GRAPH ALGEBRAS AND THE GELFAND-KIRILLOV DIMENSION

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ABSTRACT. We study some properties of the Gelfand-Kirillov dimension in a non-necessarily unital context. In particular, its Morita invariance when the algebras have local units, and its commutativity with direct limits. We then give some applications in the context of graph algebras, which embraces, among some others, path algebras and Cohn and Leavitt path algebras. In particular we determine the GK-dimension of these algebras in full generality, so extending the main result in [6].

1. INTRODUCTION AND PRELIMINARY RESULTS

In this paper, we study two general facts concerning the Gelfand-Kirillov dimension for non-necessarily unital algebras. The first one is the fact that the Gelfand-Kirillov dimension is a Morita invariant property for algebras with local units (Theorem 2.6). Although this property is known to hold for unital algebras (as seen, for example in [19, (iii) of Proposition 2.9]), the case of algebras with local units was lacking in the literature. Moreover, the proof of this fact does not follow straightforward from the unital case, nor from the passage through the unitization of the algebra. So we paved our own way. Our proof is also applicable in the unital context, giving an alternative insight to the usual one in the literature. We emphasize that there exist Morita invariant properties in the unital context that no longer hold in the non-unital one (see Section 2 for more details). The second fact concerning the Gelfand-Kirillov dimension is that it commutes with direct limits of algebras (Theorem 3.1). This will let us to obtain some interesting consequences in the context of graph algebras (a common name under which we embrace classical path algebras, Cohn and Leavitt path algebras and relative Cohn path algebras), as we will be able to determine the Gelfand-Kirillov dimension of graph algebras associated to arbitrary graphs. In particular, we extend the main result in [6, Theorem 5], which characterizes the GK-dimension of Leavitt path algebras over finite graphs.

In the first section, we introduce the basic definitions and results concerning the Gelfand-Kirillov dimension and the Morita theory that we will need. In Section 2, we prove the Morita invariance of the Gelfand-Kirillov dimension and give an application in the context of Leavitt path algebras. The third section deals with the commutativity of the Gelfand-Kirillov dimension with direct limits and we give again applications in the context of graph algebras. It is in this last section where we have chosen to include the definition of these algebras, in order not to mess up the exposition.

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In this paper, rings and algebras are not assumed to be unital unless otherwise stated. In general, A will denote an algebra over a field K and R will denote a ring. We will be concerned with the Gelfand-Kirillov dimension, defined as follows. Given a field K and a finitely generated K-algebra A, the Gelfand-Kirillov dimension of A (GK-dimension) is defined to be

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \frac{\log(\dim V^n)}{\log n},$$

where V is a finite dimensional subspace of A that generates A as an algebra over K. This definition is independent of the choice of V. If A does not happen to be finitely generated over K, the GK-dimension of A is defined to be

$$\mathrm{GKdim}(A) = \sup\{\mathrm{GKdim}(B) \mid B \text{ is a finitely generated subalgebra of } A\}.$$

We will prove and make use of the fact that, if A is an algebra and A^1 denotes its unitization, then $\mathrm{GKdim}(A) = \mathrm{GKdim}(A^1)$ (this is Proposition 2.2). Also, we will use the known fact that the inequality $\mathrm{GKdim}(A) \leq \mathrm{GKdim}(B)$ holds whenever A is a subalgebra of B, or if B is a factor algebra of A. See [16] for a general treatment of the GK-dimension. We stress here that the usual construction of the unitization of a ring R (as can be found for example in [15, Section 2.17]) consists in taking $R\oplus\mathbb{Z}$ and enlarging the product conveniently. In this paper, we consider a slightly different construction, which we denote by $A \oplus K$ and consists in the following: when $\mathrm{Char}(K) = 0$, we are considering $(A \oplus \mathbb{Z}) \otimes K \cong A \oplus K$, and if $\mathrm{Char}(K) = p > 0$ we are considering $(A \oplus \mathbb{Z}_p) \otimes K \cong A \oplus K$, where $\mathbb{Z}_p$ is the prime field of K. In other words, we are considering the natural scalar extension of the usual unitization in both cases.

We recall here some of the main aspects of Morita equivalence for idempotent rings. A ring is said to be *idempotent* if $R^2 = R$, and an algebra A is said to be *idempotent* if the underlying ring is an idempotent ring.

Let R and S be two rings, ${}_RN_S$ and ${}_SM_R$ two bimodules and $(-,-) : N \times M \to R$, $[-,-] : M \times N \to S$ two maps. Then the following conditions are equivalent:

(i) $\begin{pmatrix} R & N \\ M & S \end{pmatrix}$ is a ring with componentwise sum and product given by:

$$\begin{pmatrix} r_1 & n_1 \\ m_1 & s_1 \end{pmatrix}\begin{pmatrix} r_2 & n_2 \\ m_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1r_2 + (n_1,m_2) & r_1n_2 + n_1s_2 \\ m_1r_2 + s_1m_2 & [m_1,n_2] + s_1s_2 \end{pmatrix}$$

(ii) $[-,-]$ is S-bilinear and R-balanced, $(-,-)$ is R-bilinear and S-balanced and the following associativity conditions hold:

$$(n,m)n' = n[m,n'] \quad \text{and} \quad [m,n]m' = m(n,m'),$$

for all $m,m' \in M$ and $n,n' \in N$.
That $[-,-]$ is S-bilinear and R balanced and that $(-,-)$ is R-bilinear and S-balanced is equivalent to having bimodule maps $\varphi : N \otimes_S M \to R$ and $\psi : M \otimes_R N \to S$, given by

$$\varphi(n\otimes m) = (n,m) \quad \text{and} \quad \psi(m\otimes n) = [m,n]$$

so that the associativity conditions above read

$$\varphi(n \otimes m)n' = n\psi(m \otimes n') \quad \text{and} \quad \psi(m \otimes n)m' = m\varphi(n \otimes m) = (n, m').$$

A *Morita context* is a sextuple $(R, S, N, M, \varphi, \psi)$ satisfying one of the (equivalent) conditions given above. The associated ring (in condition (i)) is called the *Morita ring of the context.* By abuse of notation, we write (R, S, N, M) for a Morita context, omitting the maps φ and ψ. We will identify R, S, N and M with their natural images in the Morita ring associated to the context. The Morita context is said to be *surjective* if the maps φ and ψ are surjective.

Let R be an idempotent ring. We denote by R-Mod the full subcategory of the category of all left R-modules whose objects are the "unital" nondegenerate modules. Here, a left R-module is said to be *unital* if $RM = M$ and *nondegenerate* if $Rm = 0$ implies $m = 0$ for any fixed $m \in M$. When R is a unital ring, R-Mod is the usual category of left R-modules.

We can find in [17] that, if R and S are arbitrary rings such that there exists a surjective Morita context involving them, then the categories R-Mod and S-Mod are equivalent. For idempotent rings, the converse statement is satisfied, and a proof can be found in [13, Proposition 2.3], so we can state the following result.

Theorem 1.1. *Let R and S be idempotent rings. Then the categories R-Mod and S-Mod are equivalent if and only if there exists a surjective Morita context (R, S, N, M).*

Given two idempotent rings R and S, we will say that they are *Morita equivalent* if the categories R-Mod and S-Mod are equivalent. For any rings or algebras R and S, we will write $R \approx S$ if R and S are Morita equivalent. The following result will be key to prove the main result of Section 2, namely, Theorem 2.6.

Theorem 1.2. ([20], Corollary 1.8) *Let R and S be two idempotent rings which are Morita equivalent. Then for every idempotent $e \in R$ there exist a positive integer n and an idempotent f in $M_n(S)$ such that $eRe \cong fM_n(S)f$.*

Observe that for unital rings, we recover the classical Morita characterization theorem (as stated for example in [18, Proposition 18.33 (3)]).

2. The GK-dimension is a Morita invariant property for algebras with local units

A proof of the fact that the GK-dimension is a Morita invariant property for unital algebras can be found in [19, (iii) of Proposition 2.9]. We extend this result to include algebras with local units. The result proven here should not be taken for granted, because there exist ring-theoretic properties that are Morita invariant in the unital case that are no longer Morita invariant when we pass to the non-unital case. A good example of this situation is von Neumann regularity (see [7]). Moreover, although some results for non-unital rings can be obtained passing through the unitization, we cannot apply this technique here: a ring R and its unitization R^1 are not in general Morita equivalent. For example, if we consider R a simple ring and embed it into R^1, then R sits as a two-sided ideal in R^1, so R^1 is not simple, and being simple is a Morita invariant property (see [13, Proposition 3.6 (iii)]). We will work in the context of algebras with local units.

Definitions 2.1. A ring R has a set of *local units* F in case F is a set of idempotents such that for any finite subset $\{r_1, ..., r_n\} \subseteq R$, there exists $f \in F$ for which $fr_if = r_i$ for all $1 \leq i \leq n$, i.e., a set of idempotents $F \subseteq R$ is a set of local units for R in case each finite subset of R is contained in a (unital) subring of the form fRf for some $f \in F$. An algebra has local units when the underlying ring has local units.

Examples of rings with local units are abundant. For instance, Cohn and Leavitt path algebras are such (see [1]), and von Neumann regular rings are as well. We note that non-degenerated left modules over a ring with local units fall into the category R-Mod, as rings with local units are idempotent rings. We give a proof of the following fact, which will be used later.

Proposition 2.2. Let A be a K-algebra and denote by A^1 its unitization. Then $\mathrm{GKdim}(A) = \mathrm{GKdim}(A^1)$.

Proof. Let A be a finitely generated K-algebra, with finite dimensional generating subspace V. Consider the unitization A^1 of A. Then A^1 is finitely generated and has $V \oplus K$ as a finite dimensional generating subspace. Now

$$\dim_K(\sum_{i=1}^{n}(V \oplus K)^i) = \dim_K(\sum_{i=1}^{n} V^i \oplus K) = \dim_K(\sum_{i=1}^{n} V^i) + 1$$

shows that the growth functions of A and A^1 differ in a constant, and therefore these algebras have asymptotically equivalent growth.
The claim for not necessarily finitely generated algebras follows from the definition of the GK-dimension for these algebras. □

As Theorem 1.2 points out, to prove the Morita invariance of the GK-dimension, we need to check that it is well-behaved when we take matrices and corners (a corner of an algebra A is a unital subalgebra of the form eAe, where e is an idempotent in A). Therefore, the first step towards our goal is to prove that the GK-dimension of an algebra and of its algebra of matrices is the same.

Proposition 2.3. Let A be a K-algebra. Then $\mathrm{GKdim}(A) = \mathrm{GKdim}(M_n(A))$ for every $n \in \mathbb{N}$.

Proof. Assume that A is a unital K-algebra. Then $M_n(A) \cong A \otimes_K M_n(K)$, and applying [16, Proposition 3.12][1] we get that

$$\mathrm{GKdim}(M_n(A)) = \mathrm{GKdim}(A \otimes_K M_n(K)) = \mathrm{GKdim}(A) + \mathrm{GKdim}(M_n(K)) = \mathrm{GKdim}(A),$$

because $M_n(K)$ is finite dimensional, and therefore its GK-dimension is zero.

If A is not unital, we can embed A into its unitization A^1; moreover, we can embed $M_n(A)$ as a subalgebra of $M_n(A^1)$. The unital case and Proposition 2.2 apply, and we get that

$$\mathrm{GKdim}(A) \leq \mathrm{GKdim}(M_n(A)) \leq \mathrm{GKdim}(M_n(A^1)) = \mathrm{GKdim}(A^1) = \mathrm{GKdim}(A),$$

implying the statement. □

[1]Let C, D be unital K-algebras. If $\mathrm{GKdim}(C) \leq 2$, then $\mathrm{GKdim}(C \otimes D) = \mathrm{GKdim}(C) + \mathrm{GKdim}(D)$.

We state and prove the Morita invariance of the GK-dimension for unital algebras. Our proof, which relies on the definition of the GK-dimension and the Morita Theorem ([18, Proposition 18.33 (3)]), is not the same as the one given in [19, (iii) of Proposition 2.9]. The proof given in this book does not extend to the non-unital case as the authors use, firstly, the fact that if $R \approx S$, then $S \cong End(P_R)$, with P_R a progenerator; this cannot be straightforward extensible when the underlying rings are not necessarily unital. Nevertheless, we can see satisfactory progress in [13], precisely the discussion after Corollary 2.9, and then Theorem 2.10. The second obstacle is the usage of a proposition which explicitly identifies $End(R^n)$ with $M_n(R)$, an identification that cannot be done in the case of rings lacking an identity, as the first one is a unital ring, but the second one is not. This is why we take this different approach.

Lemma 2.4. *Let A and B be unital algebras. If $A \approx B$, then* $\mathrm{GKdim}(A) = \mathrm{GKdim}(B)$.

Proof. Because A is Morita equivalent to B, using the characterization given in [18, Proposition 18.33 (3)], we can find an integer $n \geq 1$ and a full idempotent $e \in M_n(B)$ such that $A \cong eM_n(B)e$. But $eM_n(B)e$ is a subalgebra of $M_n(B)$, therefore, by definition, $\mathrm{GKdim}(A) = \mathrm{GKdim}(eM_n(B)e) \leq \mathrm{GKdim}(M_n(B)) = \mathrm{GKdim}(B)$. By the same reasoning, because B is Morita equivalent to A, we also have that $\mathrm{GKdim}(B) \leq \mathrm{GKdim}(A)$, so we have the equality $\mathrm{GKdim}(A) = \mathrm{GKdim}(B)$. □

The following lemma is a crucial step in the proof of the main theorem in this section. We have chosen to save it outside the proof because of its intrinsic interest, and because it can be applied in more general contexts (as it is a result not only for algebras with local units, but for idempotent algebras).

Lemma 2.5. Let A and B be idempotent algebras. If $A \approx B$, then $\mathrm{GKdim}(eAe) \leq \mathrm{GKdim}(B)$ for every idempotent $e \in A$.

Proof. Let A and B be algebras under the assumed conditions. By Theorem 1.2, for an idempotent e in A we can find a positive integer n and an idempotent f in $M_n(B)$ such that $eAe \cong fM_n(B)f$. Now $\mathrm{GKdim}(eAe) = \mathrm{GKdim}(fM_n(B)f) \leq \mathrm{GKdim}(M_n(B)) = \mathrm{GKdim}(B)$ (by Proposition 2.3). □

We are now in a position to conclude the general case for algebras with local units. Given an algebra A, denote by $\mathcal{S}_{fg}(A) = \{B \subseteq A \mid B \text{ is a f.g. subalgebra of } A\}$.

Theorem 2.6. (The GK-dimension is a Morita invariant property for algebras with local units.) *Let A and B be algebras with local units. If $A \approx B$, then* $\mathrm{GKdim}(A) = \mathrm{GKdim}(B)$.

Proof. Let $A \approx B$ be algebras with local units. In particular, they are idempotent algebras and we can apply Lemma 2.5. Then, for any finitely generated subalgebra C of A we have that $\mathrm{GKdim}(C) \leq \mathrm{GKdim}(B)$. Indeed, assume that C is generated by the elements $\{a_1, ..., a_n\}$ of A. Then there exists a local unit $e \in A$ such that $a_1, ..., a_n \in eAe$, and therefore $C \subseteq eAe$. Hence $\mathrm{GKdim}(C) \leq \mathrm{GKdim}(eAe) \leq \mathrm{GKdim}(B)$. Taking the supremum over the finitely generated subalgebras of A, by the very definition of the Gelfand-Kirillov dimension we obtain that

$$\mathrm{GKdim}(A) = \sup_{C \in \mathcal{S}_{\mathrm{fg}}(A)} \{\mathrm{GKdim}(C)\} \leq \mathrm{GKdim}(B).$$

The roles of A and B are symmetric, so we conclude that $\mathrm{GKdim}(A) = \mathrm{GKdim}(B)$, as we claimed. □

It is still an open question if the same is true for more general algebras than those having local units.

We now give an application of Theorem 2.6 in the context of Leavitt path algebras. These algebras were considered for the first time in the papers [2] and [8], and they constitute by now an active area of research. We postpone its definition, as well as that of some other related algebras, for the next section, in order not to interrupt the exposition.

Recall that given an arbitrary graph E, a *desingularization* of E is a graph F in which we add an infinite tail to every sink and infinite emitter of E. This was introduced in the context of Leavitt path algebras in the paper [3], and previously in [12]. The point of this construction is that we obtain a new graph which is always row-finite, has no sinks, and we have the following theorem.

Theorem 2.7. ([3, Theorem 5.6]) *Let E be an arbitrary graph and let F be a desingularization of E. Then $L_K(E)$ and $L_K(F)$ are Morita equivalent.*

Moreover, in [5, Theorem 17] it is proven that an arbitrary graph E admits a desingularization if and only if E contains no uncountable emitters. If we combine these arguments, jointly with Theorem 2.6, we can state the following corollary.

Corollary 2.8. *Let E be an arbitrary graph such that for every infinite emitter $v \in E^0$, we have that $|s^{-1}(v)|$ is countable. Then there exists a row-finite graph F such that*

$$\mathrm{GKdim}(L_K(E)) = \mathrm{GKdim}(L_K(F)).$$

3. The GK-dimension commutes with direct limits. Applications.

In this section we prove the commutativity of the Gelfand-Kirillov dimension with direct limits of algebras, without any additional hypothesis on the algebras. Then, we pave the way towards the goal of establishing a classificatory theorem, determining the GK-dimension of graph algebras associated to arbitrary graphs. This extends the recent paper [6], where the authors consider Leavitt path algebras associated to finite graphs. In our terminology, graph algebras include classical path algebras, Leavitt and Cohn path algebras and Cohn path algebras relative to certain subsets of regular vertices of the considered graph. We give examples that highlight the relationship among the GK-dimensions of these algebras.

Theorem 3.1. The Gelfand-Kirillov dimension commutes with direct limits.

Proof. Assume that $A = \bigcup_{i\in I} A_i$ is a directed union of algebras. We have that $A_i \subseteq A$ for all $i \in I$, which implies that

$$\mathrm{GKdim}(A_i) \leq \mathrm{GKdim}(A) \quad \text{for all} \quad i \in I.$$

So,

$$\sup_{i\in I} \mathrm{GKdim}(A_i) \leq \mathrm{GKdim}(A) = \mathrm{GKdim}(\lim_{i\in I} A_i).$$

Let us see the converse inequality. By the very definition of the GK-dimension:

$$\mathrm{GKdim}(\lim_{i\in I} A_i) = \mathrm{GKdim}(A) := \sup_{B\in\mathcal{S}_{\mathrm{fg}}(A)} \{\mathrm{GKdim}(B)\} \leq \lim_{i\in I} \mathrm{GKdim}(A_i)$$

because for each finitely generated subalgebra $B \subseteq A$, there exists an index $i_B \in I$ such that $B \subseteq A_{i_B}$ and consequently $\mathrm{GKdim}(B) \leq \mathrm{GKdim}(A_{i_B})$. □

This result has applications in diverse contexts. In fact, it will be the very first step to determine the Gelfand-Kirillov dimension of the graph algebras, that we introduce now. A reference that covers all the definitions that follow is [1, Chapter 1]. When we mention a graph, we always mean a *directed* graph.

Notation 3.2. Let $E = (E^0, E^1, r, s)$ be a (directed) graph. Then E^0 is the set of *vertices*, E^1 is the set of *edges* and $r, s : E^1 \to E^0$ are, respectively, the *range* and *source* functions.

Definition 3.3. Let K be a field and let E be a graph. The (*classical*) *path algebra of E with coefficients in K*, denoted KE, is defined as the K-algebra generated by the set $E^0 \cup E^1$, and relations

(V) $uv = \delta_{u,v}u$ for every $u, v \in E^0$,
(E1) $s(e)e = er(e) = e$ for every $e \in E^1$.

Definition 3.4. Let K be a field and let E be a graph. The *Cohn path algebra of E with coefficients in K*, denoted by $C_K(E)$, is the K-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$, where $(E^1)^* = \{e^* \mid e \in E^1\}$, satisfying the following relations:

(V) $vw = \delta_{v,w}v$ for all $v, w \in E^0$.
(E1) $s(e)e = er(e) = e$ for all $e \in E^1$.
(E2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
(CK1) $e^*f = \delta_{e,f}r(e)$ for all $e, f \in E^1$.

Recall that given a graph E, a vertex $v \in E^0$ is said to be *regular* if $0 < |s^{-1}(v)| < \infty$. We denote the set of regular vertices of the graph E by $\mathrm{Reg}(E)$.

Definition 3.5. Let K be a field and let E be a graph. The *Leavitt path algebra of E with coefficients in K*, denoted by $L_K(E)$, is the K-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$, where $(E^1)^* = \{e^* \mid e \in E^1\}$, subject to the following relations:

(V) $vw = \delta_{v,w}v$ for all $v, w \in E^0$.
(E1) $s(e)e = er(e) = e$ for all $e \in E^1$.
(E2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
(CK1) $e^*f = \delta_{e,f}r(e)$ for all $e, f \in E^1$.
(CK2) $v = \sum_{\{e\in E^1 \mid s(e)=v\}} ee^*$ for every $v \in \mathrm{Reg}(E)$.

It will be useful to introduce the definition of an extended graph, as Leavitt and Cohn path algebras can be seen as appropriate quotients of the path algebras associated to the extended graphs.

Definition 3.6. Let $E = (E^0, E^1, r, s)$ be an arbitrary graph. We define the *extended graph of E* as the new graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$, where $(E^1)^* = \{e^* \mid e \in E^1\}$, and the functions r' and s' are defined as

$$r'|_{E^1} = r,\ s'|_{E^1} = s,\ r'(e^*) = s(e),\ \text{and } s'(e^*) = r(e) \text{ for every } e \in E^1.$$

In other words, each edge $e^* \in (E^1)^*$ has orientation the reverse of that of its counterpart $e \in E^1$.

The relationship of the two latter algebras with the extended graph is the following: $C_K(E)$ and $L_K(E)$ are the quotients of the path algebra associated to the extended graph, $K\widehat{E}$, by the two-sided ideal generated by relations (CK1) and (CK1),(CK2), respectively. From this point of view, one can justify that the Leavitt path algebra is the quotient of $C_K(E)$ by the two-sided ideal of $C_K(E)$ generated by (CK2) (see [1, Remark 1.5.2]).

Definition 3.7. ([1, Definition 1.5.9]) Let K be a field, let E be a graph and let $X \subseteq \mathrm{Reg}(E)$. Set I^X as the K-algebra two-sided ideal of $C_K(E)$ generated by the idempotents $\{q_v := v - \sum_{e\in s^{-1}(v)} ee^* \mid v \in X\}$. The *Cohn path algebra of E relative to X*, denoted $C_K^X(E)$ is defined as the quotient K-algebra $C_K^X(E)/I^X$.

It is immediate that $C_K(E) = C_K^{\emptyset}(E)$ and that $L_K(E) = C_K^{\mathrm{Reg}(E)}(E)$.

Now that we have defined the bunch of algebras that we are going to work with, we will use Theorem 3.1 jointly with some other facts in order to determine the Gelfand-Kirillov dimension of the graph algebras associated to arbitrary graphs.

Theorem 3.8. *Let E be an arbitrary graph. Then* $\mathrm{GKdim}(L_K(E)) = \varinjlim \mathrm{GKdim}(L_K(E_i))$, *where $L_K(E) = \varinjlim L_K(E_i)$ is the direct limit of certain Leavitt path algebras $L_K(E_i)$ associated to finite graphs.*

Proof. By [1, Theorem 1.3.12], we have that $L_K(E)$ is the direct limit of unital subalgebras, each of which is isomorphic to the Leavitt path algebra of a finite graph. We express this as $L_K(E) = \varinjlim L_K(E_i)$. Now apply Theorem 3.1. □

It is interesting to note that the Gelfand-Kirillov dimension of a Leavitt path algebra is always a natural number (see Theorem 3.21). The same will happen for the graph algebras considered here. The following proposition shows that determining the GK-dimension of $L_K(E)$ gives bounds for the GK-dimension of the previous mentioned algebras.

Proposition 3.9. Let E be an arbitrary graph, and consider $X \subseteq \mathrm{Reg}(E)$. Then:

$$\mathrm{GKdim}(KE) \leq \mathrm{GKdim}(L_K(E)) \leq \mathrm{GKdim}(C_K^X(E)) \leq \mathrm{GKdim}(C_K(E)) \leq \mathrm{GKdim}(K\widehat{E}).$$

Proof. Let $X \subseteq \mathrm{Reg}(E)$. By [1, Lemma 2.1.8], we have an injective algebra inclusion of KE into $L_K(E) = C_K^{\mathrm{Reg}(E)}(E)$. On the other hand, $C_K^X(E)$ is an epimorphic image of $C_K(E)$, which in turn is an epimorphic image of $K\widehat{E}$. Note also that $X \subseteq X' \subseteq \mathrm{Reg}(E)$ implies that $C_K^{X'}(E)$ is an epimorphic image of $C_K^X(E)$, so the chain of inequalities follows. □

At this point, it is worth noting that we cannot close the circle of inequalities by showing that $\mathrm{GKdim}(K\widehat{E}) \leq \mathrm{GKdim}(KE)$ because it is false in general. Examples are given in Examples 3.17.

We are now in a position to determine, for a completely arbitrary graph E, the precise relationship among the GK-dimensions of KE, $L_K(E)$, $C_K(E)$ and $K\widehat{E}$. The first step will be to give the explicit GK-dimension of a path algebra, in order to determinate the GK-dimension of the closing links of the inequality in Proposition 3.9. The proof is an adaptation of [6, Theorem 5]. Afterwards, we extend it to include arbitrary graphs.

We will introduce previous graph-theoretic definitions.

Definitions 3.10. Let E be any graph. Recall that a path $\mu = e_1 \cdots e_n$, where $e_i \in E^1$, is called a *cycle* if $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$. For a cycle c, the vertex $s(c)$ is called the *base of the cycle* or it is said that c is *based at* $s(c)$. An edge e is called an *exit* for the cycle c if $s(e) = s(e_i)$ for some $i \in \{1, \ldots, n\}$ and $e \neq e_i$. A cycle c is said to be an *exclusive cycle* if it is disjoint with every other cycle; equivalently, no vertex on c is the base of a different cycle other than a rotated of c. In other case, we will say that c is a *non-exclusive cycle*. We will say that E satisfies *Condition* (EXC) if every cycle of E is an exclusive cycle.

The notion of exclusive cycle appears for the first time, up to our knowledge, in [21, Definition 3.11] under the name of *cycle without* (K). The name of exclusive cycle was given in [9].

Definitions 3.11. Let E be any graph.

(i) For two cycles c, d we write $c \Rightarrow d$ if there exists a path that starts at a vertex in c^0 and ends at a vertex in d^0.

(ii) A sequence of distinct cycles $c_1, ..., c_k$ is a *chain of cycles of length* k if $c_1 \Rightarrow ... \Rightarrow c_k$. We will say that such a chain has an *exit* if the cycle c_k has an exit.

(iii) A path α of E *passes through the chain of cycles* $c_1 \Rightarrow ... \Rightarrow c_k$ if there exists $n_0, n_1 \in \{1, ..., k\}$ such that

$$\alpha^0 \cap c_i^0 \neq \emptyset \text{ for every } i \text{ with } n_0 \leq i \leq n_1, \text{ and } \alpha^0 \cap c_i^0 = \emptyset \text{ otherwise},$$

where μ^0 denotes the set of vertices that appear in the path $\mu = e_1 ... e_n$, i.e.,

$$\mu^0 = \{s(e_i), r(e_i) \mid i = 1, ..., n\}.$$

The following theorem can be found in the classical paper [22], or more recently as [14, Corollary 7.3]. Nevertheless, we give a proof.

Theorem 3.12. *Let E be a finite graph.*

(i) *The path algebra KE has finite GK-dimension if and only if E satisfies Condition* (EXC)*;*

(ii) *Assume that E satisfies Condition* (EXC)*. If t denotes the maximal length of chains of cycles in E, then* $\mathrm{GKdim}(KE) = t$.

Proof. Assume that E is a finite graph satisfying Condition (EXC), in which we can find a chain of cycles of maximal length t. As in the proof of [6, Theorem 5], and with the same notation, denote by P the set of paths that can be built from edges not belonging to a cycle, which is finite. Any monomial $\alpha \in KE$ with $l(\alpha) = n$ can be written as

$$\alpha = \alpha_1'(a_1 \alpha_{c_1}^{l_1} b_1) \cdots \alpha_t'(a_t \alpha_{c_t}^{l_t} b_t) \alpha_{t+1}',$$

where α passes through the chain of cycles $c_1 \Rightarrow \cdots \Rightarrow c_t$, with $\alpha_i' \in P$, $a_i \alpha_{c_i}^{l_i} b_i$ being a path in the cycle c_i, $|c_i^0| = m_i$, $l(a_i), l(b_i) \leq m_i - 1$ and $l(\alpha) \leq n$. This shows that the growth function behaves like a polynomial function of degree t, so $\mathrm{GKdim}(KE) = t$. $\square$

It is possible the passage from finite graphs to arbitrary graphs. For this, we use [10, Theorem 2.12]. There, the authors express a Leavitt path algebra as a limit, $L_K(E) = \varinjlim L_K(E_i)$, for conveniently chosen finite subgraphs E_i of E. Following the proof of the above mentioned theorem, we see that the results it appeals to, and the way to carry out the construction, is completely valid if we replace $L_K(E)$ by KE. So we can state the following result.

Proposition 3.13. Let E be an arbitrary graph. Then $KE = \varinjlim KE_i$, where $\{E_i\}_{i\in I}$ are appropriate finite subgraphs of E.

Applying Theorem 3.1, we obtain the following result.

Theorem 3.14. *Let E be an arbitrary graph. Then* $\mathrm{GKdim}(KE) = \varinjlim \mathrm{GKdim}(KE_i)$*, where $\{E_i\}_{i\in I}$ are appropriate finite subgraphs of E.*

As Cohn path algebras are quotients of path algebras over extended graphs, as we have explained, the following natural step is to determine the GK-dimension of such algebras. The proof will be a consequence of the fact that Cohn path algebras are always Leavitt path algebras ([1, Theorem 1.5.18]).

Theorem 3.15. *Let E be a finite graph.*

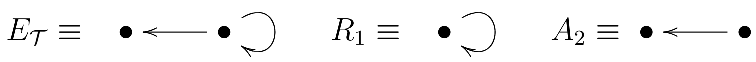
(i) *The Cohn path algebra $C_K(E)$ has finite GK-dimension if and only if E satisfies Condition* (EXC)*;*
(ii) *Assume that E satisfies Condition* (EXC)*. If t denotes the maximal length of chains of cycles in E, then* $\mathrm{GKdim}(C_K(E)) = 2t$*.*

Proof. Assume that E is a finite graph satisfying Condition (EXC). By [1, Theorem 1.5.18], we can find a graph F such that $C_K(E) \cong L_K(F)$. We take into account that the graph F is built from E without adding new cycles, so the maximal length of chains of cycles in F is the same than that of E, namely, t. This happens because the new edges added end at a sink. Moreover, if $c_1 \Rightarrow \cdots \Rightarrow c_t$ is a chain of cycles in E of maximal length, then we are adding in the graph F a new vertex v and an edge e such that $s(e) \in c_t^0$ and $r(e) = v$, so that chains of cycles in F always have exits. Therefore, by [6, Theorem 5] we have that $\mathrm{GKdim}(C_K(E)) = \mathrm{GKdim}(L_K(F)) = \max\{2t-1, 2t\} = 2t$, as we claimed. □

Remark 3.16. Cohn path algebras with finite GK-dimension are always even-dimensional. A natural question would be the following: can every even-dimensional Leavitt path algebra $L_K(E)$ be seen as the Cohn path algebra $C_K(F)$ associated to some graph F? The answer is no, and this is the content of Example 3.22.

For the chain of inequalities given in Proposition 3.9, we can now give examples of strict inequalities at each link, proceeding in order and applying Theorems 3.12 and 3.15.

Examples 3.17. Consider the following graphs:



Then:

$$1 = \text{GKdim}(KE_{\mathcal{T}}) \ < \text{GKdim}(L_K(E_{\mathcal{T}})) = 2$$
$$1 = \text{GKdim}(L_K(R_1)) < \text{GKdim}(C_K(R_1)) = 2$$
$$0 = \text{GKdim}(C_K(A_2)) < \text{GKdim}(K\widehat{A_2}) = \infty.$$

An application of [1, Theorem 1.6.9] let us now determine the GK-dimension of Cohn path algebras associated to arbitrary graphs. See [1, Definition 1.6.7] for the definition of a complete subgraph of a graph E.

Theorem 3.18. *Let E be an arbitrary graph. Then* $\text{GKdim}(C_K(E)) = \varinjlim \text{GKdim}(C_K(E_i))$, *where $\{E_i\}_{i\in I}$ are appropriate complete finite subgraphs of E.*

The following theorem contains as a particular case that of Cohn path algebras.

Theorem 3.19. *Let E be a finite graph and let $X \subseteq \text{Reg}(E)$.*

(i) *The Cohn path algebra of E relative to X, $C_K^X(E)$, has finite GK-dimension if and only if E satisfies Condition* (EXC)*;*
(ii) *Assume that E satisfies Condition* (EXC). *Denote by t the maximal length of chains of cycles in E and by t' the maximal length of chains of cycles with exits of E. Then:*
  (a) *If X contains some vertex from the last cycle c_t of a chain of cycles of maximal length, say $c_1 \Rightarrow \cdots \Rightarrow c_t$, then* $\text{GKdim}(C_K^X(E)) = 2t$.
  (b) *Otherwise,* $\text{GKdim}(C_K^X(E)) = \max\{2t-1, 2t'\}$.

Proof. Let $X \subseteq \text{Reg}(E)$ and build $C_K^X(E) = C_K(E)/I^X$. We have to distinguish two cases.

Case 1 Assume that X contains some vertex from the last cycle c_t of a chain of cycles of maximal length, say $c_1 \Rightarrow \cdots \Rightarrow c_t$. Then, by [1, Theorem 1.5.18], we have that $C_K^X(E) \cong L_K(E(X))$, where we are adding new edges and vertices in such a way that the maximal length of a chain of cycles in the graph $E(X)$ is the same. Moreover, we are adding at least an exit for the cycle c_t, so $\text{GKdim}(C_K^X(E)) = \text{GKdim}(L_K(E(X))) = 2t$.

Case 2 If X does not contain a vertex from the last cycle of any chain of cycles of maximal length, then $C_K^X(E) \cong L_K(E(X))$, where the maximal length of a chain of cycles in $E(X)$ is still t, and the length of chains of cycles with an exit of $E(X)$ is still t', as in the statement. Therefore, $\text{GKdim}(C_K^X(E)) = \text{GKdim}(L_K(E(X))) = \max\{2t-1, 2t'\}$. □

Observe that this result is coherent with the fact that Cohn path algebras are always even-dimensional: these are always included in the first case because $X = \text{Reg}(E)$. We now pass to the case of an arbitrary graph, appealing to [1, Theorem 1.6.9].

Theorem 3.20. *Let E be an arbitrary graph, and let $X \subseteq \text{Reg}(E)$. Then* $\text{GKdim}(C_K^X(E)) = \varinjlim \text{GKdim}(C_K^X(E_i))$, *where $\{E_i\}_{i\in I}$ are appropriate complete finite subgraphs of E.*

Now we summarize all the information about the GK-dimension of graph algebras in a single statement.

Theorem 3.21. (The GK-dimension of graph algebras) *Let E be an arbitrary graph and let $X \subseteq \text{Reg}(E)$. Then:*

(i) *The graph algebras KE, $L_K(E)$, $C_K(E)$ and $C_K^X(E)$ have finite GK-dimension if and only if E satisfies Condition* (EXC) *and the maximal length of chains of cycles in E is finite.*

(ii) *Assume that E satisfies Condition* (EXC) *and that the maximal length of chains of cycles in E is finite, say t. Denote by t' the maximal length of chains of cycles in E with an exit. Then:*

(a) $\mathrm{GKdim}(KE) = t$.

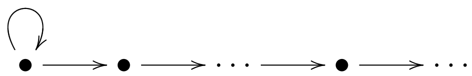
(b) $\mathrm{GKdim}(L_K(E)) = \max\{2t-1, 2t'\}$.

(c) $\mathrm{GKdim}(C_K(E)) = 2t$.

(d) $$\mathrm{GKdim}(C_K^X(E)) = \begin{cases} 2t & \text{if } X \text{ contains a vertex from the last cycle of a chain of cycles of maximal length,} \\ \max\{2t-1, 2t'\} & \text{otherwise.} \end{cases}$$

We end this section with the promised example of an even-dimensional Leavitt path algebra that is not a Cohn path algebra.

Example 3.22. (A Leavitt path algebra with GK-dimension 2 that is not a Cohn path algebra.) Let E be the graph:



There are different proofs of this fact. We have chosen the following, which we think is the shortest. By Theorem 3.21, we have that $\mathrm{GKdim}(L_K(E)) = 2$. We will show that $L_K(E)$ cannot be realized as $C_K(F)$ for some graph F. Assume that F is such that $C_K(F) \cong L_K(E)$. By [1, Theorem 1.5.18], there exits a graph $G = G(F)$ built from F such that $L_K(G) \cong C_K(F) \cong L_K(E)$. Using [11, Theorem 5.2] and [1, Theorem 2.6.10] we have that $\mathrm{Soc}(L_K(E)) \cong M_{\mathbb{N}}(K)$, as the socle of a Leavitt path algebra coincides with the ideal generated by the line points. Now $L_K(E)$ does not contain 1-dimensional ideals, as these are generated by isolated vertices in E by [4, Proposition 2.3], so neither does $L_K(G)$. This implies that G does not have isolated vertices, and therefore neither does F, as these would pass to G by [1, Proposition 1.5.21]. Now because $L_K(E)$ is not unital, $L_K(G)$ is also not unital and therefore G has an infinite number of vertices. This implies that F has an infinite quantity of vertices too. But as F does not have isolated vertices, we necessarily find distinct vertices $u \neq v$ and not necessarily distinct w_1, w_2, all in F^0, and edges $e, f \in F^1$ with $s(f) = u, r(f) = w_1$ and $s(e) = v, r(e) = w_2$, giving rise to two distinct sinks u', v' in G that give rise to two non-trivial direct summands in $\mathrm{Soc}(L_K(G))$, a contradiction.

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