Novel Results for the κ–μ Extreme Fading Distribution: Generation of White Samples and Capacity Analysis

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***Abstract*—We provide new analytical results for the κ–μ extreme (κ–μ-E) fading distribution, which is useful to model propagation conditions more severe than Rayleigh fading. First, we calculate a closed-form expression for the cumulative distribution function in terms of the first-order Marcum Q-function, which allows us to accurately generate κ–μ-E distributed random variables using the inversion method. Then, we investigate the ergodic capacity in this scenario. Strikingly, we observe that the capacity in the high-SNR regime scales differently than in all conventional fading models.**

***Index Terms*—Capacity, channel simulation, fading channels.**

I. Introduction

THE characterization of fading in the presence of severe propagation conditions through new channel models has raised the interest of many authors [1]–[5]. These extreme propagation conditions may occur in different scenarios such as indoor propagation, enclosed environments, vehicle-to-vehicle communications or in the context of wireless sensor networks [4], and in general refer to fading conditions worse than Rayleigh. One of such models is the κ–μ extreme (κ–μ-E) distribution [5] as a limiting case of the general and popular κ–μ fading model [2]. This model has been validated through field measurement campaigns in different indoor environments [5], and has found application in diverse scenarios [6]–[8].

The statistical characterization of the κ–μ-E fading distribution poses some open challenges. For instance, the cdf of this fading model is given in terms of the zero-*th* order Marcum Q_n-function, i.e. $Q_0(\cdot,\cdot)$. Since the implementation of the Marcum Q_n-function is often restricted to positive integer values of n (e.g. Matlab), the computation of the cdf is not straightforward. This lack of tractability also has an impact on the complexity associated with the generation of κ–μ-E samples for channel simulation purposes.

Another open problem is related to how much information can be transmitted in the very severe fading conditions modeled by this distribution. Recent results have shown that the loss in ergodic capacity for most popular fading models is constant in the high-SNR regime, compared to the AWGN case [9]–[11]. However, to the best of our knowledge the capacity limits of the κ–μ-E fading channel are largely unknown. In this paper, we aim to shed new light on these issues.

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There are several contributions in our work: Although a closed-form expression for the κ–μ-E cdf is available, we here derive an alternative closed-form expression for this cdf in terms of the first-order Marcum Q-function. Besides allowing for an easy computation with readily available mathematical software packages, an additional advantage of the new cdf expression is that it can be easily inverted [12], [13]; hence, it enables the generation of κ–μ-E variates using the inversion method, which is reportedly more efficient that the rejection method [14].

We also investigate the ergodic capacity of κ–μ-E fading channels. In general, the ergodic capacity in fading channels other than Rayleigh and Nakagami-m is given in terms of Meijer G-functions [15], [16] or generalized hypergeometric functions [17]. However, in the high signal-to-noise ratio (SNR) regime [9], [18] the ergodic capacity is known to behave as $C \approx \log_2 \bar{\gamma} - b$, where $\bar{\gamma}$ is the average SNR and b is a constant value independent of $\bar{\gamma}$ that only depends on the fading distribution. Since $b = 0$ in the absence of fading (i.e. AWGN channel), this parameter b is usually regarded as the capacity loss due to fading. This capacity loss has been computed for most popular fading models such [9]–[11], thus enabling a better understanding on how fading severity translates into a capacity loss. We will analyze the ergodic capacity in κ–μ-E fading channels for the first time in the literature. We show that in the high-SNR regime, the capacity scales as $C \approx a \cdot log_2\bar{\gamma} - b$ with $a \le 1$; interestingly, this result cannot be obtained by specializing the results in [11] to the κ–μ-E regime. Thus, the unique characteristics of this fading distribution to model severe propagation conditions are translated into capacity in a different way than for other fading models such as Rayleigh, Rice, Nakagami-m, Hoyt, Weibull, η–μ, κ–μ or Two-Wave with diffuse power, for which $a = 1$.

II. The κ–μ-E Cumulative Distribution Function

The κ–μ-E distribution was originally introduced in Yacoub's reference paper [2], and later formalized in [5], where an in-depth statistical characterization assessed by field measurement was presented. Let us begin by considering a κ–μ distributed random variable R with *rms* value given by $\Omega = E\{R^2\}$. The pdf for the normalized random variable $\rho = R/\Omega$ is given as in [2] by

$$f_P(\rho) = \frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp(\kappa\mu)}\rho^{\mu}\exp\left(-\mu(1+\kappa)\rho^2\right) \times I_{\mu-1}(2\mu\rho\sqrt{\kappa(1+\kappa)}), \quad (1)$$

where $I_\alpha(\cdot)$ is the modified Bessel function of the first kind and order α. The cdf for the normalized random variable ρ is therefore expressed as

$$F_P(\rho) = 1 - Q_\mu\left(\sqrt{2\kappa\mu}, \rho\sqrt{2(1+\kappa)\mu}\right)\rho \geq 0, \quad (2)$$

where $Q_n(\cdot,\cdot)$ is the generalized Marcum Q-function of order n [19]. Letting $\kappa \to \infty$ and $\mu \to 0$ while keeping the product $\kappa \cdot \mu = 2$ m constant, a κ–μ-E distributed random variable is obtained.[1] By doing so, the cdf of the normalized random variable ρ is then given in [5, eq. 7] as

$$F_P(\rho) = 1 - Q_0(2\sqrt{m}, 2\rho\sqrt{m});\ \rho \geq 0. \tag{3}$$

While given in compact form, (3) has some problems from a practical perspective. Since the computation of the Marcum Q-function for values of n other than positive integers is usually not supported by conventional mathematical packages (e.g. Matlab), the evaluation of the κ–μ-E cdf is more complicated than its general counterpart (2). As pointed out in [5], this also complicates the generation of κ–μ-E random variates, as well as further manipulations required to derive other statistics of interest. For these reasons, an alternative expression for the cdf in terms of an infinite series was also given in [5, eq. 10].

We now show that a simple and more tractable expression for the κ–μ-E cdf can be easily derived. Admitting that the definition of the Marcum Q_n-function is valid for all $n \in \mathbb{Z}$, we can use [20, eq. 9] $Q_n(a,b) = 1 - Q_{1-n}(b,a)$ to express

$$F_P(\rho) = Q_1(2\rho\sqrt{m}, 2\sqrt{m})u(\rho), \tag{4}$$

where $u(\cdot)$ denotes the Heaviside step function. Even though the derivation of (4) is remarkably simple, this expression for the κ–μ-E cdf is new to the best of our knowledge, and can be easily evaluated. Besides, it also admits further integrations using known integral results for $Q_1(\cdot,\cdot)$ function. The κ–μ-E pdf can be directly obtained by differentiating (4) as

$$f_P(\rho) = \frac{\partial Q_1(2\rho\sqrt{m}, 2\sqrt{m})}{\partial\rho}u(\rho) + Q_1(0, 2\sqrt{m})\frac{du(\rho)}{d\rho}. \tag{5}$$

Using the known expressions for the partial derivatives of the first-order Marcum Q-function [21] and knowing that $Q_1(0, 2\sqrt{m}) = e^{-2m}$, the original pdf given in [5, eq. 6] is obtained by direct differentiation of the cdf, as

$$f_P(\rho) = 4mI_1(4m\rho)e^{-2m(1+\rho^2)} + e^{-2m}\delta(\rho), \tag{6}$$

where $\delta(\cdot)$ denotes the Dirac impulse function.

An additional benefit from this new expression is the fact that simple asymptotic approximations for the cdf can be derived when $\rho \to \infty$. By successively using the equivalence given in [22, eq. 6], the asymptotic relationship between the generalized Marcum Q-function and the Gaussian Q-function given in [23, p. 100], and the asymptotic relationship for the modified Bessel function of the first kind and the Gaussian Q-function given in [24, eq. 28], we obtain the following asymptotically exact approximations for the cdf:

$$F_P(\rho)_{\rho\to\infty} \approx 1 - \rho^{-1/2}Q\left(2\sqrt{m})\rho - 1)\right) \tag{7}$$

$$\approx 1 - \frac{1}{2(\rho-1)\sqrt{2\pi\rho m}}e^{2m(\rho-1)^2}. \tag{8}$$

[1] As discussed in [5], this corresponds to the case of having very strong line-of-sight components, but scarce multipaths. The parameter m is related to the variance of ρ as $m^{-1} = \mathrm{var}\{\rho^2\}$, and has a similar interpretation to the fading severity parameter of the Nakagami-m fading model.

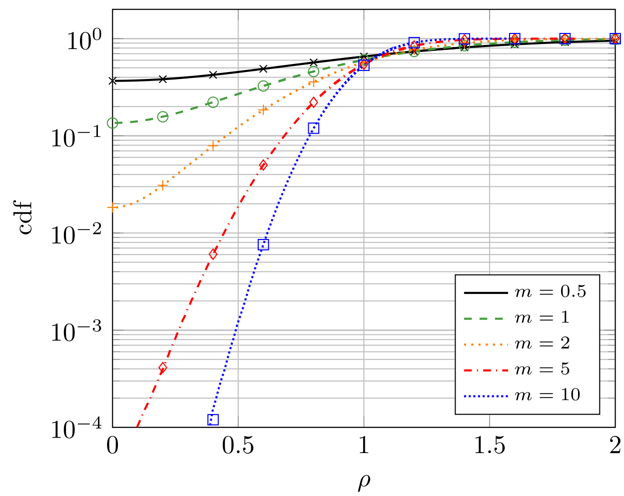

Fig. 1. Empirical (markers) vs analytical (lines) cdf of the κ–μ-E distribution.

III. Generation of κ–μ-E Samples

The generation of κ–μ variates for channel simulation purposes is usually performed by using the rejection method [14], [25]. While this approach is reportedly less efficient than the inversion method, the latter is usually infeasible when the target cdf cannot be easily inverted. Fortunately, for the specific case of the κ–μ-E distribution we have just shown that its cdf is given in terms of the first-order Marcum Q-function, for which inversion algorithms are readily available [12], [13]. Hence, we will use an inversion-based algorithm for generating a set **z** of κ–μ-E white samples, which is now described:

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(1) Define distribution parameter m
(2) Generate set r of uniformly distributed samples U[0,1]
forall the r_i ∈ r do
    if r_i < e^{-2m} then
        z_i = 0;
    else
        z_i = sqrt(Q^{-1}(r_i, 2m)/2m) [13];
    end
end
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In this algorithm, we used [13] to invert the function $\mathcal{Q}(x,b) = Q_1(\sqrt{2x}, \sqrt{2b})$, which can be accurately computed through iterative methods with fast convergence due to its monotonicity and convexity. In Fig. 1, the empirical cdfs obtained by generating 2^{20} samples are compared to the analytical κ–μ-E cdf, for different values of m.

IV. Ergodic Capacity in κ–μ-E Fading

The ergodic (or average) capacity in κ–μ fading channels was investigated in [16], where an exact expression for this metric was obtained in terms of an infinite summation of Meijer G-functions. Letting $\kappa \to \infty$ and $\mu \to 0$ while keeping $\kappa \cdot \mu = 2$ m in [16, eq. 6] allows for evaluating the capacity, but it fails to provide a first intuition of how capacity behaves in this type of fading due to its complicated form.

In order to shed some light on this issue, we here perform an asymptotic analysis of the ergodic capacity. Firstly, we focus on the low-SNR regime, for which the capacity is known to behave as [26, eq. 12] $C_{\bar{\gamma}\downarrow\downarrow} \approx \log_2 e \cdot \mathbb{E}\{\gamma\}$. This yields

$$C_{\bar{\gamma}\downarrow\downarrow} \approx \log_2 e \cdot \bar{\gamma}, \tag{9}$$

which is independent of m, and more interestingly, coincides with the low-SNR capacity in the AWGN case.

The asymptotic capacity analysis in the high-SNR regime following the approach introduced in [9]. Specifically, the ergodic capacity in this situation is approximated by [9, eq. 8]

$$C_{\bar{\gamma}\uparrow\uparrow} \approx \log_2 e \cdot \frac{\partial}{\partial k}\mathbb{E}[\gamma^n]|_{k=0}. \tag{10}$$

Therefore, the capacity analysis in the high-SNR regime requires for the calculation of the SNR moments of the κ–μ-E distribution. The instantaneous SNR γ at the receiver under κ–μ-E fading has the following pdf [8, eq. 22]:

$$f_\gamma(\gamma) = \frac{2m}{\sqrt{\gamma\bar{\gamma}}}e^{-2m(1+\frac{\gamma}{\bar{\gamma}})}I_1\left(4m\sqrt{\frac{\gamma}{\bar{\gamma}}}\right) + \frac{e^{-2m}}{2\sqrt{\gamma\bar{\gamma}}}\delta(\gamma), \tag{11}$$

where $\bar{\gamma} = \mathbb{E}[\gamma]$ is the average SNR. Hence, the k^{th} moment of γ can be calculated as

$$\mathbb{E}[\gamma^k] = \underbrace{\frac{\bar{\gamma}^k}{(2m)^k}}_{G_1(k)}\underbrace{e^{-2m}\Gamma(k)_1\bar{F}_1(k,0,2m)}_{G_2(k)}, \tag{12}$$

where $\Gamma(\cdot)$ is the Gamma function and $_1\bar{F}_1(\cdot,\cdot,\cdot)$ is the regularized confluent hypergeometric function [27]. We have defined $G_1(k)$ and $G_2(k)$ in (12) for convenience of calculation. Specifically, we use these definitions to express

$$\frac{\partial}{\partial k}\mathbb{E}[\gamma^n]|_{k=0} = \frac{\partial}{\partial k}G_1(k)|_{k=0}G_2(0) + G_1(0)\frac{\partial}{\partial k}G_2(k)|_{k=0}. \tag{13}$$

The first term in (13) is given by

$$\frac{\partial}{\partial k}G_1(k)|_{k=0}G_2(0) = (1-e^{-2m})(\log\bar{\gamma} - \log 2m), \tag{14}$$

where log denotes the natural logarithm, and we used $\lim_{k\to 0} G_2(k) = 1 - e^{-2m}$. Noting that $G_1(0) = 1$, the calculation of the second term in (13) is

$$\nu_m \Leftrightarrow \frac{\partial}{\partial k}G_2(k)|_{k=0} = e^{-2m}\frac{\partial}{\partial k}\{\Gamma(k)_1\bar{F}_1(k,0,2m)\}|_{k=0}, \tag{15}$$

where the notation ν_m was used for the sake of compactness. Expressing $_1\bar{F}_1$ in series form, we have

$$_1\bar{F}_1(k,0,2m) = \sum_{n=0}^{\infty}\frac{(k)_n}{\Gamma(n)}\frac{(2m)^n}{n!} = \sum_{n=1}^{\infty}\frac{(k)_n}{\Gamma(n)}\frac{(2m)^n}{n!}, \tag{16}$$

where $(k)_n$ denotes the Pochhammer symbol. We also used the fact that the first term in the summation equals zero as $\Gamma(n)|_{n=0} \to \infty$. Furthermore, since $\Gamma(k)\cdot(k)_n = \Gamma(k+n)$, we obtain

$$\Gamma(k)_1\bar{F}_1(k,0,2m) = \sum_{n=1}^{\infty}\frac{\Gamma(k+n)}{\Gamma(n)}\frac{(2m)^n}{n!}. \tag{17}$$

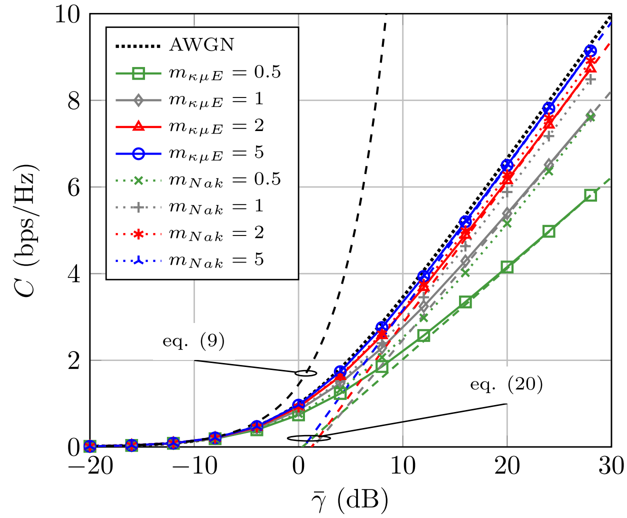

Fig. 2. Ergodic capacity of the κ–μ-E fading channel. AWGN ($m \to \infty$) and Nakagami-m cases included as references. Exact results obtained using [16], asymptotic results given by (9) and (20).

Hence, the derivative in (15) is given by

$$\nu_m = e^{-2m}\sum_{n=1}^{\infty}\Psi(n)\frac{(2m)^n}{n!}, \tag{18}$$

where $\Psi(\cdot)$ is the Digamma function. Using the simplified expression of $\Psi(n)$ for $n \in \mathbb{N}$ given by $\Psi(n) = \sum_{i=1}^{n-1}\frac{1}{i} - \gamma_e$ in [28, eq. 6.3.2], where γ_e is the Euler-Mascheroni constant, and after some algebra, we obtain

$$\nu_m = \log 2m - E_i(-2m) + e^{-2m}(2\gamma_e + \log 2m - E_i(2m)), \tag{19}$$

where $E_i(\cdot)$ is the Exponential Integral function [28, eq. 5.1.2]. Thus, combining (12)–(19) yields

$$C_{\bar{\gamma}\uparrow\uparrow} \approx a \cdot \log_2\bar{\gamma} - b, \tag{20}$$

where $a = 1 - e^{-2m}$ and

$$b = \frac{e^{-2m}E_i(2m) + E_i(-2m) - 2e^{-2m}(\gamma_e + \log 2m)}{\log(2)}; \tag{21}$$

One important observation must be made here: the slope of the capacity as a function of the average SNR $\bar{\gamma}$ is lower than the unity for finite m, i.e. $a < 1$; $\forall m < \infty$. This behavior is unique, compared to existing results in the literature for the ergodic capacity in fading channels [9]–[11], [18], [29], for which $a = 1$ regardless of the parameters of each fading model. Specifically, the recent analysis in [11] studies the asymptotic capacity in the high-SNR regime under η–μ and κ–μ fading, which include the popular Nakagami-m, Hoyt, Rayleigh and Rician models as particular cases. However, the effect on capacity here observed cannot be deduced by specializing the results in [11] to the κ–μ-E regime.

A direct implication of this result is that the parameter b cannot be regarded as the asymptotic capacity loss with respect to the AWGN case. In other words, the asymptotic capacity loss

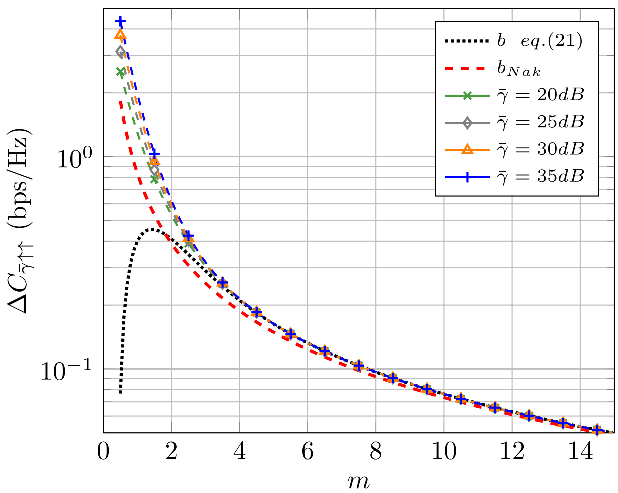

Fig. 3. Capacity loss in κ-μ-E fading given by (22) as a function of m, for different values of $\bar{\gamma}$. Nakagami-m case $\Delta C_{\bar{\gamma}\uparrow\uparrow}^{\mathrm{Nak}} = \log_2 e(\log m - \Psi(m))$ given in [9].

for the κ–μ-E fading channels *is not independent* of $\bar{\gamma}$; instead, it can be expressed as

$$\Delta C_{\bar{\gamma}\to\infty} \Leftrightarrow C_{\bar{\gamma}\to\infty}^{\mathrm{AWGN}} - C_{\bar{\gamma}\to\infty}^{\kappa-\mu-E} = e^{-2m} \log_2 \bar{\gamma} + b. \quad (22)$$

The ergodic capacity is evaluated in Fig. 2 for different values of m. The result for the Nakagami-m fading channel [9] is included as a reference, since its m parameter has a similar interpretation. We observed that reducing m dramatically changes the slope of capacity in κ–μ-E fading. Hence, the capacity loss compared to the AWGN and Nakagami-m cases grows with $\bar{\gamma}$ more pronouncedly as m is decreased. Conversely, as m is increased the severity of fading is reduced, and the capacity loss is practically negligible.

In Fig. 3, the asymptotic capacity loss in (22) is represented as a function of m. When m takes low values, we see that $\Delta C_{\bar{\gamma}\to\infty}$ depends on $\bar{\gamma}$ in κ–μ-E fading, due to $a < 1$. Conversely, as m increases then the capacity loss is mainly due to (21), and tends to behave similarly to the capacity loss in Nakagami-m fading, which is always independent of $\bar{\gamma}$.

V. Conclusion

We provided new insights for the κ–μ-E fading distribution. Our new cdf expression is easy to evaluate, and allows for efficiently generating κ–μ-E distributed white samples. Our analysis reveals that the blockage events inherent to the κ–μ-E fading channel affect the scaling laws of capacity in the high-SNR regime, whereas no differences are observed with the AWGN case in the low-SNR regime.

References

[1] G. D. Durgin, T. S. Rappaport, and D. A. De Wolf, "New analytical models and probability density functions for fading in wireless communications," *IEEE Trans. Commun.*, vol. 50, no. 6, pp. 1005–1015, Jun. 2002.
[2] M. D. Yacoub, "The κ–μ distribution and the η–μ distribution," *IEEE Antennas Propag. Mag.*, vol. 49, no. 1, pp. 68–81, Feb. 2007.
[3] J. Frolik, "On appropriate models for characterizing hyper-rayleigh fading," *IEEE Trans. Wireless Commun.*, vol. 7, no. 12, pp. 5202–5207, Dec. 2008.
[4] D. W. Matolak and J. Frolik, "Worse-than-rayleigh fading: Experimental results and theoretical models," *IEEE Commun. Mag.*, vol. 49, no. 4, pp. 140–146, Apr. 2011.
[5] G. S. Rabelo and M. D. Yacoub, "The κ–μ extreme distribution," *IEEE Trans. Commun.*, vol. 59, no. 10, pp. 2776–2785, Oct. 2011.
[6] P. C. Sofotasios and S. Freear, "The κ–μ extreme/gamma distribution: A physical composite fading model," in *Proc. IEEE WCNC*, Mar. 2011, pp. 1398–1401.
[7] P. C. Sofotasios and S. Freear, "The α–κ–μ Extreme distribution: Characterizing non-linear severe fading conditions," in *Proc. ATNAC*, Nov. 2011, pp. 1–4.
[8] P. C. Sofotasios *et al.*, "Energy detection based spectrum sensing over κ–μ and κ–μ extreme fading channels," *IEEE Trans. Veh. Technol.*, vol. 62, no. 3, pp. 1031–1040, Mar. 2013.
[9] F. Yilmaz and M.-S. Alouini, "Novel asymptotic results on the high-order statistics of the channel capacity over generalized fading channels," in *Proc. IEEE 13th Int. Workshop SPAWC*, Jun. 2012, pp. 389–393.
[10] M. Rao, F. J. Lopez-Martinez, M.-S. Alouini, and A. Goldsmith, "MGF approach to the analysis of generalized two-ray fading models," *IEEE Trans. Wireless Commun.*, vol. 14, no. 5, pp. 2548–2561, May 2015.
[11] L. Moreno-Pozas, F. J. Lopez-Martinez, J. F. Paris, and E. Martos-Naya, "The κ–μ shadowed fading model: Unifying the κ–μ and η–μ distributions," arXiv preprint arXiv:1504.05764.
[12] C. W. Helstrom, "Approximate inversion of marcum's q-function," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 34, no. 1, pp. 317–319, Jan. 1998.
[13] A. Gil, J. Segura, and N. M. Temme, "The asymptotic and numerical inversion of the marcum q-function," *Stud. Appl. Math.*, vol. 133, no. 2, pp. 257–278, Aug. 2014.
[14] R. Cogliatti, R. A. A. de Souza, and M. D. Yacoub, "Practical, highly efficient algorithm for generating κ–μ and η–μ variates and a near-100% efficient algorithm for generating α–μ variates," *IEEE Commun. Lett.*, vol. 16, no. 11, pp. 1768–1771, Nov. 2012.
[15] N. C. Sagias, G. S. Tombras, and G. K. Karagiannidis, "New results for the shannon channel capacity in generalized fading channels," *IEEE Commun. Lett.*, vol. 9, no. 2, pp. 97–99, Feb. 2005.
[16] D. B. da Costa and M. D. Yacoub, "Average channel capacity for generalized fading scenarios," *IEEE Commun. Lett.*, vol. 11, no. 12, pp. 949–951, Dec. 2007.
[17] E. J. Leonardo, D. Benevides da Costa, U. S. Dias, and M. D. Yacoub, "The ratio of independent arbitrary α–μ random variables and its application in the capacity analysis of spectrum sharing systems," *IEEE Commun. Lett.*, vol. 16, no. 11, pp. 1776–1779, Nov. 2012.
[18] W. C. Y. Lee, "Estimate of channel capacity in Rayleigh fading environment," *IEEE Trans. Veh. Technol.*, vol. 39, no. 3, pp. 187–189, Aug. 1990.
[19] A. H. Nuttall, "Some integrals involving the q-function," Naval Underwater Systems Center Newport., Washington, DC, USA, Tech. Rep., pp. 1–41, 1972.
[20] C. O'Driscoll and C. Murphy, "A simplified expression for the probability of error for binary multichannel communications," *IEEE Trans. Commun.*, vol. 57, no. 1, pp. 32–35, Jan. 2009.
[21] W. K. Pratt, "Partial differentials of marcum's q function," *Proc. IEEE*, vol. 56, no. 7, pp. 1220–1221, Jul. 1968.
[22] A. H. Nuttall, "Some Integrals Involving the (Q sub M)-function," Naval Underwater Systems Center, New London, CT, USA, Tech. Rep. 779 846, May 1974.
[23] M. K. Simon and M.-S. Alouini, *Digital Communication Over Fading Channels*. Hoboken, NJ, USA: Wiley-IEEE Press, 2004.
[24] F. J. Lopez-Martinez and J. M. Romero-Jerez, "Asymptotically exact approximations for the symmetric difference of generalized Marcum Q-functions," *IEEE Trans. Veh. Technol.*, vol. 64, no. 5, pp. 2154–2159, May 2015.
[25] R. A. A. de Souza, A. M. O. Ribeiro, and D. A. Guimaraes, "On the efficient generation of α–κ–μ and α–η–μ white samples with applications," *Int. J. Antennas. Propag.*, vol. 2015, p. 13, 2015, Art. ID. 873890.
[26] M. Di Renzo, F. Graziosi, and F. Santucci, "Channel capacity over generalized fading channels: A novel MGF-based approach for performance analysis and design of wireless communication systems," *IEEE Trans. Veh. Technol.*, vol. 59, no. 1, pp. 127–149, Jan. 2010.
[27] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 7th ed. New York, NY, USA: Academic, 2007.
[28] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*. New York, NY, USA: Dover, 1964.
[29] A. Lozano, A. M. Tulino, and S. Verdu, "High-SNR power offset in multiantenna communication," *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4134–4151, Dec. 2005.