{-2}]] \subset [\mathcal{F}_{-2},[\mathcal{F}_{-1},\mathcal{F}_1]] \subset$
$\subset [\mathcal{F}_{-1},\mathcal{F}_{-1}] \subset [[a,\operatorname{Ker} a],[a,\operatorname{Ker} a]] \subset [a,[a,[\operatorname{Ker} a,\operatorname{Ker} a]]]$,
and similarly $[a,[[k_1,k_2],[a,[a,y]]]] \in [a,[a,[\operatorname{Ker} a,\operatorname{Ker} a]]$,
and $[a,[[k_1,k_2],a]] \in [a,[a,[\operatorname{Ker} a,\operatorname{Ker} a]]]$,

so (1) and (2) give

$$\{\mathcal{I}^-, V^+, V^-\} \subset \mathcal{I}^-,$$

and (3) implies

$$\{V^+, \mathcal{I}^-, V^+\} \subset \mathcal{I}^+.$$

Now we are going to prove that $(\operatorname{Ker}\Psi_1, \operatorname{Ker}\Psi_{-1}) \subset \operatorname{Ann}_V(\mathcal{I})$. We will use the characterization of the annihilator of an ideal as those elements z such that $\{z, I, z\} = 0$, which holds when V is nondegenerate [14, Proposition 1.7]. Remember that V is nondegenerate because we are assuming that L is nondegenerate.

If $u \in \operatorname{Ker}\Psi_1$ ($u \in \mathcal{F}_{-2}$), then $[u, \operatorname{Ker} a] \subset \mathcal{F}_{-2}$ and, therefore for every $k_1, k_2 \in \operatorname{Ker} a$

$$\{u,[k_1,k_2],u\} = [[u,[k_1,k_2]],u] = [[[u,k_1],k_2],u] + [[k_1,[u,k_2]],u] \in [[\mathcal{F}_{-2},\mathcal{F}_1],\mathcal{F}_{-2}] = 0,$$

which shows that u belongs to the annihilator of $\mathcal{I}$.

Take $\bar{x} = x + \operatorname{Ker} a \in \operatorname{Ker}\Psi_{-1}$, which means that $[x,[a,\operatorname{Ker} a]] \subset \mathcal{F}_0$. To see that $\bar{x}$ belongs to the annihilator of $\mathcal{I}$ let us see that

$$\{\bar{x},[a,[a,[\operatorname{Ker} a,\operatorname{Ker} a]]],\bar{x}\} = \bar{0},$$

which is equivalent to prove that $\operatorname{ad}_a^2[x,[[a,[a,[\operatorname{Ker} a,\operatorname{Ker} a]]],x]] = 0$: for any $k_1, k_2 \in \operatorname{Ker} a$

$$[a,[a,[x,[x,[a,[a,[k_1,k_2]]]]]]] \subset [a,[a,[x,[x,[[a,\operatorname{Ker} a],[a,\operatorname{Ker} a]]]]]] \subset$$
$$\subset [a,[a,[x,[\mathcal{F}_0,[a,\operatorname{Ker} a]]]]] \subset [a,[a,[x,[a,\operatorname{Ker} a]]]] \subset [a,[a,\mathcal{F}_0]] = 0.$$

□

Corollary 2.5. *Let L be a strongly prime Lie algebra over a ring of scalars Φ with $\frac{1}{2},\frac{1}{3} \in \Phi$ and let $a \in L$ be a Jordan element. If $\operatorname{Ker} a$ is not a subalgebra of L, i.e., $[\operatorname{Ker} a, \operatorname{Ker} a] \not\subset \operatorname{Ker} a$, then L_a is a special strongly prime Jordan algebra.*

Proof. The Jordan pair $V = (\Phi a + [a,[a,L]],\ L/\operatorname{Ker} a)$ is the subquotient of L determined by the abelian inner ideal $\Phi a + [a,[a,L]]$, so it is strongly prime by [13, 3.5(iii)]. If $\operatorname{Ker} a$ is not a subalgebra of L then the ideal $\mathcal{I}$ of V is nonzero. Therefore, it has zero annihilator, thus (Ψ_1, Ψ_{-1}) is a monomorphism of Jordan pairs between V and the special Jordan pair

$$(\operatorname{Hom}(\mathcal{F}_1/\mathcal{F}_0,\ \mathcal{F}_{-1}/\mathcal{F}_{-2}), \operatorname{Hom}(\mathcal{F}_{-1}/\mathcal{F}_{-2},\ \mathcal{F}_1/\mathcal{F}_0))^{(+)},$$

i.e., V is special.

As we have mentioned in 2.2, the Jordan algebra L_a can be regarded as the a-homotope of the Jordan pair V at a. Thus the speciality of V implies that of L_a. Moreover, L_a is strongly prime by [8, 2.2]. □

3. Speciality of subquotients of a Lie algebra

In this section we will deal with subquotients associated to abelian inner ideals B of L. Recall that a Φ-module B of L is an abelian inner ideal if $[B,[B,L]]\subset B$ and $[B,B]=0$. The kernel of an abelian inner ideal is

$$\operatorname{Ker} B=\{x\in L \mid [B,[B,x]]=0\}.$$

Associated to an abelian inner ideal B of L we can consider the subquotient $(B, L/\operatorname{Ker} B)$, which is a Jordan pair with products

$$\{b_1,\bar{x},b_2\}=[[b_1,x],b_2] \qquad \{\bar{x},b_1,\bar{y}\}=\overline{[[x,b_1],y]}$$

for every $b_1,b_2\in B$ and every $\bar{x},\bar{y}\in L/\operatorname{Ker} B$, see [13, 3.2].

In the following proposition we define a specialization homomorphism.

Proposition 3.1. *Let L be a Lie algebra over a ring of scalars Φ and let B be an abelian inner ideal of L. Let us consider the Φ-submodules*

$$V_{-1}:=[B,\operatorname{Ker} B]/A_1 \quad \textit{where} \quad A_1:=[[B,\operatorname{Ker} B],[B,\operatorname{Ker} B]]+[B,[B,L]]$$
$$V_1:=\operatorname{Ker} B/A_2 \quad \textit{where} \quad A_2:=[[B,\operatorname{Ker} B],\operatorname{Ker} B]+[B,L]$$

Then the pair of maps (Ψ_{-1},Ψ_1) defined by

$$\begin{array}{rccc}\Psi_{-1}: & B & \to & \operatorname{Hom}(V_1,V_{-1})\\ & b & \mapsto & \Psi_b\end{array} \quad \textit{where} \quad \begin{array}{rccc}\Psi_b: & V_1 & \to & V_{-1}\\ & \overline{z} & \mapsto & \overline{[b,z]}\end{array}$$

$$\begin{array}{rccc}\Psi_1: & L/\operatorname{Ker} B & \to & \operatorname{Hom}(V_{-1},V_1)\\ & \overline{a} & \mapsto & \Psi_{\overline{a}}\end{array} \quad \textit{where} \quad \begin{array}{rccc}\Psi_{\overline{a}}: & V_{-1} & \to & V_1\\ & \overline{w} & \mapsto & \overline{[a,w]}\end{array}$$

where $\bar{a}=a+\operatorname{Ker} B$, $\bar{z}=z+A_2$ and $\bar{w}=w+A_1$, is a homomorphism of Jordan pairs between

$$(B,L/\operatorname{Ker} B) \quad \textit{and} \quad (\operatorname{Hom}(V_1,V_{-1}),\operatorname{Hom}(V_{-1},V_1))^{(+)}.$$

Proof. Let us first see that V_{-1} and V_1 are well defined:
• $A_1\subset[B,\operatorname{Ker} B]$: for any $b,b'\in B$ and any $z,z'\in\operatorname{Ker} B$, $[[b,z],[b',z']]=[b,[z,[b',z']]]\in[B,\operatorname{Ker} B]$. Moreover, $[B,[B,L]]\subset[B,\operatorname{Ker} B]$ because $[B,L]\subset\operatorname{Ker} B$.
• $A_2\subset\operatorname{Ker} B$: for any $b\in B$ and $z,z'\in\operatorname{Ker} B$ we have that $[[b,z'],z]\in\operatorname{Ker} B$ because, by the Jacoby identity, for any $b',b''\in B$ we have $[b',[b'',[[b,z'],z]]]=0$. Moreover, $[B,L]\subset\operatorname{Ker} B$.

Now we will show that the maps Ψ_{-1} and Ψ_1 are well defined:
(1). Let us prove that for any $b\in B$ the linear map $\Psi_b: V_1\to V_{-1}$ is well defined: Let $\overline{z}\in V_1$, then $[b,z]\in[B,\operatorname{Ker} B]$ and, by above, if $z\in A_2$, $[b,z]\in A_1$.
(2). Let us prove that for any $\overline{a}\in L/\operatorname{Ker} B$ the linear map $\Psi_{\overline{a}}: V_{-1}\to V_1$ does not depend on the representative element of the equivalence class $\overline{a}$ and is well defined: Let $\overline{w}\in V_{-1}$, then $[a,w]\in\operatorname{Ker} B$, because $[L,[B,\operatorname{Ker} B]]\subset\operatorname{Ker} B$ and,

- If $a\in\operatorname{Ker} B$, $[a,w]\in[\operatorname{Ker} B,[B,\operatorname{Ker} B]]\subset A_2$.
- If $w\in A_1$ and $a\in L$,
  - If $w\in[[B,\operatorname{Ker} B],[B,\operatorname{Ker} B]]$,

$$[a,[[B,\operatorname{Ker} B],[B,\operatorname{Ker} B]]]\subset[[a,[B,\operatorname{Ker} B]],[B,\operatorname{Ker} B]]\subset A_2.$$

  - If $w\in[B,[B,L]]$,

$$[a,[B,[B,L]]]\subset[B,L]\subset A_2.$$

Now, let us prove that the map (Ψ_{-1}, Ψ_1) is a homomorphism of Jordan pairs. Let us consider $b, b' \in B$, $a, c \in L$, $z \in \operatorname{Ker} B$ and $w \in [B, \operatorname{Ker} B]$, and let us denote $\bar{a} = a + \operatorname{Ker} B$, $\bar{c} = c + \operatorname{Ker} B \in L/\operatorname{Ker} B$, $\bar{z} = z + A_2 \in \operatorname{Ker} B/A_2$, and $\bar{w} = w + A_1 \in [B, \operatorname{Ker} B]/A_1$:

$$\begin{aligned}
\Psi_{[b,[\overline{a},b']]}(\overline{z}) &= \overline{[[b,[a,b']],z]} = \overline{[[b,z],[a,b']]} + \overline{[b,[[a,b'],z]]} \\
&= \overline{[[[b,z],a],b']} + \overline{[a,[[b,z],b']]} + \overline{[b,[[a,z],b']]} + \overline{[b,[a,[b',z]]]} \\
&=_{(1)} \overline{[b',[a,[b,z]]]} + \overline{0} - \overline{[b,[b',[a,z]]]} + \overline{[b,[a,[b',z]]]} \\
&=_{(2)} \overline{[b',[a,[b,z]]]} + \overline{0} + \overline{[b,[a,[b',z]]]} = \{\Psi_b, \Psi_{\overline{a}}, \Psi_{b'}\}(\overline{z}) \\
\Psi_{[\overline{a},[b,\overline{c}]]}(\overline{w}) &= \overline{[[a,[b,c]],w]} = \overline{[[a,w],[b,c]]} + \overline{[a,[[b,c],w]]} \\
&= \overline{[[[a,w],b],c]} + \overline{[b,[[a,w],c]]} + \overline{[a,[[b,w],c]]} + \overline{[a,[b,[c,w]]]} \\
&= \overline{[c,[b,[a,w]]]} - \overline{[b,[c,[a,w]]]} - \overline{[a,[c,[b,w]]]} + \overline{[a,[b,[c,w]]]} \\
&=_{(3)} \overline{[c,[b,[a,w]]]} + \overline{0} - \overline{[a,[c,[b,w]]]} + \overline{[a,[b,[c,w]]]} \\
&=_{(4)} \overline{[c,[b,[a,w]]]} + \overline{0} + \overline{[a,[b,[c,w]]]} = \{\Psi_{\overline{a}}, \Psi_b, \Psi_{\overline{c}}\}(\overline{w})
\end{aligned}$$

where $_{(1)}$ holds because $[[b,z],b'] = 0$, $_{(2)}$ is true since $[b,[b',[a,z]]] \in [B,[B,L]]$, we have $_{(3)}$ because $[b,[c,[a,w]]] \in [B,L]$, and $_{(4)}$ is due to $[b,w] \in [B,[B,\operatorname{Ker} B]] = 0$. □

Lemma 3.2. *The pair*

$$\mathcal{I} = (\mathcal{I}^+, \mathcal{I}^-) = ([B,[B,[\operatorname{Ker} B, \operatorname{Ker} B]]], \overline{[\operatorname{Ker} B, \operatorname{Ker} B]})$$

is an ideal of the Jordan pair $V = (V^+, V^-) = (B, L/\operatorname{Ker} B)$.

Proof. This proof is analogous to the first part of the proof of Proposition 2.4, where we showed that $\mathcal{I} = ([a,[a,[\operatorname{Ker} a, \operatorname{Ker} a]]], ([\operatorname{Ker} a, \operatorname{Ker} a] + \operatorname{Ker} a)/\operatorname{Ker} a)$ is an ideal of the Jordan pair $V = (\Phi a + [a,[a,L]], L/\operatorname{Ker} a)$. We include it here for the sake of completeness.

Set $K = \operatorname{Ker} B$.

- $\{\mathcal{I}^+, V^-, V^+\} \subset \mathcal{I}^+$: for any $b_1, b_2, b_3 \in B$, $k_1, k_2 \in K$, $\bar{x} \in L/K$,

$$\begin{aligned}
\{[b_1,&[b_2,[k_1,k_2]]], \bar{x}, b_3\} = [[b_1,[b_2,[k_1,k_2]]],x],b_3] = [[b_1,[b_2,[k_1,k_2]]],[x,b_3]] = \\
&= [[b_1,[x,b_3]],[b_2,[k_1,k_2]]] + [b_1,[[b_2,[x,b_3]],[k_1,k_2]]] + \\
&+ [b_1,[b_2,[[k_1,k_2],[x,b_3]]]] \in [B,[B,[K,K]]
\end{aligned}$$

because $[[k_1,k_2],[x,b_3]] = [[k_1,[x,b_3]],k_2] + [k_1,[k_2,[x,b_3]]] \in [K,K]$.

- $\{V^-, \mathcal{I}^+, V^-\} \subset \mathcal{I}^-$: Let $b_1, b_2 \in B$, $k_1, k_2 \in K$, $\bar{x}, \bar{y} \in L/K$

$$\begin{aligned}
\{\bar{x},&[b_1,[b_2,[k_1,k_2]]], \bar{y}\} = \overline{[[x,[b_1,[b_2,[k_1,k_2]]]],y]} \in \overline{[L,[L,[B,[B,[K,K]]]]]} \subset \\
&\subset \overline{[L,[[L,B],[B,[K,K]]]]} + \overline{[L,[B,[[L,B],[K,K]]} \subset \\
&\subset \overline{[L,[B,[K,K]]} + \overline{[L,[B,[[L,B],[K,K]]]]} + \overline{[[L,B],[[L,B],[K,K]]]} + \overline{[B,L]} \subset \\
&\subset \overline{[[L,B],[K,K]]} + \overline{[B,L]} + \overline{[[L,B],[[L,B],[K,K]]]} \subset \overline{[K,K]}
\end{aligned}$$

since $[[L,B],[K,K]] \subset [K,K]$ and $\overline{[B,L]} = \bar{0}$ because B is an abelian inner ideal.

- $\{\mathcal{I}^-, V^+, V^-\} \subset \mathcal{I}^-$: Let $b_1 \in B$, $k_1, k_2 \in K$, $\bar{x} \in L/K$

$$\{\overline{[k_1,k_2]}, b_1, \bar{x}\} = \overline{[[[k_1,k_2],b_1],x]} = \overline{[[k_1,k_2],[b_1,x]]} + \overline{[[[k_1,k_2],x],b_1]} \in$$
$$\in \overline{[[K,K],[B,L]]} \subset \overline{[K,K]} \text{ using again that } \overline{[B,L]} = \bar{0}.$$

- $\{V^+, \mathcal{I}^-, V^+\} \subset \mathcal{I}^+$: Let $b_1, b_2 \in B$, $k_1, k_2 \in K$,

$$\{b_1, \overline{[k_1,k_2]}, b_2\} = [[b_1,[k_1,k_2]],b_2] \in [B,[B,[K,K]]].$$

□

In the following proposition we will consider powers of Jordan ideals of the Jordan pair $V = (B, L/\operatorname{Ker} B)$. For any Jordan ideal $\mathcal{J} = (\mathcal{J}^+, \mathcal{J}^-)$ of V we will define

$$\mathcal{J}^3 = ((\mathcal{J}^3)^+, (\mathcal{J}^3)^-) = (\{\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}^+\}, \{\mathcal{J}^-, \mathcal{J}^+, \mathcal{J}^-\}),$$

and $\mathcal{J}^{3^{m+1}} = ((\mathcal{J}^{3^{m+1}})^+, (\mathcal{J}^{3^{m+1}})^-)$ with

$$(\mathcal{J}^{3^{m+1}})^+ = \{(\mathcal{J}^{3^m})^+, (\mathcal{J}^{3^m})^-, (\mathcal{J}^{3^m})^+\}$$
$$(\mathcal{J}^{3^{m+1}})^- = \{(\mathcal{J}^{3^m})^-, (\mathcal{J}^{3^m})^+, (\mathcal{J}^{3^m})^-\}$$

for $m \geq 1$.

Proposition 3.3. *Consider the ideal $\mathcal{J} = (\mathcal{J}^+, \mathcal{J}^-) = (\operatorname{Ker}\Psi_{-1}, \operatorname{Ker}\Psi_1)$ of $V = (B, L/\operatorname{Ker} B)$. Then*

$$(\mathcal{J}^{3^m})^+ \subset \{b \in B \mid [b, \operatorname{Ker} B] \subset [B, \operatorname{Ker} B]^{m+2} + B\}.$$

Moreover, if L is nondegenerate and $[B, \operatorname{Ker} B]$ is nilpotent with $[B, \operatorname{Ker} B]^n = 0$, then

$$(\mathcal{J}^{3^m})^+ \subset \operatorname{Ann}_V(\mathcal{I})^+ \text{ for } m+2 \geq \tfrac{n}{2},$$

where $\mathcal{I} = ([B,[B,[\operatorname{Ker} B, \operatorname{Ker} B]]], \overline{[\operatorname{Ker} B, \operatorname{Ker} B]})$.

Proof. Let us denote $K = \operatorname{Ker} B$. We will prove by induction on $m \geq 0$ that $(\mathcal{J}^{3^m})^+ \subset \{b \in B \mid [b,K] \subset [B,K]^{m+2} + B\}$:
- If $m = 0$ then $\mathcal{J}^+ = \operatorname{Ker}\Psi_{-1}$ and every $b \in \operatorname{Ker}\Psi_{-1}$ satisfies $[b,K] \subset [[B,K],[B,K]] + [B,[B,L]] \subset [B,K]^2 + B$.
- Now suppose that it is true for $m-1$ and take $b_1, b_2 \in (\mathcal{J}^{3^{m-1}})^+$ and $\bar{a} \in (\mathcal{J}^{3^m})^-$. By hypothesis $[b_1,K] + [b_2,K] \subset [B,K]^{m+1} + B$. We will show that $b_3 = \{b_1, \bar{a}, b_2\} \in \{b \in B \mid [b,K] \subset [B,K]^{m+2} + B\}$: for every $k \in K$,

$$[b_3,k] = [[[b_1,a],b_2],k] = [[b_1,k],[a,b_2]] + [b_1,[[a,k],b_2]] + [[b_1,a],[b_2,k]] \in$$
$$\in [[b_1,k],[a,b_2]] + [[b_1,a],[b_2,k]] + B \subset$$
$$\subset [[B,K]^{m+1},[a,b_2]] + [[b_1,a],[B,K]^{m+1}] + B \subset$$
$$\subset [[[B,K]^{m+1},a],b_2] + [b_1,[a,[B,K]^{m+1}]] + B \subset$$
$$\subset^{(1)} [[[B,K],K],b_2] + [b_1,[[B,K],K]] + B \subset$$
$$\subset [[B,K],[K,b_2]] + [[B,K],[b_1,K]] + B \subset$$
$$\subset [B,K]^{m+2} + B$$

where $^{(1)}$ holds because $[[B,K]^{m+1},L] = [[[B,K],[B,K]^m],L] \subset [K,[B,K]^m] + [[B,K],K] \subset [[B,K],K]$.

Now suppose that L is nondegenerate and that there exists n such that $[B,K]^n = 0$. Recall that V is nondegenerate by [13, 3.5(ii)]. Then if $b \in (\mathcal{J}^{3^m})^+$ for $m+2 \geq n/2$, for every $k_1, k_2 \in K$ we have that

$$[b,[b,[k_1,k_2]]] = 2[[b,k_1],[b,k_2]] \subset [[B,K]^{m+2} + B, [B,K]^{m+2} + B] \subset [B,K]^n = 0,$$

so $(\mathcal{J}^{3^m})^+ \subset \mathrm{Ann}_V(\mathcal{I})^+$ by the characterization of annihilators of ideals for nondegenerate Jordan pairs. □

Corollary 3.4. *If L is strongly prime, $[B, \operatorname{Ker} B]$ is nilpotent and $\operatorname{Ker} B$ is not a subalgebra of L, then the subquotient $(B, L/\operatorname{Ker} B)$ is a special strongly prime Jordan pair.*

Proof. The Jordan pair $(B, L/\operatorname{Ker} B)$ is strongly prime by [13, 3.5(ii)]. If $\operatorname{Ker} B$ is not a subalgebra of L, the ideal

$$\mathcal{I} = ([B,[B,[\operatorname{Ker} B, \operatorname{Ker} B]]], \overline{[\operatorname{Ker} B, \operatorname{Ker} B]})$$

of V is nonzero. Thus the annihilator of $\mathcal{I}$ in V is zero, hence

$$\mathcal{J} = (\operatorname{Ker} \Psi_{-1}, \operatorname{Ker} \Psi_1) = 0$$

by 3.3, i.e., the homomorphism (Ψ_{-1}, Ψ_1) is a monomorphism of Jordan pairs, proving that $(B, L/\operatorname{Ker} B)$ is a special Jordan pair. □

The nilpotency of $[B, \operatorname{Ker} B]$ given in Corollary 3.4 is not a too restrictive condition. Indeed, if $L = R^-$ for a prime ring, or if L is finite dimensional, this is always the case for any inner ideal B of L, as can be seen in the following results.

Proposition 3.5. *Let R be a prime ring, $\hat{R}$ the central closure of R and let B be an abelian inner ideal of R^-. Then*

(a) *For every $b \in B$ there exists a unique λ_b in the extended centroid $C(R)$ of R such that $(b - \lambda_b)^2 = 0$.*
(b) *$B' := \{b - \lambda_b \mid b \in B\}$ is an abelian inner ideal of $\hat{R}^-$ such that $B'B' = \{0\}$.*
(c) *$B^* := B + B'RB'$ is an abelian inner ideal of $\hat{R}^-$ such that* $\operatorname{Ker} B = \operatorname{Ker} B^*$ *and $B' \operatorname{Ker} B B' = 0$.*
(d) *$[[B^*, \operatorname{Ker} B], [B^*, \operatorname{Ker} B]] \subset B^*$ so $[B^*, \operatorname{Ker} B]$ and $[B, \operatorname{Ker} B]$ are nilpotent of index less than or equal to 3.*
(e) *The chain*

$$\mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$$

given by

$$\mathcal{F}_{-2} = B^*, \quad \mathcal{F}_{-1} = [B^*, \operatorname{Ker} B] + B^*,$$
$$\mathcal{F}_0 = [B^*, \hat{R}] + [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]], \quad \mathcal{F}_1 = \operatorname{Ker} B, \quad \mathcal{F}_2 = \hat{R}$$

is a filtration of $\hat{R}^-$.

Proof. (a) is [3, Theorem 3.2].

(b) For any $b \in B$, let us denote by $b' := b - \lambda_b$ where λ_b is the unique element of $C(R)$ given in (a). It is clear that $B' = \{b - \lambda_b \mid b \in B\}$ satisfies $[B',[B',\hat{R}]] \subset B'$. Let us see that $B'B' = 0$: For every $b, c \in B$, and $x \in R$ we have that

$$0 = [b,c] = [b',c'] = b'c' - c'b'$$

and

$$0 = [c,[b,[b,x]]] = -2[c', b'xb'] = -2(c'b'xb' - b'xb'c').$$

Therefore, by [3, Corollary 2.14], there exists $\lambda_1 \in C(R)$ such that $c'b' = \lambda_1 b'$. Arguing similarly, there exists $\lambda_2 \in C(R)$ such that $b'c' = \lambda_2 c'$. Then $0 = \lambda_1 b'b' = b'c'b' = \lambda_2 c'b'$, and since $C(R)$ is a field, $c'b' = b'c' = 0$.

Finally, B' is a submodule of $\hat{R}$ because for every $b, c \in B$, $(b + c - \lambda_b - \lambda_c)^2 = (b' + c')^2 = b'^2 + c'^2 + 2b'c' = 0$, so $b' + c' = (b + c)' \in B'$.

(c) For every $b, c \in B$, $b'_1, b'_2, c'_1, c'_2 \in B'$ and $x, y, u \in R$ we have that

$$[b + b'_1xb'_2, [c + c'_1yc'_2, u]] = [b' + b'_1xb'_2, [c' + c'_1yc'_2, u]] = -(b' + b'_1xb'_2)u(c' + c'_1yc'_2) - (c' + c'_1yc'_2)u(b' + b'_1xb'_2) \in B'RB'$$

Moreover, if $z \in \operatorname{Ker} B$,

$$0 = [b, [c, z]] = [b', [c', z]] = -b'zc' - c'zb'$$

so

$$b'zc' = -c'zb' \qquad (*)$$

and

$$0 = -\frac{1}{2}[c, [[b, [b, x]], z]] = -\frac{1}{2}[c', [[b', [b', x]], z]] = [c', [b'xb', z]] = = -c'zb'xb' - b'xb'zc' = b'zc'xb' + b'xc'zb' \quad \text{(by } (*)\text{)},$$

so $b'zc'xb' = -b'xc'zb'$ for every $x \in R$. By [3, Corollary 2.14], there exists $\mu_1 \in C(R)$ such that $b'zc' = \mu_1 b'$, and if we change b' by c' in the above argument, there exists μ_2 with $b'zc' = -c'zb' = \mu_2 c'$. If $\{b', c'\}$ are linearly independent over $C(R)$, $b'zc' = 0$, but if $c' = \alpha b'$ for some $\alpha \in C(R)$ then

$$0 = \alpha[b, [b, z]] = \alpha[b', [b', z]] = \alpha(-2b'zb') = -2(\alpha b')zb' = -2c'zb' = 2b'zc'$$

giving again $b'zc' = 0$. Now, if $z \in \operatorname{Ker} B$, for every $b, c \in B$, $b'_1, b'_2, c'_1, c'_2 \in B'$ we have that

$$[b + b'_1xb'_2, [c + c'_1yc'_2, z]] = [b' + b'_1xb'_2, [c' + c'_1yc'_2, z]] = 0$$

and $\operatorname{Ker} B \subset \operatorname{Ker} B^*$. The containment $\operatorname{Ker} B^* \subset \operatorname{Ker} B$ is trivial.

(d) $[B^*, [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]]] = [[B^*, \operatorname{Ker} B], [B^*, \operatorname{Ker} B]]$. Take $b, c \in B^*$ and $u, v \in \operatorname{Ker} B$. By (a) for every $b \in B^*$ there exists $\lambda_b \in C(R)$ such that $b - \lambda_b \in B' + B'RB'$; let us denoted it by b'. Then

1. $[[b, u], [c, u]] = [[b', u], [c', u]] = b'uc'u - b'u^2c' - u'bc'u + ub'uc' - c'ub'u + c'u^2b' + uc'b'u - uc'ub' = b'(-u^2)c' + c'u^2b' \in B^*$, because $B'B' = 0$ by (b) and $B' \operatorname{Ker} BB' = 0$ by (c).
2. $2[[b, u], [b, v]] = [b, [b, [u, v]]] \in B^*$.
3. $2([[b, u], [c, v]] + [[c, u], [b, v]]) = [b+c, [b+c, [u, v]]] - 2[b, [b, [u, v]]] - 2[c, [c, [u, v]]] \in B^*$ by (2). So $[[b, u], [c, v]] - [[b, v], [c, u]] = [[b, u], [c, v]] + [[c, u], [b, v]] \in B^*$.
4. $[[b, u], [c, v]] + [[b, v], [c, u]] = [[b, u+v], [c, u+v]] - [[b, u], [c, u]] - [[b, v], [c, v]] \in B^*$ using (1).
5. $[[b, u], [c, v]] - [[b, v], [c, u]] \in B^*$ and $[[b, u], [c, v]] + [[b, v], [c, u]] \in B^*$, so $[[b, u], [c, v]] \in B^*$, i.e., $[[B^*, \operatorname{Ker} B], [B^*, \operatorname{Ker} B]] \subset B^*$.

Therefore

$$[[B^*, \operatorname{Ker} B], [[B^*, \operatorname{Ker} B], [B^*, \operatorname{Ker} B]]] \subset [[B^*, \operatorname{Ker} B], B^*] = 0.$$

(e) Let us prove that the chain

$$\mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$$

given by

$$\mathcal{F}_{-2} = B^*, \quad \mathcal{F}_{-1} = [B^*, \operatorname{Ker} B] + B^*,$$

$$\mathcal{F}_0 = [B^*, \hat{R}] + [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]], \quad \mathcal{F}_1 = \operatorname{Ker} B, \quad \mathcal{F}_2 = \hat{R}$$

is a filtration:

(1) $[\mathcal{F}_{-2}, \mathcal{F}_{-2}] = [B^*, B^*] = 0$, because B^* is an abelian inner ideal of $\hat{R}^-$.
(2) $[\mathcal{F}_{-2}, \mathcal{F}_{-1}] = [B^*, [B^*, \operatorname{Ker} B]] = 0$ because $\operatorname{Ker} B = \operatorname{Ker} B^*$.
(3) $[\mathcal{F}_{-2}, \mathcal{F}_0] = [B^*, [B^*, \hat{R}] + [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]]] \subset B^*$ by (d).
(4) $[\mathcal{F}_{-2}, \mathcal{F}_1] \subset \mathcal{F}_{-1}$ and $[\mathcal{F}_{-2}, \mathcal{F}_2] \subset \mathcal{F}_0$ follow by definition of $\mathcal{F}_{-1}$ and $\mathcal{F}_0$.
(5) $[\mathcal{F}_{-1}, \mathcal{F}_{-1}] = [[B^*, \operatorname{Ker} B] + B^*, [B^*, \operatorname{Ker} B] + B^*] \subset B^* = \mathcal{F}_{-2}$ by (d).
(6) $[\mathcal{F}_{-1}, \mathcal{F}_0] \subset \mathcal{F}_{-1}$:

- $[[B^*, [B^*, \hat{R}]] \subset B^*$
- $[B^*, [\operatorname{Ker} B, [\operatorname{Ker} B, B^*]]] \subset B^*$ by (d)
- $[[B^*, \operatorname{Ker} B], [B^*, \hat{R}]] \subset [B^*, \operatorname{Ker} B]$.
- $[[B^*, \operatorname{Ker} B], [[B^*, \operatorname{Ker} B], [B^*, \operatorname{Ker} B]]] = 0$ implies $[\operatorname{Ker} B, [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]]] \subset \operatorname{Ker} B$, so

$$\begin{aligned}[][[B^*, \operatorname{Ker} B], &[\operatorname{Ker} B, [B^*, \operatorname{Ker} B]]] \subset [B^*, [\operatorname{Ker} B, [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]]]] \\ &+ [[B^*, [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]]], \operatorname{Ker} B] \subset [B^*, \operatorname{Ker} B]\end{aligned}$$

(7) $[\mathcal{F}_{-1}, \mathcal{F}_1] = [[B^*, \operatorname{Ker} B] + B^*, \operatorname{Ker} B] \subset [B^*, R] + [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]] = \mathcal{F}_0$.
(8) $[\mathcal{F}_{-1}, \mathcal{F}_2] = [[B^*, \operatorname{Ker} B] + B^*, R] \subset \operatorname{Ker} B$.
(9) $[\mathcal{F}_0, \mathcal{F}_0] \subset \mathcal{F}_0$:

$$\begin{aligned}&[[B^*, \hat{R}] + [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]],\ [B^*, \hat{R}] + [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]]] \subset \\ &\subset [[B^*, \hat{R}], [B^*, \hat{R}]] + [[B^*, \hat{R}], [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]]] + \\ &+ [[\operatorname{Ker} B, [B^*, \operatorname{Ker} B]], [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]]] \subset \\ &\subset [B^*, \hat{R}] + [[[B^*, \hat{R}], \operatorname{Ker} B], [B^*, \operatorname{Ker} B]] + [\operatorname{Ker} B, [[B^*, \hat{R}], [B^*, \operatorname{Ker} B]] + \\ &+ [[\operatorname{Ker} B, [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]]], [B^*, \operatorname{Ker} B]] + \\ &+ [\operatorname{Ker} B, [[B^*, \operatorname{Ker} B], [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]]]] \subset \\ &\subset [B^*, \hat{R}] + [\operatorname{Ker} B, [\operatorname{Ker} B, B^*]] = \mathcal{F}_0.\end{aligned}$$

(10) $[\mathcal{F}_0, \mathcal{F}_1] = [[B^*, \hat{R}] + [\operatorname{Ker} B, [B^*, \operatorname{Ker} B]], \operatorname{Ker} B] \subset \operatorname{Ker} B = \mathcal{F}_1$.
(11) $[\mathcal{F}_0, \mathcal{F}_2] \subset \mathcal{F}_2$, $[\mathcal{F}_1, \mathcal{F}_1] \subset \mathcal{F}_2$ and $[\mathcal{F}_1, \mathcal{F}_2] \subset \mathcal{F}_2$ because $\mathcal{F}_2 = \hat{R}$. □

The next lemma is a consequence of [6, Theorem 5.4]. We will show that in the context of centrally closed prime rings, every abelian inner ideal $\mathcal{B}$ of the quotient $R/Z(R)$ comes from an abelian inner ideal B' of R^- satisfying $B'B' = 0$.

Lemma 3.6. *Let R be a centrally closed prime associative ring and let $\mathcal{B}$ be an abelian inner ideal of $R/Z(R)$. Then there exists an abelian inner ideal B' of R^- such that $B'B' = 0$ and $\pi(B') = \mathcal{B}$, where $\pi : R^- \to R^-/Z(R)$ denotes the canonical projection. In particular, $B' \cap Z(R) = 0$.*

Proof. Let us prove that $B := \pi^{-1}(\mathcal{B})$ is an inner ideal of R^-: $\pi([B, [B, R]]) \subset [\mathcal{B}, [\mathcal{B}, R/Z(R)]] \subset \mathcal{B}$, so $[B, [B, R]] \subset B$. Moreover, B is abelian: if $a \in B$, $\mathrm{ad}_a^3(R) \subset Z(R)$, and by [3, Lemma 3.1] $\mathrm{ad}_a^3(R) = 0$; now for any $b \in B$, $[a, b] \in Z(R)$ hence $0 = \mathrm{ad}_a^3(b^3) = 6[a, b]^3$, which implies, by primeness of R, that $[a, b] = 0$.

Finally, let B' be the abelian inner ideal of R^- associated to B, given in Proposition 3.5(b), for which we also have $\pi(B') = \mathcal{B}$. It is clear that $B'B' = 0$ implies $B' \cap Z(R) = 0$. □

Corollary 3.7. *Let R be a prime ring and let B be an abelian inner ideal of the strongly prime Lie algebra $L := R^-/Z(R)$. Then if $\operatorname{Ker} B$ is not a subalgebra of L, the subquotient $(B, L/\operatorname{Ker} B)$ is a strongly prime special Jordan pair.*

Proof. The Lie algebra L is a strongly prime by [9, Lemma 4.2] and the subquotient $(B, L/\operatorname{Ker} B)$ is strongly prime by [13, 3.5(ii)]. Let us consider the central closure $\hat{R}$ of R, $\hat{L} := \hat{R}^-/Z(\hat{R})$, and let $\hat{B}$ be the scalar extension of B on $\hat{L}$. By 3.6, we can suppose that $\hat{B} = \pi(B')$ where B' is an abelian inner ideal of $\hat{R}^-$ with $B'B' = 0$ and where π denotes the canonical projection of $\hat{R}$ onto $\hat{R}/Z(\hat{R})$. By Proposition 3.5(d) we have that $[B', \operatorname{Ker} B']$ is a nilpotent subalgebra of $\hat{R}^-$ and therefore $\pi([B', \operatorname{Ker} B']) = [\hat{B}, \operatorname{Ker} \hat{B}]$ is a nilpotent subalgebra of $\hat{L}$. Then Corollary 3.4 applies and $(\hat{B}, \hat{L}/\operatorname{Ker} \hat{B})$ is special, so also $(B, L/\operatorname{Ker} B)$ is special. □

3.8. Remark. Let L be a nondegenerate Lie algebra with $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \in \Phi$. For every nonzero abelian inner ideal B of finite length of L there exists a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ such that $B = L_n$ [13, Corollary 6.2]. This is always the case when L is nodegenerate finite dimensional. With respect to this grading, $\operatorname{Ker} B = L_{-n+1} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ and $[B, \operatorname{Ker} B] \subset L_1 \oplus \cdots \oplus L_n$ is nilpotent.

When the grading is a 3-grading, $L = L_{-n} \oplus L_0 \oplus L_n$, $\operatorname{Ker} B$ is always a subalgebra. In this situation Corollary 3.4 gives no information about the speciality of the subquotient $(B, L/\operatorname{Ker} B) \cong (L_{-n}, L_n)$. If we review the list of abelian inner ideals of the simple finite dimensional Lie algebras over an algebraically closed field of characteristic zero given in [4], only when $L = E_6$ or E_7 one finds exceptional subquotients: if L is E_6, the subquotient associated to the abelian inner ideal $B_{\{1\}}$ (with the notation of [4]) is isomorphic to the exceptional BiCayley pair; if L is E_7, the subquotient associated to the abelian inner ideal $B_{\{7\}}$ (with the notation of [4]) is isomorphic to the exceptional Albert pair. The rest of subquotients of abelian inner ideals of the simple finite dimensional Lie algebras over an algebraically closed field of characteristic zero are all special.

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