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On prescribed characteristic polynomials

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Let $\mathbb{F}$ be a field. We show that given any nth degree monic polynomial $q(x) \in \mathbb{F}[x]$ and any matrix $A \in \mathbb{M}_n(\mathbb{F})$ whose trace coincides with the trace of $q(x)$ and consisting in its main diagonal of k 0-blocks of order one, with $k < n - k$, and an invertible non-derogatory block of order $n - k$, we can construct a square-zero matrix N such that the characteristic polynomial of $A + N$ is exactly $q(x)$. We also show that the restriction $k < n - k$ is necessary in the sense that, when the equality $k = n - k$ holds, not every characteristic polynomial having the same trace as A can be obtained by adding a square-zero matrix. Finally, we apply our main result to decompose matrices into the sum of a square-zero matrix and some other matrix which is either diagonalizable, invertible, potent or torsion.

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1. Introduction and basic facts

Throughout the current article, we denote by $\mathbb{M}_n(\mathbb{F})$ the matrix ring consisting of all square matrices of size $n \times n$ over an arbitrary field $\mathbb{F}$.

The problem of completing a matrix, when some of its entries are fixed, such that the resulting matrix satisfies a prescribed characteristic polynomial is classical and has been studied by several authors; see, for more detailed information, the survey by G. Cravo [2]. Back in 1958, H.K. Farahat and W. Ledermann showed that any matrix of the form

$$A = \left(\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & C & \\ 0 & & & \end{array}\right) \in \mathbb{M}_n(\mathbb{F}),$$

where $C \in \mathbb{M}_{n-1}(\mathbb{F})$ is non-derogatory, can be modified by adding a matrix of the form

$$N = \left(\begin{array}{c|ccc} a & u_1 & \dots & u_{n-1} \\ \hline v_1 & & & \\ \vdots & & \mathbf{0}_{n-1,n-1} & \\ v_{n-1} & & & \end{array}\right) \in \mathbb{M}_n(\mathbb{F})$$

such that the characteristic polynomial of $A + N$ is any prescribed monic polynomial of degree n (cf. [7, Theorem 3.4]). Furthermore, this problem was generalized by G.N. Oliveira who raised the question of when a block matrix of the form

$$\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right),$$

where part of its blocks are known, can be completed to achieve any prescribed characteristic polynomial, and addressed it in several of his works (see [8], [9], [10], [11]).

In particular, in his work [8], Oliveira proved that any matrix of the form

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,n-k} \\ \hline \mathbf{0}_{n-k,k} & A_{22} \end{array}\right) \in \mathbb{M}_n(\mathbb{F}),$$

where $A_{22} \in \mathbb{M}_{n-k}(\mathbb{F})$ is fixed, can be completed by adding a matrix of the form

$$N = \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & \mathbf{0}_{n-k,n-k} \end{array}\right) \in \mathbb{M}_n(\mathbb{F})$$

such that the characteristic polynomial of $A + N$ is any prescribed monic polynomial $q(x)$ of degree n if, and only if, the product of certain elementary divisors of the matrix A_{22} divides $q(x)$.

The prescription of several entries of a matrix together with a fixed characteristic polynomial has been recently used by S. Breaz and G. Călugăreanu when showing that every square matrix with nilpotent trace over a general ring can be expressed as the sum of three nilpotent matrices; see [1].

With the aim of addressing decomposition problems involving square-zero matrices, in this work we study the problem of modifying a matrix of the form

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,n-k} \\ \hline \mathbf{0}_{n-k,k} & A_{22} \end{array}\right) \in \mathbb{M}_n(\mathbb{F}),$$

with fixed $A_{22} \in \mathbb{M}_{n-k}(\mathbb{F})$, by adding to it a square-zero matrix N such that the characteristic polynomial of $A+N$ is any prescribed polynomial. Since square-zero matrices always have zero trace, the trace of $A+N$ equals the trace of A, and by this modification of A we can only get characteristic polynomials having the same trace as A. We will show in the sequel that, when $k < n-k$ and A_{22} is invertible and non-derogatory, given any monic degree n polynomial $q(x)$ with the same trace as A there exists a square-zero matrix N such that the characteristic polynomial of $A+N$ is precisely $q(x)$. The first and last k rows of such matrix N will be non-zero in general, so this is not exactly a completion problem, because, in order to get the desired characteristic polynomial by adding a square-zero matrix to A, we modify the last k rows of A_{22}.

We also illustrate that the restriction $k < n-k$ cannot be generally ignored in the sense that, when the equality $k = n-k$ holds, not every characteristic polynomial having the same trace as A can be obtained by adding a square-zero matrix.

Further, in the last section of this work, we will apply this result to the problem of decomposing matrices into the sum of a square-zero matrix and a matrix satisfying some "good" condition, where "good" stands for the properties of being either diagonalizable, invertible, n-torsion or n-potent. This fits in our general project of decomposing square matrices into the sum of a square-zero matrix and another matrix satisfying some properties such as been diagonalizable, potent, invertible, etc. (see, for instance, [3], [4], [5] and [6]).

2. Main theorem

Recall that the *trace* of a monic polynomial

$$q(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{F}[x]$$

is $-a_{n-1} \in \mathbb{F}$. If such polynomial is the characteristic polynomial of a certain matrix, the trace coincides with the trace (i.e., the sum of the diagonal elements) of such matrix. Recall that a matrix N is *nilpotent* if there exists $k \in \mathbb{N}$ with $N^k = 0$, and it is called a *square-zero* matrix when $N^2 = 0$.

Remark 2.1. Given a non-derogatory matrix A, there always exists a square-zero matrix N such that the characteristic polynomial of $A+N$ is any given monic polynomial whose trace coincides with the trace of A. In fact, we can suppose that A is the companion matrix of a certain polynomial

$$p(x) = x^n + u_{n-1}x^{n-1} + u_{n-2}x^{n-2} + \cdots + u_0$$

thus:

$$A = \begin{pmatrix} 0 & \dots & 0 & -u_0 \\ 1 & & 0 & \vdots \\ & \ddots & & -u_{n-2} \\ 0 & & 1 & -u_{n-1} \end{pmatrix}.$$

Given any polynomial

$$q(x) = x^n + u_{n-1}x^{n-1} + v_{n-2}x^{n-2} + \cdots + v_0$$

(where the trace of $q(x)$ coincides with the trace of A) it is enough to consider the square-zero matrix

$$N = \begin{pmatrix} 0 & 0 & u_0 - v_0 \\ 0 & 0 & \vdots \\ \vdots & \vdots & u_{n-2} - v_{n-2} \\ 0 & 0 & 0 \end{pmatrix}$$

to get that

$$A + N = \begin{pmatrix} 0 & & 0 & -v_0 \\ 1 & & 0 & \vdots \\ & \ddots & & -v_{n-2} \\ 0 & & 1 & -u_{n-1} \end{pmatrix}$$

has characteristic polynomial equal to $q(x)$.

The goal of this section is to generalize the previous result to any invertible non-derogative matrix completed with k rows/columns of zeros. In order to do that, we begin with a technical lemma.

Lemma 2.2. *Let $\mathbb{F}$ be a field and let $a_1, \dots, a_n, u_2, \dots, u_n \in \mathbb{F}$. Then the determinant of the matrix*

$$A = \begin{pmatrix} a_1+x & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & a_n \\ 0 & x & 0 & \cdots & 0 & 0 & u_2 \\ 0 & 0 & x & \cdots & 0 & 0 & u_3 \\ 0 & 0 & 0 & \ddots & 0 & 0 & u_4 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x & u_{n-1} \\ x & 0 & 0 & \cdots & 0 & 0 & x+u_n \end{pmatrix}$$

is $|A| = x^n + u_n x^{n-1} + (a_1 - a_n)x^{n-1} + a_1 u_n x^{n-2} + \sum_{i=2}^{n-1} a_i u_i x^{n-2}$.

Proof. By $R(i)$ and $C(i)$, we respectively denote the i^{th}-row and the i^{th}-column of the involved matrix. To perform the calculations, we will assume that our matrix is a matrix with coefficients in the field of fractions of $\mathbb{F}[x]$.

For $s = 2, \ldots, n-1$, we add $-\frac{a_s}{x}R(s)$ to row $R(1)$ as follows:

$$|A| = \begin{vmatrix} a_1+x & 0 & 0 & \cdots & 0 & 0 & a_n - \sum_{s=2}^{n-1} \frac{a_s u_s}{x} \\ 0 & x & 0 & \cdots & 0 & 0 & u_2 \\ 0 & 0 & x & \cdots & 0 & 0 & u_3 \\ 0 & 0 & 0 & \ddots & 0 & 0 & u_4 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x & u_{n-1} \\ x & 0 & 0 & \cdots & 0 & 0 & x+u_n \end{vmatrix}.$$

By the expansion of the determinant using $C(2), \ldots, C(n-2)$, we get:

$$\begin{aligned} |A| &= x^{n-2} \begin{vmatrix} a_1+x & a_n - \sum_{s=2}^{n-1} \frac{a_s u_s}{x} \\ x & x+u_n \end{vmatrix} \\ &= x^{n-2}(a_1+x)(x+u_n) - x^{n-1}(a_n - \sum_{s=2}^{n-1} \frac{a_s u_s}{x}) \\ &= x^n + u_n x^{n-1} + (a_1 - a_n)x^{n-1} + a_1 u_n x^{n-2} + \sum_{i=2}^{n-1} a_i u_i x^{n-2}. \quad \square \end{aligned}$$

We are now prepared to establish the following chief result.

Theorem 2.3. *Let* $\mathbb{F}$ *be a field, let* $n, k \in \mathbb{N}$ *with* $k < n-k$*, and consider the block matrix*

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,n-k} \\ \hline \mathbf{0}_{n-k,k} & A_{22} \end{array}\right) \in \mathbb{M}_n(\mathbb{F})$$

consisting of k rows and columns of zeros and an invertible non-derogatory matrix A_{22}. Then, for any monic polynomial $q(x)$ of degree n whose trace coincides with the trace of A, there exists a square-zero matrix N such that the characteristic polynomial of $A+N$ coincides with $q(x)$.

Proof. The case $k=0$ was treated in Remark 2.1, so we can suppose that $k>0$. Under an appropriate change of the existing basis, we can also suppose that A_{22} is a companion matrix $C(p(x))$ for a polynomial of the form $p(x)=x^{n-k}+u_{n-k-1}x^{n-k-1}+\cdots+u_0$.

Take arbitrary $a_{ij}\in\mathbb{F}$, $i=1,\dots,k$, $j=0,\dots,n-2k$, and define the matrix $N:=(\alpha_{ij})\in\mathbb{M}_n(\mathbb{F})$ like this:

$$\alpha_{i,j}:=\begin{cases}-a_{i,n-k+1} & \text{if } \ i=1,\dots,k \text{ and } j=k,\\ -a_{i,j} & \text{if } \ i=1,\dots,k \text{ and } j=k+1,\dots,n-k+1,\\ a_{n-i+1,n-k+1} & \text{if } \ i=n-k+1,\dots,n \text{ and } j=k,\\ a_{n-i+1,j} & \text{if } \ i=n-k+1,\dots,n \text{ and } j=k+1,\dots,n-k+1,\\ -1 & \text{if } \ i=1,\dots,k-1 \text{ and } j=i,n-i+1,\\ 1 & \text{if } \ i=n-k+1,\dots,n \text{ and } j=i,n-i+1,\\ 0 & \text{otherwise.}\end{cases}$$

Let us first show that $N^2=0$: in fact, if we denote by $\{e_1,\dots,e_n\}$ the vectors of the canonical basis of $\mathbb{F}^n$, and by $f_N:\mathbb{F}^n\to\mathbb{F}^n$ the endomorphism whose matrix with respect to the canonical basis is N, one can check that

$$f_N(e_j)\in\operatorname{span}\{e_1-e_n,e_2-e_{n-1},\dots,e_k-e_{n-k+1}\},\qquad j=1,\dots,n$$

and $f_N(e_i-e_{n-i+1})=0$, $i=1,\dots,k$, so $f_N^2=0$, i.e., N is really a square-zero matrix, as claimed.

Let us calculate the characteristic polynomial of the matrix $A+N$, i.e., let us calculate the determinant of the matrix $M=x\operatorname{Id}_n-(A+N)$. We will perform several elementary transformations on the rows and columns of M in order to obtain a matrix of the form of Lemma 2.2, and then we will use the formula for the determinant of such matrix proved in Lemma 2.2. To this purpose, we denote by $R(i)$ and $C(i)$ the respective i^{th}-row and the i^{th}-column of the involved matrices. For our calculations, we will assume that our matrix is a matrix with coefficients in the field of fractions of the polynomial ring possessing coefficients in $\mathbb{F}$.

Since it is not possible to explicitly write the generic matrix M, we will represent the transformations on the example $n=10$ and $k=4$. In this concrete case, we have:

$$A = \left(\begin{array}{cccc|cccccc} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&-u_0\\ 0&0&0&0&1&0&0&0&0&-u_1\\ 0&0&0&0&0&1&0&0&0&-u_2\\ 0&0&0&0&0&0&1&0&0&-u_3\\ 0&0&0&0&0&0&0&1&0&-u_4\\ 0&0&0&0&0&0&0&0&1&-u_5 \end{array}\right),$$

$$N = \begin{pmatrix} -1&0&0&-a_{1,7}&-a_{1,5}&-a_{1,6}&-a_{1,7}&0&0&-1\\ 0&-1&0&-a_{2,7}&-a_{2,5}&-a_{2,6}&-a_{2,7}&0&-1&0\\ 0&0&-1&-a_{3,7}&-a_{3,5}&-a_{3,6}&-a_{3,7}&-1&0&0\\ 0&0&0&-a_{4,7}&-a_{4,5}&-a_{4,6}&-a_{4,7}&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&a_{4,7}&a_{4,5}&a_{4,6}&a_{4,7}&0&0&0\\ 0&0&1&a_{3,7}&a_{3,5}&a_{3,6}&a_{3,7}&1&0&0\\ 0&1&0&a_{2,7}&a_{2,5}&a_{2,6}&a_{2,7}&0&1&0\\ 1&0&0&a_{1,7}&a_{1,5}&a_{1,6}&a_{1,7}&0&0&1 \end{pmatrix}, \text{ and}$$

$$M = \begin{pmatrix} x+1&0&0&a_{1,7}&a_{1,5}&a_{1,6}&a_{1,7}&0&0&1\\ 0&x+1&0&a_{2,7}&a_{2,5}&a_{2,6}&a_{2,7}&0&1&0\\ 0&0&x+1&a_{3,7}&a_{3,5}&a_{3,6}&a_{3,7}&1&0&0\\ 0&0&0&x+a_{4,7}&a_{4,5}&a_{4,6}&a_{4,7}&0&0&0\\ 0&0&0&0&x&0&0&0&0&u_0\\ 0&0&0&0&-1&x&0&0&0&u_1\\ 0&0&0&-a_{4,7}&-a_{4,5}&-a_{4,6}&x-a_{4,7}&0&0&u_2\\ 0&0&-1&-a_{3,7}&-a_{3,5}&-a_{3,6}&-a_{3,7}-1&x-1&0&u_3\\ 0&-1&0&-a_{2,7}&-a_{2,5}&-a_{2,6}&-a_{2,7}&-1&x-1&u_4\\ -1&0&0&-a_{1,7}&-a_{1,5}&-a_{1,6}&-a_{1,7}&0&-1&x-1+u_5 \end{pmatrix}.$$

(1) For $s = 1, 2, \ldots.k$, we add row $R(s)$ to row $R(n - s + 1)$:

$$\begin{pmatrix} x+1&0&0&a_{1,7}&a_{1,5}&a_{1,6}&a_{1,7}&0&0&1\\ 0&x+1&0&a_{2,7}&a_{2,5}&a_{2,6}&a_{2,7}&0&1&0\\ 0&0&x+1&a_{3,7}&a_{3,5}&a_{3,6}&a_{3,7}&1&0&0\\ 0&0&0&x+a_{4,7}&a_{4,5}&a_{4,6}&a_{4,7}&0&0&0\\ 0&0&0&0&x&0&0&0&0&u_0\\ 0&0&0&0&-1&x&0&0&0&u_1\\ 0&0&0&x&0&-1&x&0&0&u_2\\ 0&0&x&0&0&0&-1&x&0&u_3\\ 0&x&0&0&0&0&0&-1&x&u_4\\ x&0&0&0&0&0&0&0&-1&x+u_5 \end{pmatrix}$$

(2) For $s = 2, ...k$, we add $-\sum_{i=2}^{s} \frac{1}{x^{i-2}} C(n-s+i-1)$ to column $C(s)$:

$$\begin{pmatrix} x+1 & 0 & 0 & 0 & a_{1,5} & a_{1,6} & a_{1,7} & 0 & 0 & 1 \\ 0 & x & -1/x & -1/x^2 & a_{2,5} & a_{2,6} & a_{2,7} & 0 & 1 & 0 \\ 0 & 0 & x & -1/x & a_{3,5} & a_{3,6} & a_{3,7} & 1 & 0 & 0 \\ 0 & 0 & 0 & x & a_{4,5} & a_{4,6} & a_{4,7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & u_0 \\ 0 & 0 & 0 & 0 & -1 & x & 0 & 0 & 0 & u_1 \\ 0 & 0 & 0 & 0 & 0 & -1 & x & 0 & 0 & u_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & x & 0 & u_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & x & u_4 \\ x & 1 & 1/x & 1/x^2 & 0 & 0 & 0 & 0 & -1 & x+u_5 \end{pmatrix}$$

(3) For $s = 2, ...k$, we add $-\frac{1}{x^{s-1}} C(1)$ to column $C(s)$:

$$\begin{pmatrix} x+1 & -(x+1)/x & -(x+1)/x^2 & -(x+1)/x^3 & a_{1,5} & a_{1,6} & a_{1,7} & 0 & 0 & 1 \\ 0 & x & -1/x & -1/x^2 & a_{2,5} & a_{2,6} & a_{2,7} & 0 & 1 & 0 \\ 0 & 0 & x & -1/x & a_{3,5} & a_{3,6} & a_{3,7} & 1 & 0 & 0 \\ 0 & 0 & 0 & x & a_{4,5} & a_{4,6} & a_{4,7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & u_0 \\ 0 & 0 & 0 & 0 & -1 & x & 0 & 0 & 0 & u_1 \\ 0 & 0 & 0 & 0 & 0 & -1 & x & 0 & 0 & u_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & x & 0 & u_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & x & u_4 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & x+u_5 \end{pmatrix}$$

(4) For $s = k+1, \dots, n-1$, we add $\frac{1}{x} R(s)$ to row $R(s+1)$:

$$\begin{pmatrix} x+1 & -(x+1)/x & -(x+1)/x^2 & -(x+1)/x^3 & a_{1,5} & a_{1,6} & a_{1,7} & 0 & 0 & 1 \\ 0 & x & -1/x & -1/x^2 & a_{2,5} & a_{2,6} & a_{2,7} & 0 & 1 & 0 \\ 0 & 0 & x & -1/x & a_{3,5} & a_{3,6} & a_{3,7} & 1 & 0 & 0 \\ 0 & 0 & 0 & x & a_{4,5} & a_{4,6} & a_{4,7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & u_0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & \sum_{i=0}^{1} u_i/x^{1-i} \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & \sum_{i=0}^{2} u_i/x^{2-i} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & \sum_{i=0}^{3} u_i/x^{3-i} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & \sum_{i=0}^{4} u_i/x^{4-i} \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x+\sum_{i=0}^{5} u_i/x^{5-i} \end{pmatrix}$$

Let us denote by $v_s := \sum_{i=0}^{s} \frac{u_i}{x^{s-i}}$, $s = 0, \dots, n-k-1$.

(5) For $s = 3, \dots, k$, we add $\sum_{i=2}^{s-1} \frac{C(i)}{x^{s-i+1}}$ to column $C(s)$:

$$\begin{pmatrix} x+1 & -(x+1)/x & -(x+1)(1/x^2+1/x^3) & -(x+1)(1/x^3+2/x^4+1/x^5) & a_{1,5} & a_{1,6} & a_{1,7} & 0 & 0 & 1 \\ 0 & x & 0 & 0 & a_{2,5} & a_{2,6} & a_{2,7} & 0 & 1 & 0 \\ 0 & 0 & x & 0 & a_{3,5} & a_{3,6} & a_{3,7} & 1 & 0 & 0 \\ 0 & 0 & 0 & x & a_{4,5} & a_{4,6} & a_{4,7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & v_0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & v_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & v_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & v_4 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x+v_5 \end{pmatrix}$$

Let us now prove by induction on k that, for any $t \leq k$, in the above matrix the coefficient in position $(1,t)$ is $-z_t$, where

$$z_t := (x+1)\left(\sum_{r=0}^{t-2} \frac{\binom{t-2}{r}}{x^{t-1+r}}\right).$$

Indeed, for $k=1$ there is nothing to prove. Let us suppose that

$$z_i := (x+1)\left(\sum_{r=0}^{i-2} \frac{\binom{i-2}{r}}{x^{i-1+r}}\right)$$

for $i = 1, \dots, t-1$, then one computes that

$$\begin{aligned} z_t &= (x+1)/x^{t-1} + z_2/x^{t-1} + \cdots + z_{t-2}/x^3 + z_{t-1}/x^2 \\ &= (x+1)\left(1/x^{t-1} + \sum_{i=0}^{t-3}\sum_{j=0}^{i} \frac{\binom{i}{j}}{x^{t+j}}\right) = (x+1)\left(1/x^{t-1} + \sum_{i=0}^{t-3}\sum_{j=i}^{t-3-i} \frac{\binom{j}{i}}{x^{t+i}}\right) \\ &= (x+1)\left(1/x^{t-1} + \sum_{i=0}^{t-3} \frac{\sum_{j=i}^{t-3-i}\binom{j}{i}}{x^{t+i}}\right) = (x+1)\left(1/x^{t-1} + \sum_{i=0}^{t-3} \frac{\binom{t-2}{i+1}}{x^{t+i}}\right) \\ &= (x+1)\left(1/x^{t-1} + \sum_{i=1}^{t-2} \frac{\binom{t-2}{i}}{x^{t-1+i}}\right) = (x+1)\left(\sum_{r=0}^{t-2} \frac{\binom{t-2}{r}}{x^{t-1+r}}\right). \end{aligned}$$

(6) For $s = k+1, \dots, n-k+1$, we add $-\sum_{i=2}^{s-1} \frac{a_{i,s}}{x} C(i)$ to column $C(s)$:

$$\begin{pmatrix} x+1 & -z_2 & -z_3 & -z_4 & a_{1,5}+\sum_{i=2}^{k} a_{i,5}z_i/x & a_{1,6}+\sum_{i=2}^{k} a_{i,6}z_i/x & a_{1,7}+\sum_{i=2}^{k} a_{i,7}z_i/x & 0 & 0 & 1 \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & v_0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & v_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & v_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & v_4 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x+v_5 \end{pmatrix}$$

Let us denote by

$$w_s := a_{1,s} + \sum_{i=2}^{k} a_{i,s} z_i/x,$$

$s = k+1, \dots, n-k+1$.
(7) For $s = 2, \dots, k-1$, we add $-\frac{1}{x}C(s)$ to column $C(n-s+1)$:

$$\begin{pmatrix} x+1 & -z_2 & -z_3 & -z_4 & w_5 & w_6 & w_7 & z_3/x & z_2/x & 1 \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & v_0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & v_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & v_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & v_4 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x+v_5 \end{pmatrix}$$

Thereby, we have obtained a matrix equivalent to M which is of the form required in Lemma 2.2. Therefore, we get:

$$|M| = x^n + v_{n-k-1}x^{n-1} + v_{n-k-1}x^{n-2} + \sum_{s=k+1}^{n-k+1} w_s v_{s-k-1} x^{n-2}$$
$$+ \sum_{i=1}^{k-2} \frac{z_{k-i}}{x} v_{n-2k+i} x^{n-2}.$$

Next, substituting the $v's$, $w's$ and $z's$ in the above expression, we conclude that

$$|M| = x^n + \sum_{i=0}^{n-k-1} u_i x^{k+i} + \sum_{i=0}^{n-k-1} u_i x^{k+i-1}$$
$$+ \sum_{s=k+1}^{n-k+1} \left(a_{1,s} + \sum_{i=2}^{k} \sum_{r=0}^{i-2} \binom{i-2}{r} a_{i,s} \frac{1}{x^{i+r-2}} \right.$$
$$\left. + \sum_{i=2}^{k} \sum_{r=0}^{i-2} \binom{i-2}{r} a_{i,s} \frac{1}{x^{i+r-1}} \right) \left(\sum_{j=0}^{s-k-1} u_j / x^{s-k-1-j} \right) x^{n-2}$$
$$+ \sum_{i=1}^{k-2} \frac{(1+x)}{x} \left(\sum_{r=0}^{k-i-2} \frac{\binom{k-i-2}{r}}{x^{k-i-1+r}} \right) \left(\sum_{j=0}^{n-2k+i} u_j / x^{n-2k+i-j} \right) x^{n-2}$$

$$= x^n + \sum_{i=0}^{n-k-1} u_i x^{k+i} + \sum_{i=0}^{n-k-1} u_i x^{k+i-1} \tag{1}$$

$$+ \sum_{s=k+1}^{n-k+1} \sum_{j=0}^{s-k+1} a_{1,s} u_j x^{n+k+j-s-1} \tag{2}$$

$$+ \sum_{s=k+1}^{n-k+1} \sum_{i=2}^{k} \sum_{r=0}^{i-2} \sum_{j=0}^{s-k+1} \binom{i-2}{r} a_{i,s} u_j x^{n+k+j-i-r-s-1} \tag{3}$$

$$+ \sum_{s=k+1}^{n-k+1} \sum_{i=2}^{k} \sum_{r=0}^{i-2} \sum_{j=0}^{s-k+1} \binom{i-2}{r} a_{i,s} u_j x^{n+k+j-i-r-s} \tag{4}$$

$$+ \sum_{i=1}^{k-2} \sum_{r=0}^{k-i-2} \sum_{j=0}^{n-2k+i} \binom{k-i-2}{r} u_j x^{k+j-r-2} \tag{5}$$

$$+ \sum_{i=1}^{k-2} \sum_{r=0}^{k-i-2} \sum_{j=0}^{n-2k+i} \binom{k-i-2}{r} u_j x^{k+j-r-1}. \tag{6}$$

From now on, we are going to focus on the elements $a_{i,s}$ and we are going to determine the monomial of smallest degree, among the monomials appearing in the determinant of M, where each $a_{i,s}$ appears. Recall that $u_0 \neq 0$.

• $\underline{i = 1}$. The element $a_{1,s}$ appears only in the summands labeled by (2)

$$\sum_{s=k+1}^{n-k+1} \sum_{j=0}^{s-k+1} a_{1,s} u_j x^{n+k+j-s-1}. \tag{2}$$

To get the monomial of smallest degree, the index j has to be equal to 0:

$$\sum_{s=k+1}^{n-k+1} a_{1,s} u_0 x^{n+k-s-1},$$

hence each $a_{1,s}$ appears in the monomial $a_{1,s} u_0 x^{n+k-s-1}$, $s = k+1, \ldots, n-k+1$.

Let us represent the degrees of such monomials in the following table:

degree	$s = k+1$	$\ldots$	$\ldots$	$s = n-k+1$
$i = 1$	$n-2$	$\ldots$	$\ldots$	$2k-2$

• $\underline{i > 1}$. Set $i > 1$ and $s \in \{k+1, \ldots, n-k+1\}$ and let us determine the monomial of smallest degree where $a_{i,s}$ appears: it only appears in summands labeled by (3) and (4) in the expansion of the determinant of M and, consequently, it is in the summand labeled by (3) where it appears with the smallest degree (notice that the summand labeled by (4) coincides with the summand (3) multiplied by x):

$$\sum_{s=k+1}^{n-k+1} \sum_{i=2}^{k} \sum_{r=0}^{i-2} \sum_{j=0}^{s-k+1} \binom{i-2}{r} a_{i,s} u_j x^{n+k+j-i-r-s-1}. \tag{3}$$

There, the index j has to be as small as possible (whence $j = 0$) and r has to be as large as possible (so $r = i - 2$):

$$\sum_{s=k+1}^{n-k+1} \sum_{i=2}^{k} \binom{i-2}{i-2} a_{i,s} u_0 x^{n+k-2i-s+1},$$

hence each $a_{i,s}$ appears in the monomial $u_0 x^{n+k-2i-s+1}$. Representing the degrees of such monomials in a table we get

degree	$s = k+1$	$\dots$	$\dots$	$s = n-k+1$
$i > 1$	$n - 2i$	$\dots$	$\dots$	$2k - 2i$

Let us now collect all the information about the degrees in a single table:

degree	$s = k+1$	$s = k+2$	$\dots$	$s = n-k-1$	$s = n-k$	$s = n-k+1$
$i = 1$	$n-2$	$n-3$	$\dots$	$2k$	$2k-1$	$2k-2$
$i = 2$	$n-4$	$n-5$	$\dots$	$2k-2$	$2k-3$	$2k-4$
$i = 3$	$n-6$	$n-7$	$\dots$	$2k-4$	$2k-5$	$2k-6$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$i = k-2$	$n-2k+4$	$n-2k+3$	$\dots$	6	5	4
$i = k-1$	$n-2k+2$	$n-2k+1$	$\dots$	4	3	2
$i = k$	$n-2k$	$n-2k-1$	$\dots$	2	1	0

If we concentrate on the last row and the first two columns of this table, we obtain all the possible degrees from 0 to $n - 2$.

For $i = 1, \dots, k-1$ and $s = k+1, k+2$, as well as for $i = k$ and $s = k+1, \dots, n-k+1$, let us define the polynomial $f_{i,s}$ consisting of the sum of all the monomials of the determinant of M which contain the variable $a_{i,s}$. Thus, $\{f_{i,s}\}$ is a family of $n - 2$ polynomials of smallest degree from 0 to $n - 2$, which implies that the set $\{f_{i,s}\}$ forms a basis of the vector space of polynomials of degree less than or equal to $n - 2$ over $\mathbb{F}$.

Therefore, for any monic polynomial $q(x)$ of degree n whose trace coincides with the trace of $p(x)$, we can choose appropriate elements $a_{i,s}$ and use such basis to get $q(x)$ as characteristic polynomial of $A + N$, which completes the proof. □

We will show now that the restriction $k < n - k$ in the previous theorem is unremovable. Before giving a counter-example when $n = 2k$, let us study how square-zero matrices $N \in \mathbb{M}_{2k}(\mathbb{F})$ must be in order to get an invertible matrix $A + N$ for

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,k} \\ \hline \mathbf{0}_{k,k} & C \end{array}\right) \in \mathbb{M}_{2k}(\mathbb{F}),$$

where C is invertible and non-derogatory.

The following assertion is pivotal for our purpose mentioned above.

Proposition 2.4. *For any matrix of the form*

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,k} \\ \hline \mathbf{0}_{k,k} & C(p(x)) \end{array}\right) \in \mathbb{M}_{2k}(\mathbb{F}),$$

where $C(p(x))$ is the companion matrix of a degree k polynomial whose zero-coefficient is non-zero and any square-zero matrix $N \in \mathbb{M}_{2k}(\mathbb{F})$ such that $A+N$ is invertible, there exists a basis of $\mathbb{F}^n$ with respect to which the matrix A remains the same and N has the form

$$N = \left(\begin{array}{c|c} X & -X^2 \\ \hline \mathrm{Id}_k & -X \end{array}\right)$$

for some $X \in \mathbb{M}_k(\mathbb{F})$.

Proof. Suppose that

$$C(p(x)) = \begin{pmatrix} 0 & \dots & 0 & -u_0 \\ 1 & & 0 & \vdots \\ & \ddots & & -u_{n-2} \\ 0 & & 1 & -u_{n-1} \end{pmatrix} \in \mathbb{M}_k(\mathbb{F})$$

with $u_0 \neq 0$ and

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,k} \\ \hline \mathbf{0}_{k,k} & C(p(x)) \end{array}\right) \in \mathbb{M}_{2k}(\mathbb{F}).$$

Let $N = (n_{i,j})$ be any square-zero matrix such that $A+N$ is invertible. Then, the matrix $(A+N)^2 = A^2 + AN + NA$ is also invertible and has the form

$$(A+N)^2 = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & * \\ \hline M & * \end{array}\right),$$

where M is given by:

$$M = \begin{pmatrix} -u_0 n_{2k,1} & -u_0 n_{2k,2} & \dots & \dots & -u_0 n_{2k,k} \\ n_{k+1,1} - u_1 n_{2k,1} & n_{k+1,2} - u_1 n_{2k,2} & \dots & \dots & n_{k+1,k} - u_1 n_{2k,k} \\ n_{k+2,1} - u_2 n_{2k,1} & n_{k+2,2} - u_2 n_{2k,2} & \dots & \dots & n_{k+2,k} - u_2 n_{2k,k}) \\ \vdots & \vdots & & & \vdots \\ n_{2k-1,1} - u_{k-1} n_{2k,1} & n_{2k-1,2} - u_{k-1} n_{2k,2} & \dots & \dots & n_{2k-1,2} - u_{k-1} n_{2k,2} \end{pmatrix}.$$

Since $u_0 \neq 0$, from the form of M we obtain that the block matrix

$$N_{21} = \begin{pmatrix} n_{k+1,1} & n_{k+1,2} & \cdots & n_{k+1,k} \\ n_{k+2,1} & n_{k+2,2} & \cdots & n_{k+2,k} \\ \vdots & & & \vdots \\ n_{2k,1} & n_{2k,2} & & n_{2k,k} \end{pmatrix} \text{ in } N = \left(\begin{array}{c|c} N_{11} & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right)$$

is invertible too.

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{F}^n$, and consider the subspace S of $\mathbb{F}^n$ generated by $e_1, \ldots, e_k$; since the block N_{21} is invertible, there exist linear combinations of the first k-columns of N that give rise to $e_{k+1}, e_{k+2}, \ldots, e_{2k}$ modulo S, i.e., there exist scalars $\alpha_{ij} \in \mathbb{F}$ such that

$$(\alpha_{1i} N^{(1)} + \cdots + \alpha_{ki} N^{(k)}) + S = e_{k+i} + S, \quad i = 1, \ldots, k,$$

where $N^{(1)}, \ldots, N^{(k)}$ denote the first k-columns of N. Define the basis

$$\{e'_1, \ldots, e'_k, e_{k+1}, \ldots, e_{2k}\},$$

where

$$e'_i = \alpha_{1i} e_1 + \cdots + \alpha_{ki} e_k, \quad i = 1, \ldots, k.$$

With respect to this basis, the matrix N is of the form

$$N = \left(\begin{array}{c|c} * & * \\ \hline \mathrm{Id}_k & * \end{array}\right)$$

and the matrix A remains as

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,k} \\ \hline \mathbf{0}_{k,k} & C(p(x)) \end{array}\right) \in \mathbb{M}_{2k}(\mathbb{F}).$$

Suppose that

$$N = \left(\begin{array}{c|c} X & Y \\ \hline \mathrm{Id}_k & Z \end{array}\right).$$

From the condition $N^2 = 0$, we easily infer that $Y = -X^2$ and $Z = -X$, i.e.,

$$N = \left(\begin{array}{c|c} X & -X^2 \\ \hline \mathrm{Id}_k & -X \end{array}\right)$$

for some $X \in \mathbb{M}_k(\mathbb{F})$. $\square$

We are now ready to give the promised counter-example.

Remark 2.5. Let $\mathbb{F} = \mathbb{R}$, the field of real numbers. Consider the matrix

$$A = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \in \mathbb{M}_4(\mathbb{F}).$$

We claim that there does not exist any square-zero matrix N such that the characteristic polynomial of $A + N$ is $x^4 + 1$. Otherwise, according to Proposition 2.4, we can suppose that N is of the form

$$N = \left(\begin{array}{c|c} X & -X^2 \\ \hline \mathrm{Id}_2 & -X \end{array}\right) \text{ for some } X = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}.$$

It is readily checked that the characteristic polynomial of $A + N$ is

$$p_{A+N}(x) = x^4 + (n_{12} - n_{21} + 1)x^2 - (n_{11} + n_{22})x + n_{11}n_{22} - n_{12}n_{21}$$

and, if this polynomial equals $x^4 + 1$, then

$$\begin{cases} n_{12} - n_{21} + 1 = 0 \\ n_{11} + n_{22} = 0 \\ n_{11}n_{22} - n_{12}n_{21} = 1 \end{cases}$$

so that $n_{22} = -n_{11}$ and $n_{21} = n_{12} + 1$ would imply $1 = -n_{11}^2 - n_{12}(n_{12} + 1) = -n_{11}^2 - n_{12}^2 - n_{12}$; therefore,

$$n_{11}^2 = -n_{12}^2 - n_{12} - 1,$$

but, for any real number, the left hand side of this equality is positive, while the right hand side of this equality is less than or equal to $-3/4$, which is an absurd.

Besides, one also observes that:

Remark 2.6. In Remark 2.1, we showed how to modify an invertible non-derogatory matrix A by adding a square-zero matrix N to obtain a matrix $A+N$ with any prescribed characteristic polynomial with the same trace as A. In addition, by construction, the resulting matrix $A + N$ was again non-derogatory. With that idea in mind, we close this section by posing the following problem:

Question. Given a matrix of the form

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,n-k} \\ \hline \mathbf{0}_{n-k,k} & A_{22} \end{array}\right) \in \mathbb{M}_n(\mathbb{F})$$

consisting of k rows and columns of zeros and an invertible non-derogatory matrix A_{22}, and a prescribed polynomial $q(x)$ of degree n whose trace coincides with the trace of A, is it possible to find a square-zero matrix such that $A + N$ is *non-derogatory* and the characteristic polynomial of $A + N$ coincides with $q(x)$?

3. Some consequences

In the present section, some results related to our project of decomposing a square matrix A into the sum of a square-zero matrix N and another matrix satisfying some fundamental properties such as been diagonalizable, invertible, torsion, etc. (see, for example, [3], [4], [5] and [6]) can be obtained by forcing that $A - N$ verifies a certain prescribed characteristic polynomial. Specifically, the key point in [5] when showing that any nilpotent matrix of rank at least $\frac{n}{2}$ can be expressed as the sum of a torsion matrix and a square-zero matrix was the fact that the original nilpotent matrix could be written as the direct sum of blocks, each of them expressed as the sum of a square-zero matrix and a matrix satisfying a certain characteristic polynomial that forced this last matrix to be torsion.

So, with the aid of Theorem 2.3, we can elementarily prove the following three corollaries.

Corollary 3.1. *Let $\mathbb{F}$ be a field, let $n, k \in \mathbb{N}$ with $k < n - k$, and consider the block matrix*

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,n-k} \\ \hline \mathbf{0}_{n-k,k} & A_{22} \end{array}\right) \in \mathbb{M}_n(\mathbb{F})$$

consisting of k rows and columns of zeros and an invertible non-derogatory matrix A_{22}. Then, if the field $\mathbb{F}$ has enough elements, there exists a square-zero matrix N such that $A + N$ is diagonalizable (with n different eigenvalues).

Proof. If $\mathbb{F}$ has enough elements, we can take different elements $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that $-(\alpha_1 + \cdots + \alpha_n)$ equals the trace of A. Then fix the polynomial $q(x) = (x - \alpha_1) \cdots (x - \alpha_n)$ and use Theorem 2.3 to find a square-zero matrix N such that the characteristic polynomial of $A + N$ is $q(x)$. Since $A + N$ has n different eigenvalues, $A + N$ is diagonalizable, as asserted. □

Corollary 3.2. *Let $\mathbb{F}$ be a field, let $n, k \in \mathbb{N}$ with $k < n-k$, and consider the block matrix*

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,n-k} \\ \hline \mathbf{0}_{n-k,k} & A_{22} \end{array}\right) \in \mathbb{M}_n(\mathbb{F})$$

consisting of k rows and columns of zeros and an invertible matrix A_{22}. Then, there exists an invertible matrix T and a square-zero matrix N such that $A = T + N$.

Proof. Without loss of generality, assume that A_{22} is the direct sum of the invertible and non-derogatory companion matrices of its elementary divisors. Combine the 0's in the main diagonal of A with each of these invertible and non-derogatory companion matrices and utilize Theorem 2.3 to get characteristic polynomials with nonzero zero-degree coefficients by adding appropriate square-zero matrices, as desired. □

Recall that a matrix $P \in \mathbb{M}_n(\mathbb{F})$ is *m-potent* if $P^m = P$ for some $1 < m \in \mathbb{N}$; a matrix $T \in \mathbb{M}_n(\mathbb{F})$ is *k-torsion* (or just a *k-root of the unity*) if $T^k = \mathrm{Id}$ for some $k \in \mathbb{N}$.

Corollary 3.3. *Let $\mathbb{F}$ be a field, let $n, k \in \mathbb{N}$ with $k < n-k$, and consider the block matrix*

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,n-k} \\ \hline \mathbf{0}_{n-k,k} & A_{22} \end{array}\right) \in \mathbb{M}_n(\mathbb{F})$$

consisting of k rows and columns of zeros and an invertible non-derogatory matrix A_{22}. If the trace of A is zero, then the following two statements are true:

(i) *if $n \geq 3$, there exists an n-potent matrix P and a square-zero matrix N_1 such that $A = P + N_1$;*
(ii) *if $n \geq 2$, there exists an n-torsion matrix T and a square-zero matrix N_2 such that $A = T + N_2$;*

Proof. One observes that it suffices to prescribe the zero-trace polynomials $p_1(x) = x^n - x$ and $p_2(x) = x^n - 1$ and employ Theorem 2.3 to get square-zero matrices N_1, N_2 such that $A + N_i$ has characteristic polynomial equal to $p_i(x)$, $i = 1, 2$. Then, decompose $A = (A + N_1) - N_1$ to get (i) and $A = (A + N_2) - N_2$ to get (ii), as wanted. □

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Declaration of competing interest

None declared.

Data availability

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