WEIGHTED INEQUALITIES FOR THE ONE-SIDED GEOMETRIC MAXIMAL OPERATORS

PEDRO ORTEGA SALVADOR AND CONSUELO RAMÍREZ TORREBLANCA

Abstract. We characterize the pairs of weights (u, v) such that the one-sided geometric maximal operator G^+, defined for functions f of one real variable by

$$G^+f(x) = \sup_{h>0} \exp\left(\frac{1}{h}\int_x^{x+h} \log|f|\right),$$

verifies the weak-type inequality

$$\int_{\{x\in\mathbb{R}:G^+f(x)>\lambda\}} u \leq \frac{C}{\lambda^p}\int_0^\infty |f|^p v$$

or the strong type inequality

$$\int_{\mathbb{R}} (G^+f)^p u \leq C\int_{\mathbb{R}} |f|^p v$$

for $0 < p < \infty$.

1. Introduction and results

In 1996, X. Yin and B. Muckenhoupt ([6]) studied the weighted weak and strong type inequalities for the geometric maximal operator G defined by

$$Gf(x) = \sup_{s,h>0} \exp\left(\frac{1}{s+h}\int_{x-s}^{x+h} \log|f|\right).$$

Specifically, Yin and Muckenhoupt proved the following results:

Theorem A. *Let u, v be positive measurable functions. The following statements are equivalent:*

(i) *There exists $C > 0$ such that the inequality*

$$\int_{\{x\in\mathbb{R}:Gf(x)>\lambda\}} u \leq \frac{C}{\lambda^p}\int_{\mathbb{R}} |f|^p v$$

holds for all $p > 0$, $f \in L^p(v)$ and $\lambda > 0$.

(ii) *There exists $C > 0$ such that*

$$\left(\frac{1}{|I|}\int_I u\right)\exp\left(\frac{1}{|I|}\int_I \log(v^{-1})\right) \leq C$$

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Geometric maximal operator, geometric mean operator, weighted inequalities, weights.

This research has been supported in part by MEC, grant MTM 2005-08350-C03-02, and Junta de Andalucía, grant FQM354.

for all bounded intervals I.

Theorem B. *Let u, v be positive measurable functions. The following statements are equivalent:*

(i) *There exists $C > 0$ such that the inequality*
$$\int_{\mathbb{R}} (Gf)^p u \leq C \int_{\mathbb{R}} |f|^p v$$
holds for all $p > 0$ and $f \in L^p(v)$.

(ii) *There exists $C > 0$ such that*
$$\int_I G(v^{-1}\chi_I)u \leq C|I|$$
for all bounded intervals I.

Theorem C. *Let w be a positive measurable function. The following statements are equivalent:*

(i) *There exists $C > 0$ such that the inequality*
$$\int_{\mathbb{R}} (Gf)^p w \leq C \int_{\mathbb{R}} |f|^p w$$
holds for all $p > 0$ and $f \in L^p(w)$.

(ii) *There exists $C > 0$ such that*
$$\int_{\{x\in\mathbb{R}:Gf(x)>\lambda\}} w \leq \frac{C}{\lambda^p} \int_{\mathbb{R}} |f|^p w$$
holds for all $p > 0$, $\lambda > 0$ and $f \in L^p(w)$.

(iii) *$w \in A_\infty$, i.e., there exists $C > 0$ such that*
$$\left(\frac{1}{|I|}\int_I w\right) \exp\left(\frac{1}{|I|}\int_I \log(w^{-1})\right) \leq C$$
for all bounded intervals I.

Really, the equivalence of (i) and (iii) in Theorem C was previously proved by X. Shi ([5]).

The goal of this paper is to characterize the weighted inequalities for the one-sided geometric maximal operators G^+ and G^- defined respectively by
$$G^+f(x) = \sup_{h>0} \exp\left(\frac{1}{h}\int_x^{x+h} \log|f|\right)$$
and
$$G^-f(x) = \sup_{h>0} \exp\left(\frac{1}{h}\int_{x-h}^{x} \log|f|\right).$$

The operators G^+ and G^- are related to G by the inequalities
$$\frac{1}{2}(G^+ + G^-) \leq G \leq G^+ + G^-$$
in such a way that the characterizations of the weighted weak and strong type inequalities for G^+ and G^- give immediately characterizations for G.

As for the two-sided operator, the characterizations of the weak and strong type inequalities for G^+ in the case of equal weights involve the one-sided A_∞ condition. This condition, called A_∞^+, was introduced by F. J. Martín-Reyes, L. Pick and A. de la Torre in [2]. A weight w belongs to the class A_∞^+ if there exist two positive constants C and δ such that for all $a<b<c$ and all measurable sets $E\subset(a,b)$ the inequality

$$\frac{|E|}{c-a}\le C\left(\frac{w(E)}{w(a,b)}\right)^{\delta}$$

holds.

The class A_∞^+ is closely related to the weighted inequalities for the one-sided maximal operator M^+ defined for functions f of one real variable by

$$M^+f(x)=\sup_{h>0}\frac{1}{h}\int_x^{x+h}|f|.$$

The weighted inequalities for M^+ were characterized by E. Sawyer in [4]. Sawyer showed that M^+ is bounded in $L^p(w)$, $p>1$, if and only if M^+ is of weak type (p,p) with respect to the measure wdx and this is also equivalent to the weight w verifies the so-called A_p^+ condition, which means that

$$\sup_{a<b<c}\left(\frac{1}{c-a}\int_a^b w\right)^{\frac{1}{p}}\left(\frac{1}{c-a}\int_b^c w^{1-p'}\right)^{\frac{1}{p'}}<\infty,$$

where p' stands for the conjugate exponent of p, i.e., $\frac{1}{p}+\frac{1}{p'}=1$.

The weighted weak type $(1,1)$ inequality for M^+ is characterized by condition A_1^+, which means

$$\sup_{h>0}\frac{1}{h}\int_{x-h}^{x} w\le Cw(x)\qquad\text{a.e.}$$

In [2], Martín-Reyes, Pick and de la Torre showed that $w\in A_\infty^+$ if and only if $w\in A_p^+$ for some $p\ge 1$. Later, D. Cruz-Uribe, C. J. Neugebauer and V. Olesen gave in [1] a new characterization of A_∞^+. They proved that $w\in A_\infty^+$ if and only if $w\in W_p^+$ for all $0<p<\infty$, where W_p^+ means that there exists $C>0$ such that if $I=(a,b)$ and $I^-=(a,\frac{a+b}{2})$ then

$$\frac{1}{|I^-|}\int_{I^-}w\le C\left(\frac{1}{|I|}\int_I w^{\frac{1}{p+1}}\right)^{p+1}.$$

As it was showed in [1], the condition W_p^+ characterizes the weighted weak type inequality

$$w(\{x\in\mathbb{R}:\mathsf{m}^+f(x)<\frac{1}{\lambda}\})\le\frac{C}{\lambda^p}\int_{\mathbb{R}}\frac{w}{|f|^p},$$

where m^+ is the one-sided minimal operator defined by

$$\mathsf{m}^+f(x)=\inf_{h>0}\frac{1}{h}\int_x^{x+h}|f|.$$

Although neither explicitly stated nor proved, the paper [1] contains the result corresponding to Theorem C for the operator G^+. It is the following one:

Theorem 1. *Let w be a positive measurable function. The following statements are equivalent:*

(i) *There exists $C > 0$ such that*

$$\int_{\mathbb{R}} (G^+ f)^p w \le C \int_{\mathbb{R}} |f|^p w$$

holds for all $p > 0$ and $f \in L^p(w)$.

(ii) *There exists $C > 0$ such that*

$$\int_{\{x\in\mathbb{R}: G^+ f(x) > \lambda\}} w \le \frac{C}{\lambda^p} \int_{\mathbb{R}} |f|^p w$$

holds for all $p > 0$, $f \in L^p(w)$ and $\lambda > 0$.

(iii) $w \in A^+_\infty$

In this paper, we focus our attention on the weak and strong type inequalities with possibly different weights. In this sense, we prove two theorems. The first one characterizes the weighted weak type inequality for G^+. Before stating it, we recall the definition of the weak L^1 norm:

$$\|f\|_{1,\infty;u} = \sup_{\lambda>0} \lambda \int_{\{x\in\mathbb{R}: |f(x)|>\lambda\}} u.$$

Theorem 2. *Let $s_0 > 1$ and let u and v be positive measurable functions. The following statements are equivalent:*

(i) *There exists $C > 0$ such that*

$$\int_{\{x\in\mathbb{R}: G^+ f(x) > \lambda\}} u \le \frac{C}{\lambda^p} \int_{\mathbb{R}} |f|^p v$$

holds for all $p > 0$, $f \in L^p(v)$ and $\lambda > 0$.

(ii) $B_{s_0} < \infty$, *where*

$$B_{s_0} = \sup_{a<t<b} (b-t)^{s_0-1} \|\chi_{(a,t)}(x)(b-x)^{-s_0}\beta(x)\|_{1,\infty;u}$$

and $\beta(x) = \exp\left(\frac{1}{b-x}\int_x^b \log(v^{-1})\right)$.

Theorem 3. *Let u and v be positive measurable functions. The following statements are equivalent:*

(i) *There exists $C > 0$ such that*

$$\int_{\mathbb{R}} (G^+ f)^p u \le C \int_{\mathbb{R}} |f|^p v$$

holds for all $p > 0$ and $f \in L^p(v)$.

(ii) *There exists $C > 0$ such that*

$$\int_I G^+(\chi_I v^{-1}) u \le C|I|$$

for all bounded intervals I.

By combining Theorems 1 and 2 we find a new characterization of A_∞^+.

Corollary 1. *Let $s_0 > 1$ and w a positive measurable function. Then $w \in A_\infty^+$ if and only if*

$$\sup_{a<t<b} (b-t)^{s_0-1} \|\chi_{(a,t)}(x)(b-x)^{-s_0}\beta(x)\|_{1,\infty;w} < \infty,$$

where $\beta(x) = \exp\left(\frac{1}{b-x}\int_x^b \log(w^{-1})\right)$.

For the proofs we will need two lemmas. The first one is a straightforward version of the sun rising lemma for geometric means.

Lemma 1. *Let f be a nonnegative locally integrable function and $\lambda > 0$. Let $O_\lambda = \{x \in \mathbb{R} : Gf(x) > \lambda\}$. Let $\{(a_j, b_j)\}_j$ be the countable collection of the connected components of O_λ. If $I_j = (a_j, b_j)$ verifies $b_j < \infty$, then*

$$\exp\left(\frac{1}{b_j - x}\int_x^{b_j} \log|f|\right) > \lambda$$

for all $x \in I_j$.

The second lemma we will apply characterizes the weighted weak type $(1,1)$ inequality for the geometric mean operator.

Lemma 2. *([3]) Let $-\infty < a < b < \infty$, u, v positive measurable functions on (a,b) and $s_0 > 1$. The following statements are equivalent:*

(i) *There exists $C > 0$ such that the inequality*

$$\int_{\{x\in(a,b):\exp\left(\frac{1}{b-x}\int_x^b \log f\right)>\lambda\}} u \le \frac{C}{\lambda}\int_a^b fv$$

holds for all $\lambda > 0$ and all positive f.

(ii)

$$\sup_{a<t<b} (b-t)^{s_0-1} \|\chi_{(a,t)}(x)(b-x)^{-s_0}\beta(x)\|_{1,\infty;u} < \infty,$$

where $\beta(x) = \exp\left(\frac{1}{b-x}\int_x^b \log(v^{-1})\right)$.

The remaining sections of the paper are devoted to the proofs of Theorems 2 and 3.

2. Proof of Theorem 2

(ii) $\Rightarrow$ (i). Let f be a compactly supported nonnegative function in $L^1(v)$ and $\lambda > 0$. Let $O_\lambda = \{x \in \mathbb{R} : G^+f(x) > \lambda\}$. Let $\{(a_j, b_j)\}_{j\in\mathbf{N}}$ be the collection of the connected components of O_λ. By Lemma 1, we know that if $x \in (a_j, b_j)$, then

$$\lambda < \exp\left(\frac{1}{b_j - x}\int_x^{b_j} \log f\right).$$

Since condition (ii) holds, we may apply Lemma 2 to each operator T_j defined by

$$T_j f(x) = \exp\left(\frac{1}{b_j - x}\int_x^{b_j} \log f\right),$$

obtaining

$$\int_{O_\lambda} u = \sum_j \int_{a_j}^{b_j} u = \sum_j \int_{\{x\in(a_j,b_j):T_jf(x)>\lambda\}} u \le \sum_j \frac{C}{\lambda}\int_{a_j}^{b_j} fv \le \frac{C}{\lambda}\int_{\mathbb{R}} fv.$$

(i) ⇒ (ii). Let $a<b$ and let f be a positive function supported on (a,b). If $z\in(a,b)$, then

$$G^+f(z) \ge \exp\left(\frac{1}{b-z}\int_z^b \log f\right).$$

Since (i) holds, we have

$$\int_{\{x\in(a,b):\exp\left(\frac{1}{b-x}\int_x^b \log f\right)>\lambda\}} u \le \int_{\{x\in\mathbb{R}:G^+f(x)>\lambda\}} u \le \frac{C}{\lambda}\int_a^b fv.$$

Hence, we have seen that the geometric mean operator

$$G_{a,b}f(x) = \exp\left(\frac{1}{b-x}\int_x^b \log f\right)$$

verifies (i) in Lemma 2 with a constant independent of a and b. Therefore, by Lemma 2, (ii) holds.

3. Proof of Theorem 3

(i) ⇒ (ii). It suffices to test *(i)* with the function $f=\chi_I v^{-1}$.

(ii) ⇒ (i). Let f be a compactly supported positive function and let k_0 be a fixed integer. Let, for every $k\in\mathbb{Z}$, $O_k=\{x\in\mathbb{R}: G^+f(x)>2^k\}$. By Lemma 1, there exists a collection $\{I_{jk}\}_j$ of pairwise disjoint intervals such that $O_k=\cup_j I_{jk}$ and if $x\in I_{jk}=(a_{jk},b_{jk})$, then

$$\exp\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}} \log f\right) > 2^k.$$

Let $E_{jk}=I_{jk}\cap\{x\in\mathbb{R}: G^+f(x)\le 2^{k+1}\}$. The sets E_{jk} are pairwise disjoint and for all k, $\cup_j E_{jk}=\{x\in\mathbb{R}: 2^k<G^+f(x)\le 2^{k+1}\}$. Then, by Jensen's inequality

$$\sum_{k\ge k_0}\int_{\{x\in\mathbb{R}:2^k<G^+f(x)\le 2^{k+1}\}} (G^+f)^p u = \sum_{k\ge k_0}\sum_j \int_{E_{jk}} (G^+f)^p u$$
$$\le 2^p \sum_{k\ge k_0}\sum_j \int_{E_{jk}} u(x)\exp\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}}\log f\right)^p dx$$
$$= 2^p \sum_{k\ge k_0}\sum_j \int_{E_{jk}} u(x)\left[\exp\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}}\log(fv^{\frac{1}{p}})\right)\right]^p$$
$$\times\exp\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}}\log(v^{-1})\right)dx$$
$$\le 2^p \sum_{k\ge k_0}\sum_j \int_{E_{jk}} u(x)\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}} fv^{\frac{1}{p}}\right)^p \exp\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}}\log(v^{-1})\right)dx.$$

Let $X = \mathbb{Z} \times \{k \in \mathbb{Z} : k \geq k_0\} \times \mathbb{R}$ and $w = \nu \times \nu \times m$, where ν stands for the counting measure in the integers and m designs the Lebesgue measure. Let $\varphi(j,k,x) = \chi_{E_{jk}}(x)u(x)\exp\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}}\log(v^{-1})\right)$ and let T be the operator defined by $Th(j,k,x) = \frac{1}{b_{jk}-x}\int_x^{b_{jk}} h$. Then the above inequality can be written as

$$\sum_{k\geq k_0}\int_{\{x\in\mathbb{R}:2^k<G^+f(x)\leq 2^{k+1}\}}(G^+f)^p u \leq 2^p\int_X\left(T(fv^{\frac{1}{p}})\right)^p\varphi dw.$$

If we show that T is bounded from $L^p(\mathbb{R},m)$ to $L^p(X,\varphi dw)$, we get

$$\sum_{k\geq k_0}\int_{\{x\in\mathbb{R}:2^k<G^+f(x)\leq 2^{k+1}\}}(G^+f)^p u \leq C\int_{\mathbb{R}} f^p v. \tag{3.1}$$

Since the operator T is sublinear and bounded from $L^\infty(\mathbb{R})$ to $L^\infty(X,\varphi dw)$, by Marcinkiewicz's interpolation Theorem it suffices to see that T is of weak type $(1,1)$. We have to prove that there exists $C>0$ such that for all h and all $\lambda>0$

$$\int_{\left\{(j,k,x)\in X:\frac{1}{b_{jk}-x}\int_x^{b_{jk}}h>\lambda\right\}}\varphi dw \leq \frac{C}{\lambda}\int_{\mathbb{R}} h.$$

Let $\lambda>0$ and for each j,k let $A_{jk}(\lambda) = E_{jk}\cap\{x\in\mathbb{R} : Th(j,k,x)>\lambda\}$. The sets $A_{jk}(\lambda)$ are pairwise disjoint. Let $s_{jk}(\lambda) = s_{jk} = \inf A_{jk}(\lambda)$ and $J_{jk}(\lambda) = J_{jk} = (s_{jk}, b_{jk})$. Two intervals of the family J_{jk} are either disjoint or one of them is contained in the other one. They also verify

$$\frac{1}{|J_{jk}|}\int_{J_{jk}} h \geq \lambda. \tag{3.2}$$

Let $\{J_i\}$ be the collection of the maximal elements of the family $\{J_{jk}\}$ (they exist since $k\geq k_0$). Then applying *(ii)* and (3.2) we have

$$\begin{aligned}
&\int_{\left\{(j,k,x)\in X:\frac{1}{b_{jk}-x}\int_x^{b_{jk}}h>\lambda\right\}}\varphi dw = \sum_{j,k}\int_{A_{jk}} u(x)\exp\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}}\log(v^{-1})\right)dx\\
&= \sum_i\sum_{\{(j,k):J_{jk}\subset J_i\}}\int_{A_{jk}} u(x)\exp\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}}\log(v^{-1})\right)dx\\
&\leq \sum_i\sum_{\{(j,k):J_{jk}\subset J_i\}}\int_{A_{jk}} u(x)G^+(\chi_{J_i}v^{-1})(x)dx \leq \sum_i\int_{J_i} u(x)G^+(\chi_{J_i}v^{-1})(x)dx\\
&\leq C\sum_i|J_i| \leq \frac{C}{\lambda}\sum_i\int_{J_i} h \leq \frac{C}{\lambda}\int_{\mathbb{R}} h.
\end{aligned}$$

So we have proved (3.1) and letting k_0 tend to $-\infty$ we obtain *(i)*.

□

References

[1] D. Cruz-Uribe, SFO, C. J. Neugebauer and V. Olesen, *The one-sided minimal operator and the one-sided reverse Hölder inequality*, Studia Math. 116(1995), no.3, 255-270.

[2] F. J. Martín-Reyes, L. Pick and A. de la Torre, A_∞^+ *condition*, Canad. J. Math 45(1993), 1231-1244.
[3] P. Ortega Salvador and C. Ramírez Torreblanca, *Weighted weak type inequalities for modified Hardy operators and geometric means operators in dimensions one and greater*, J. Math. Anal. Appl. 336(2007), no.1, 593-607.
[4] E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. 297(1986), no.1, 53-61.
[5] X. Shi, *Two inequalities related to geometric mean operators*, J. Zhejiang Teacher's College 1(1980), 21-25.
[6] X. Yin and B. Muckenhoupt, *Weighted inequalities for the maximal geometric mean operator*, Proc. Amer. Math. Soc. 124(1996), no.1, 75-81.

Pedro Ortega Salvador, Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
E-mail address: portega@uma.es

Consuelo Ramírez Torreblanca, Departamento de Matemáticas, Escuela Politécnica Superior, Universidad de Córdoba, 14071 Córdoba, Spain
E-mail address: ma1ratoc@uco.es