WEIGHTED INEQUALITIES FOR CESÀRO MAXIMAL OPERATORS IN ORLICZ SPACES

PEDRO ORTEGA SALVADOR AND CONSUELO RAMÍREZ TORREBLANCA

Abstract. Let $0 < \alpha \leq 1$ and let M_α^+ be the Cesàro maximal operator of order α defined by

$$M_\alpha^+ f(x) = \sup_{c>x} \frac{1}{(c-x)^\alpha} \int_x^c \frac{|f(s)|}{(c-s)^{1-\alpha}} ds.$$

In this work we characterize the pairs of measurable, positive and locally integrable functions (u, v) for which there exists a constant $C > 0$ such that the inequality

$$u(\{x \in \mathbf{R} : M_\alpha^+ f(x) > \lambda\}) \leq \frac{C}{\Phi(\lambda)} \int_{\mathbf{R}} \Phi(|f|)v$$

holds for all $\lambda > 0$ and every f in the Orlicz space $L_\Phi(v)$.

We also characterize the measurable, positive and locally integrable functions w such that the integral inequality

$$\int_{\mathbf{R}} \Phi(M_\alpha^+ f)w \leq C \int_{\mathbf{R}} \Phi(|f|)w$$

holds for every $f \in L_\Phi(w)$.

The discrete versions of this results allow, by techniques of transference, to prove weighted inequalities for the Cesàro maximal ergodic operator

$$M_{\alpha,T}^+ f(x) = \sup_{N\in\mathbf{N}} \frac{1}{N^\alpha} \sum_{i=0}^{N-1} \frac{|f(T^i x)|}{(N-i)^{1-\alpha}}$$

associated to an invertible measurable transformation, T, which preserves the measure.

Finally, we give sufficient conditions on w for the convergence of the sequence of Cesàro-α ergodic averages for all functions in the weighted Orlicz space $L_\Phi(w)$.

1. Introduction.

In 1979 W. Jurkat and J. Troutman ([2]) introduced the operators M_α^+ acting on measurable functions defined by

$$M_\alpha^+ f(x) = \sup_{c>x} \frac{1}{(c-x)^\alpha} \int_x^c \frac{|f(s)|}{(c-s)^{1-\alpha}} ds, \quad 0 < \alpha \leq 1.$$

If $\alpha = 1$, the operator M_α^+ is the one-sided Hardy-Littlewood maximal operator denoted usually by M^+. The pairs of weights for which this operator is of weak or strong type are well known. It can be seen in [5], [6] and [13].

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The weighted inequalities in L^p spaces for the operators M_α^+ have been studied by F. J. Martín-Reyes and A. de la Torre ([8]). The results obtained there were the following:

Theorem 1.1 ([8]). *Let u and v be positive and locally integrable functions. Let $0 < \alpha \leq 1$. Let $p \in \mathbf{R}$ with $1 < p < \infty$ and let p' be its conjugate exponent. The following statements are equivalent:*

(i) *There exists a constant $C > 0$ such that for any $f \in L^p(v)$ and any $\lambda > 0$*

$$u(\{x \in \mathbf{R} : M_\alpha^+ f(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbf{R}} |f|^p v.$$

(ii) *The pair (u, v) satisfies condition $A_{p,\alpha}^+$ ($(u,v) \in A_{p,\alpha}^+$); i.e., there exists a constant $C > 0$ such that for any three numbers $a, b, c \in \mathbf{R}$ with $a < b < c$*

$$\frac{1}{(c-a)^\alpha} \left(\int_a^b u(s)ds \right)^{\frac{1}{p}} \left(\int_b^c \frac{v^{1-p'}(s)ds}{(c-s)^{(1-\alpha)p'}} \right)^{\frac{1}{p'}} \leq C.$$

In the case of equal weights it is proved that weak and strong type for the operator M_α^+ are equivalent. It is contained in the following result:

Theorem 1.2 ([8]). *Let w be a positive and locally integrable function, $\alpha \in (0, 1]$ and $1 < p < \infty$. The following statements are equivalent:*

(i) *There exists a constant $C > 0$ such that for every $f \in L^p(w)$*

$$\int_{\mathbf{R}} (M_\alpha^+ f)^p w \leq C \int_{\mathbf{R}} |f|^p w.$$

(ii) *The function w satisfies condition $A_{p,\alpha}^+$ ($w \in A_{p,\alpha}^+$).*

Later, M.D. Sarrión characterized the weighted inequalities for the operator M_α^+ in $L^{p,q}$ spaces, obtaining consequences in Ergodic Theory ([7], [11], [12]).

It is interesting to ask wether generalizations of these theorems to Orlicz spaces are possible. The first purpose of this paper is to give affirmative answer to this question and to characterize inequalities as

$$\int_{\{x \in \mathbf{R} : M_\alpha^+ f(x) > \lambda\}} u \leq \frac{C}{\Phi(\lambda)} \int_{\mathbf{R}} \Phi(|f|) v$$

and

$$\int_{\mathbf{R}} \Phi(M_\alpha^+ f) w \leq C \int_{\mathbf{R}} \Phi(|f|) w,$$

where Φ is a N-function.

Before giving the statements of the theorems we recall the basic definitions and results about N-functions and Orlicz spaces which will be used later. Detailed treatments can be found in [3] and [9].

A function $\Phi : [0, \infty) \to \mathbf{R}$ is a N-function if

$$\Phi(t) = \int_0^t \varphi(s)ds,$$

where $\varphi : [0, \infty) \to \mathbf{R}$ is a right continuous nondeceasing function such that $\varphi(0) = 0$, $\varphi(s) > 0$ if $s > 0$ and $\lim_{s\to\infty} \varphi(s) = \infty$.

The function φ is called the density function of Φ.

Every N-function Φ is continuous, positive, strictly increasing and convex in $(0,\infty)$.

Let Φ be a N-function with density function φ. Associated with φ we have a function ψ defined by

$$\psi(t)=\sup\{s:\varphi(s)\leq t\},\quad t\geq 0.$$

The function ψ has the same properties as φ and is called the generalized inverse of φ. The N-function Ψ defined by

$$\Psi(t)=\int_0^t\psi(s)ds,\quad t\geq 0$$

is called the complementary N-function of Φ.

A pair of complementary N-functions Φ and Ψ satisfies the property

$$t\leq\Phi^{-1}(t)\Psi^{-1}(t)\leq 2t,\quad t\geq 0.$$

A N-function Φ satisfies condition Δ_2 in $[0,\infty)$ ($\Phi\in\Delta_2$) if $\sup_{s>0}\frac{\Phi(2s)}{\Phi(s)}<\infty$. We can characterize that Φ satisfies condition Δ_2 in terms of its density function or its complementary N-function as follows: $\Phi\in\Delta_2$ if and only if there exists $\alpha>1$ such that $s\varphi(s)<\alpha\Phi(s)$ for every $s>0$; if Ψ is the complementary N-function of Φ, then $\Phi\in\Delta_2$ if and only if there exists $\beta>1$ such that $\beta\Psi(s)<s\psi(s)$ for all $s>0$. Condition Δ_2 can also be expressed in the following equivalent way: for every $A>0$ there exists $B>0$ such that $\Phi(As)\leq B\Phi(s)$ for all $s\geq 0$.

Let $(X,\mathcal{M},\mu)$ be a σ-finite measure space and let Φ be a N-function. We define the Orlicz class of Φ, $\widetilde{L_\Phi}$, as follows:

$$\widetilde{L_\Phi}=\{f:X\to\mathbf{C}:f\text{ is measurable and }\int_X\Phi(|f|)d\mu<\infty\}.$$

If Ψ is the complementary N-function of Φ we define the Orlicz norm of a measurable function f defined almost everywhere by

$$\|f\|_\Phi=\sup\left\{\int_X|fg|d\mu:g\in\widetilde{L_\Psi}\text{ and }\int_X\Psi(|g|)d\mu\leq 1\right\}.$$

The Orlicz space L_Φ is defined as follows:

$$L_\Phi=\{f:f\text{ is measurable and }\|f\|_\Phi<\infty\}.$$

The number

$$|||f|||_\Phi=\inf\left\{\lambda>0:\int_X\Phi\left(\frac{|f|}{\lambda}\right)d\mu\leq 1\right\}$$

is the Luxemburg norm of f.

The Luxemburg and Orlicz norms are equivalent and the Orlicz space L_Φ is a Banach space with these norms.

Hölder inequality in L^p spaces has a natural extension to Orlicz spaces: if $f\in L_\Phi$ and $g\in L_\Psi$ then $fg\in L^1$ and

$$\int_X|fg|\leq\|f\|_\Phi\|g\|_\Psi.$$

If Φ is a N-function we can define its upper and lower indices, respectively, as follows:

$$\alpha_\Phi = \inf_{0<s<1}\frac{-\log h_\Phi(s)}{\log s} \quad \text{and} \quad \beta_\Phi = \sup_{s>1}\frac{-\log h_\Phi(s)}{\log s},$$

where $h_\Phi(s) = \sup_{t>0}\frac{\Phi^{-1}(t)}{\Phi^{-1}(st)}$. The numbers $p_\Phi = \alpha_\Phi^{-1}$ and $q_\Phi = \beta_\Phi^{-1}$ are called, respectively, the lower and upper exponent of Φ.

We will also need the following interpolation theorem (see [1]):

Theorem 1.3 ([1]). *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two σ-finite spaces. Let Φ be an N-function with complementary N-function Ψ. Suppose that Φ and Ψ satisfy condition Δ_2. Let p and q be, respectively, the lower and upper exponents of Φ. Let T be a sublinear operator which is of weak type (r, r) and of weak type (s, s), where $1 \le r < p$ and $q < s \le \infty$. Then T maps $L_\Phi(\mu)$ into $L_\Phi(\nu)$ and there exists $C > 0$ such that*

$$\int_Y \Phi(|Tf|)d\nu \le C \int_X \Phi(|f|)d\mu$$

for every $f \in L_\Phi(\mu)$.

In the proofs of the theorems we will use arguments and techniques due to Martín-Reyes and A. de la Torre ([8]) and also to P. Ortega ([10]).

Our results are the following:

Theorem 1.4. *Let Φ be an N-function with complementary N-function Ψ such that both of them satisfy condition Δ_2. Let $0 < \alpha \le 1$ and let u, v be two positive and locally integrable functions. The following statements are equivalent:*

(i) *The couple (u, v) satisfies $A^+_{\Phi,\alpha}$, i.e., there exists $K > 0$ such that the inequality*

$$\frac{\int_a^b \varepsilon u(s)ds}{(c-a)^\alpha}\varphi\left(\frac{\int_b^c \psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)\frac{1}{(c-s)^{1-\alpha}}ds}{(c-a)^\alpha}\right) \le K$$

holds for any $a, b, c \in \mathbf{R}$ with $a < b < c$ and every $\varepsilon > 0$.

(ii) *There exists a constant $C > 0$ such that the inequality*

$$u(\{x \in \mathbf{R} : M^+_\alpha f(x) > \lambda\}) \le \frac{C}{\Phi(\lambda)}\int_{\mathbf{R}} \Phi(|f|)v$$

holds for every $\lambda > 0$ and all $f \in L_\Phi(v)$.

Theorem 1.5. *Let Φ be an N-function with complementary N-function Ψ such that $\Phi, \Psi \in \Delta_2$. Let p be the lower exponent of Φ. Let $\alpha \in \mathbf{R}$ with $0 < \alpha \le 1$ and let w be a positive and locally integrable function. The following statements are equivalent:*

(i) *There exists a constant $C > 0$ such that the inequality*

$$\int_{\mathbf{R}} \Phi(M^+_\alpha f)w \le C \int_{\mathbf{R}} \Phi(|f|)w$$

holds for every $f \in L_\Phi(w)$.

(ii) *There exists a constant* $C > 0$ *such that the inequality*

$$w(\{x \in \mathbf{R} : M_\alpha^+ f(x) > \lambda\}) \leq \frac{C}{\Phi(\lambda)} \int_{\mathbf{R}} \Phi(|f|)w$$

holds for all $f \in L_\Phi(w)$ *and every* $\lambda > 0$.

(iii) $w \in A_{\Phi,\alpha}^+$.

(iv) $w \in A_{p,\alpha}^+$.

If we consider the function $\Phi(t) = t^p$, $p > 1$, then the corresponding Orlicz space is the L^p space, condition $A_{\Phi,\alpha}^+$ coincides with condition $A_{p,\alpha}^+$ and we obtain theorems 1.1 and 1.2.

The proofs of theorems 1.4 and 1.5 appear, respectively, in sections 2 and 3.

The techniques we will use in the proofs of theorems 1.4 and 1.5 also work in discrete settings. Specifically, if m_α^+ is the Cesàro maximal operator in the integers defined by

$$m_\alpha^+ f(s) = \sup_{j>s, j\in\mathbf{Z}} \frac{1}{(j-s)^\alpha} \sum_{i=s}^{j-1} \frac{|f(i)|}{(j-i)^{1-\alpha}}$$

and (u, v) is a couple of positive functions on $\mathbf{Z}$, the weighted weak type Φ-inequality for m_α^+ is characterized by condition $A_{\Phi,\alpha}^+(\mathbf{Z})$ which means that there exists a constant $C > 0$ such that for all $\varepsilon > 0$ and any $r, s, k \in \mathbf{Z}$ with $r \leq s < k$,

$$\frac{1}{(k-r)^\alpha} \left(\sum_{i=r}^{s} \varepsilon u(i) \right) \varphi \left(\frac{1}{(k-r)^\alpha} \sum_{i=s}^{k-1} \psi \left(\frac{\varepsilon^{-1} v^{-1}(i)}{(k-i)^{1-\alpha}} \right) \frac{1}{(k-i)^{1-\alpha}} \right) \leq C.$$

On the other hand, the weighted strong type Φ-inequality with equal weights for m_α^+ is characterized by $A_{\Phi,\alpha}^+$, which coincides with $A_{p,\alpha}^+$, where p is the lower exponent of Φ.

The discrete versions of theorems 1.4 and 1.5 will be applied, together with techniques of transference, to characterize the good weights for the Cesàro maximal ergodic operator.

Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a σ-finite measure space and let $T : \mathcal{X} \to \mathcal{X}$ be an invertible measure preserving transformation. Let us suppose that T is ergodic, which means that if A is a measurable set such that $\mu(T^{-1}A \triangle A) = 0$, then $\mu(A) = 0$ or $\mu(\mathcal{X} \backslash A) = 0$. The Cesàro ergodic averages of order α, $0 < \alpha \leq 1$, and the Cesàro maximal ergodic operator are defined, respectively, by

$$T_{\alpha,N}^+ f(x) = \frac{1}{N^\alpha} \sum_{i=0}^{N-1} \frac{f(T^i x)}{(N-i)^{1-\alpha}}$$

and

$$M_{\alpha,T}^+ f(x) = \sup_{N\in\mathbf{N}} T_{\alpha,N}^+(|f|)(x).$$

We will say that a pair of measurable functions (u, v) verifies condition $A_{\Phi,\alpha}^+(T)$ if there exists $C > 0$ such that

$$\frac{1}{k^\alpha} \left(\sum_{i=0}^{r} \varepsilon u(T^i y) \right) \varphi \left(\frac{1}{k^\alpha} \sum_{i=r}^{k-1} \psi \left(\frac{\varepsilon^{-1} v^{-1}(T^i y)}{(k-i)^{1-\alpha}} \right) \frac{1}{(k-i)^{1-\alpha}} \right) \leq C,$$

for all $\varepsilon > 0$, any two numbers $r, k \in \mathbf{Z}$ with $0 \leq r < k$ and almost all $y \in X$.

If $1 < p < \infty$ we will say that (u,v) verifies $A^+_{p,\alpha}(T)$ if (u,v) verifies $A^+_{\Phi,\alpha}(T)$ with $\Phi(t) = t^p$.

It is clear that to say that $(u,v) \in A^+_{\Phi,\alpha}(T)$ means that the couple (u^y, v^y), where $u^y(i) = u(T^i y)$ and $v^y(i) = v(T^i y)$, $i \in \mathbf{Z}$, satisfies condition $A^+_{\Phi,\alpha}(\mathbf{Z})$ for almost every $y \in \mathcal{X}$ with constant independent of y.

Then, the following theorems hold:

Theorem 1.6. *Let u and v be two positive functions on $\mathcal{X}$. Let Φ and Ψ be two complementary N-functions such that $\Phi, \Psi \in \Delta_2$. The following statements are equivalent:*

(i) *The pair (u,v) verifies condition $A^+_{\Phi,\alpha}(T)$.*
(ii) *There exists a constant $C > 0$ such that for all $\lambda > 0$ and every measurable functions f,*

$$u(\{x \in \mathcal{X} : M^+_{\alpha,T} f(x) > \lambda\}) \leq \frac{C}{\Phi(\lambda)} \int_{\mathcal{X}} \Phi(|f|) v \, d\mu.$$

Theorem 1.7. *Let w be a positive function on $\mathcal{X}$. Let Φ and Ψ be two complementary N-functions that satisfy condition Δ_2. Let p be the lower exponent of Φ. The following statements are equivalent:*

(i) *There exists a constant $C > 0$ such that for all measurable function f,*

$$\int_{\mathcal{X}} \Phi(M^+_{\alpha,T} f(x)) w(x) \, d\mu(x) \leq C \int_{\mathcal{X}} \Phi(|f(x)|) w(x) \, d\mu(x).$$

(ii) *There exists a constant $C > 0$ such that for all $\lambda > 0$ and every measurable function f,*

$$w(\{x \in \mathcal{X} : M^+_{\alpha,T} f(x) > \lambda\}) \leq \frac{C}{\Phi(\lambda)} \int_{\mathcal{X}} \Phi(|f|) w \, d\mu.$$

(iii) *w verifies $A^+_{\Phi,\alpha}(T)$.*
(iv) *w verifies $A^+_{p,\alpha}(T)$.*

The proofs of theorems 1.6 and 1.7 can be found, respectively, in sections 4 and 5.

As a corollary, we give sufficient conditions in order to get the a.e. convergence of the sequence of Cesàro ergodic averages for all $f \in L_\Phi(w)$.

Corollary 1.8. *Let Φ, Ψ, w and p be as in theorem 1.7. If $w \in A^+_{p,\alpha}$ (or equivalently $w \in A^+_{\Phi,\alpha}$), then the sequence of Cesàro ergodic averages $\{T^+_{k,\alpha} f\}$ converges a. e. for all $f \in L_\Phi(w)$.*

The corollary follows by standard methods: by theorem 1.7, we have the maximal inequality; on the other hand, the convergence in the dense class $L^p(d\mu) \cap L_\Phi(w d\mu)$ was established in [7] and [11]. Then, Banach principle can be applied.

Throughout the paper, the letter C will design a positive constant whose value can change.

2. Proof of Theorem 1.4.

(i) ⇒ (ii). First we will prove that there exists a constant $K > 0$ such that for every $a < b$ and any $f \geq 0$ the following inequality holds:

$$\frac{1}{(b-a)^\alpha}\int_a^b \frac{f(s)}{(b-s)^{1-\alpha}}ds \leq K\Phi^{-1}(M_u^+(\Phi(f)vu^{-1})(a)),$$

where M_u^+ is the operator defined by

$$M_u^+ g(x) = \sup_{h>0}\frac{1}{\int_x^{x+h} u}\int_x^{x+h}|g|u.$$

To prove this, we define a sequence $x_0 = b > x_1 > x_2 > \cdots > a$ by the identities

$$\int_a^{x_{i+1}} u = \int_{x_{i+1}}^{x_i} u = \frac{1}{2}\int_a^{x_i} u.$$

For every i we have

$$\int_{x_{i+1}}^{x_i}\frac{f(s)}{(b-s)^{1-\alpha}}ds = \int_{x_{i+1}}^{x_i}\left(\frac{x_i-s}{b-s}\right)^{1-\alpha}\frac{f(s)}{(x_i-s)^{1-\alpha}}\varepsilon v(s)(\varepsilon v(s))^{-1}ds.$$

The fact that the function $\cdot \to \left(\frac{x_i-\cdot}{b-\cdot}\right)^{1-\alpha}$ is decreasing and Hölder inequality give

$$\int_{x_{i+1}}^{x_i}\left(\frac{x_i-s}{b-s}\right)^{1-\alpha}\frac{f(s)}{(x_i-s)^{1-\alpha}}\varepsilon v(s)(\varepsilon v(s))^{-1}ds$$

$$\leq \left(\frac{x_i-x_{i+2}}{b-x_{i+2}}\right)^{1-\alpha}\int_{x_{i+1}}^{x_i}\frac{f(s)}{(x_i-s)^{1-\alpha}}\varepsilon v(s)(\varepsilon v(s))^{-1}ds$$

$$\leq \left(\frac{x_i-x_{i+2}}{b-x_{i+2}}\right)^{1-\alpha}\|f\chi_{(x_{i+1},x_i)}\|_{\Phi,\varepsilon v}\left\|\frac{(\varepsilon v)^{-1}}{(x_i-\cdot)^{1-\alpha}}\chi_{(x_{i+1},x_i)}\right\|_{\Psi,\varepsilon v}.$$

Let us estimate $\left\|\frac{(\varepsilon v)^{-1}}{(x_i-\cdot)^{1-\alpha}}\chi_{(x_{i+1},x_i)}\right\|_{\Psi,\varepsilon v}$. According to the definition of the Luxemburg norm,

$$\left\|\frac{(\varepsilon v)^{-1}}{(x_i-\cdot)^{1-\alpha}}\chi_{(x_{i+1},x_i)}\right\|_{\Psi,\varepsilon v}$$

$$= \inf\left\{\lambda > 0 : \int_{\mathbf{R}}\Psi\left(\frac{(\varepsilon v(s))^{-1}}{\lambda(x_i-s)^{1-\alpha}}\chi_{(x_{i+1},x_i)}(s)\right)\varepsilon v(s)ds \leq 1\right\}.$$

Since $\Psi(t) \leq t\psi(t)$ for all $t \geq 0$, then for every $\lambda > 0$

$$\int_{\mathbf{R}}\Psi\left(\frac{(\varepsilon v(s))^{-1}}{\lambda(x_i-s)^{1-\alpha}}\chi_{(x_{i+1},x_i)}(s)\right)\varepsilon v(s)ds$$

$$\leq \int_{\mathbf{R}}\frac{(\varepsilon v(s))^{-1}}{\lambda(x_i-s)^{1-\alpha}}\chi_{(x_{i+1},x_i)}(s)\psi\left(\frac{(\varepsilon v(s))^{-1}}{\lambda(x_i-s)^{1-\alpha}}\chi_{(x_{i+1},x_i)}(s)\right)\varepsilon v(s)ds$$

$$= \int_{x_{i+1}}^{x_i}\frac{1}{\lambda(x_i-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{\lambda(x_i-s)^{1-\alpha}}\right)ds.$$

Since $\Phi, \Psi \in \Delta_2$, their respective density functions φ and ψ verify $t \leq \psi \circ \varphi(t) \leq Ct$. Therefore from condition $A^+_{\Phi,\alpha}$ we deduce

$$\int_{x_{i+1}}^{x_i} \psi\left(\frac{(\lambda\varepsilon v(s))^{-1}}{(x_i-s)^{1-\alpha}}\right)\frac{1}{(x_i-s)^{1-\alpha}}ds \leq \psi\left(\frac{K(x_i-x_{i+2})^\alpha}{\int_{x_{i+2}}^{x_{i+1}}\lambda\varepsilon u}\right)(x_i-x_{i+2})^\alpha.$$

Then we can write

$$\int_{x_{i+1}}^{x_i} \frac{1}{\lambda(x_i-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{\lambda(x_i-s)^{1-\alpha}}\right)ds$$
$$\leq \lambda^{-1}\psi\left(\frac{K(x_i-x_{i+2})^\alpha}{\int_{x_{i+2}}^{x_{i+1}}\lambda\varepsilon u}\right)(x_i-x_{i+2})^\alpha.$$

We have obtained for all $\lambda > 0$

$$\int_{\mathbf{R}} \Psi\left(\frac{(\lambda\varepsilon v(s))^{-1}}{(x_i-s)^{1-\alpha}}\chi_{(x_{i+1},x_i)}(s)\right)\varepsilon v(s)ds$$
$$\leq \lambda^{-1}\psi\left(\frac{K(x_i-x_{i+2})^\alpha}{\int_{x_{i+2}}^{x_{i+1}}\lambda\varepsilon u}\right)(x_i-x_{i+2})^\alpha.$$

Let $\lambda = K(x_i-x_{i+2})^\alpha\Phi^{-1}\left(\varepsilon^{-1}\left(\int_{x_{i+2}}^{x_{i+1}}u\right)^{-1}\right)$. For this λ the previous inequality remains

$$\int_{\mathbf{R}} \Psi\left(\frac{(\lambda\varepsilon v(s))^{-1}}{(x_i-s)^{1-\alpha}}\chi_{(x_{i+1},x_i)}(s)\right)\varepsilon v(s)ds$$

$$= K^{-1}\frac{1}{\Phi^{-1}\left(\varepsilon^{-1}\left(\int_{x_{i+2}}^{x_{i+1}}u\right)^{-1}\right)}\psi\left(\frac{\left(\int_{x_{i+2}}^{x_{i+1}}\varepsilon u\right)^{-1}}{\Phi^{-1}\left(\varepsilon^{-1}\left(\int_{x_{i+2}}^{x_{i+1}}u\right)^{-1}\right)}\right)$$

$$= K^{-1}\varepsilon\left(\int_{x_{i+2}}^{x_{i+1}}u\right)\frac{\left(\int_{x_{i+2}}^{x_{i+1}}\varepsilon u\right)^{-1}}{\Phi^{-1}\left(\varepsilon^{-1}\left(\int_{x_{i+2}}^{x_{i+1}}u\right)^{-1}\right)}\psi\left(\frac{\left(\int_{x_{i+2}}^{x_{i+1}}\varepsilon u\right)^{-1}}{\Phi^{-1}\left(\varepsilon^{-1}\left(\int_{x_{i+2}}^{x_{i+1}}u\right)^{-1}\right)}\right). \quad (2.1)$$

Since $\Psi \in \Delta_2$, there exists $\beta_2 > 0$ such that $t\psi(t) \leq \beta_2\Psi(t)$ for all $t > 0$. On the other hand, as Φ and Ψ are complementary N-functions, we have $t \leq \Phi^{-1}(t)\Psi^{-1}(t)$ or, equivalently, $\Psi\left(\frac{t}{\Phi^{-1}(t)}\right) \leq t$ for all $t > 0$. Using these properties we can ensure that (2.1) is less than

$$K^{-1}\beta_2\varepsilon\left(\int_{x_{i+2}}^{x_{i+1}}u\right)\Psi\left(\frac{\left(\int_{x_{i+2}}^{x_{i+1}}\varepsilon u\right)^{-1}}{\Phi^{-1}\left(\varepsilon^{-1}\left(\int_{x_{i+2}}^{x_{i+1}}u\right)^{-1}\right)}\right)$$

$$\leq K^{-1}\beta_2\left(\varepsilon\int_{x_{i+2}}^{x_{i+1}}u\right)\left(\varepsilon\int_{x_{i+2}}^{x_{i+1}}u\right)^{-1} = K^{-1}\beta_2.$$

If we take $K \geq \beta_2$ from the beginning we get for the λ chosen

$$\int_{\mathbf{R}} \Psi\left(\frac{(\lambda\varepsilon v(s))^{-1}}{(x_i-s)^{1-\alpha}}\chi_{(x_{i+1},x_i)}(s)\right)\varepsilon v(s)ds \leq 1.$$

By the definition of the Luxemburg norm, we can affirm

$$\left\|\frac{(\varepsilon v)^{-1}}{(x_i-\cdot)^{1-\alpha}}\chi_{(x_{i+1},x_i)}\right\|_{\Psi,\varepsilon v} \leq K(x_i-x_{i+2})^{\alpha}\Phi^{-1}\left(\varepsilon^{-1}\left(\int_{x_{i+2}}^{x_{i+1}} u\right)^{-1}\right).$$

Then

$$\int_{x_{i+1}}^{x_i} \frac{f(s)}{(b-s)^{1-\alpha}}ds$$

$$\leq \left(\frac{x_i-x_{i+2}}{b-x_{i+2}}\right)^{1-\alpha}\|f\chi_{(x_{i+1},x_i)}\|_{\Phi,\varepsilon v}K(x_i-x_{i+2})^{\alpha}\Phi^{-1}\left(\varepsilon^{-1}\left(\int_{x_{i+2}}^{x_{i+1}} u\right)^{-1}\right).$$

The above inequality holds for every $\varepsilon > 0$. Let $\varepsilon = \left(\int_{x_{i+1}}^{x_i}\Phi(f)v\right)^{-1}$. For this ε

$$\|f\chi_{(x_{i+1},x_i)}\|_{\Phi,\varepsilon v} = 1.$$

Then

$$\int_{x_{i+1}}^{x_i} \frac{f(s)}{(b-s)^{1-\alpha}}ds \leq K\frac{x_i-x_{i+2}}{(b-x_{i+2})^{1-\alpha}}\Phi^{-1}\left(\frac{\int_{x_{i+1}}^{x_i}\Phi(f)v}{\int_{x_{i+2}}^{x_{i+1}} u}\right).$$

On the other hand, the choice of the sequence $\{x_i\}$ allows us to write

$$\frac{\int_{x_{i+1}}^{x_i}\Phi(f)v}{\int_{x_{i+2}}^{x_{i+1}} u} = \frac{\int_{x_{i+1}}^{x_i}\Phi(f)vu^{-1}u}{\int_{x_{i+2}}^{x_{i+1}} u} = 2\frac{\int_{x_{i+1}}^{x_i}\Phi(f)vu^{-1}u}{\int_{a}^{x_{i+1}} u}$$

$$= 4\frac{\int_{x_{i+1}}^{x_i}\Phi(f)vu^{-1}u}{\int_{a}^{x_i} u} \leq 4\frac{\int_{a}^{x_i}\Phi(f)vu^{-1}u}{\int_{a}^{x_i} u} \leq 4M_u^+(\Phi(f)vu^{-1})(a).$$

We have obtained

$$\int_{x_{i+1}}^{x_i} \frac{f(s)}{(b-s)^{1-\alpha}}ds \leq K\frac{x_i-x_{i+2}}{(b-x_{i+2})^{1-\alpha}}\Phi^{-1}(4M_u^+(\Phi(f)vu^{-1})(a)).$$

Summing up over i

$$\int_a^b \frac{f(s)}{(b-s)^{1-\alpha}}ds \leq K\Phi^{-1}(4M_u^+(\Phi(f)vu^{-1})(a))\sum_{i=0}^{\infty}\frac{x_i-x_{i+2}}{(b-x_{i+2})^{1-\alpha}}$$

$$= K\Phi^{-1}(4M_u^+(\Phi(f)vu^{-1})(a))\sum_{i=0}^{\infty}\int_{x_{i+2}}^{x_i}\frac{1}{(b-x_{i+2})^{1-\alpha}}ds$$

$$\leq K\Phi^{-1}(4M_u^+(\Phi(f)vu^{-1})(a))\sum_{i=0}^{\infty}\int_{x_{i+2}}^{x_i}\frac{1}{(b-s)^{1-\alpha}}ds$$

$$= K\Phi^{-1}(4M_u^+(\Phi(f)vu^{-1})(a))\int_a^b\frac{1}{(b-s)^{1-\alpha}}ds$$

$$= K\Phi^{-1}(4M_u^+(\Phi(f)vu^{-1})(a))(b-a)^{\alpha}.$$

We have obtained

$$\frac{1}{(b-a)^\alpha}\int_a^b \frac{f(s)}{(b-s)^{1-\alpha}}ds \le K\Phi^{-1}(4M_u^+(\Phi(f)vu^{-1})(a)).$$

Therefore, for all $a \in \mathbf{R}$

$$M_\alpha^+ f(a) \le K\Phi^{-1}(4M_u^+(\Phi(f)vu^{-1})(a)).$$

This implies for every $\lambda > 0$

$$u(\{x\in \mathbf{R} : M_\alpha^+ f(x) > \lambda\}) \le u(\{x \in \mathbf{R} : K\Phi^{-1}(4M_u^+(\Phi(f)vu^{-1})(x)) > \lambda\})$$
$$= u(\{x \in \mathbf{R} : M_u^+(\Phi(f)vu^{-1})(x) > \frac{\Phi(\frac{\lambda}{K})}{4}\}).$$

Using that M_u^+ is of weak type (1,1) we obtain

$$u(\{x\in \mathbf{R} : M_\alpha^+ f(x) > \lambda\}) \le \frac{C}{\Phi(\frac{\lambda}{K})}\int_{\mathbf{R}} \Phi(f)v,$$

i.e.,

$$u(\{x\in \mathbf{R} : M_\alpha^+ f(x) > \lambda\}) \le \frac{C}{\Phi(\lambda)}\int_{\mathbf{R}} \Phi(K|f|)v$$

and since $\Phi \in \Delta_2$, this implies

$$u(\{x\in \mathbf{R} : M_\alpha^+ f(x) > \lambda\}) \le \frac{C}{\Phi(\lambda)}\int_{\mathbf{R}} \Phi(|f|)v.$$

(ii) $\Rightarrow$ *(i)*. Let a, b, c be real numbers with $a < b < c$ and let $\varepsilon > 0$. Let us consider the function

$$f(s) = \psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)\chi_{(b,c)}(s).$$

For every $x \in (a, b)$ we have

$$M_\alpha^+ f(x) > \frac{1}{(c-a)^\alpha}\int_b^c \frac{1}{(c-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)ds.$$

This implies that

$$(a,b) \subset \left\{x \in \mathbf{R} : M_\alpha^+ f(x) > \frac{1}{(c-a)^\alpha}\int_b^c \frac{1}{(c-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)ds\right\}.$$

Using *(ii)* we obtain

$$\int_a^b \varepsilon u(s)ds$$
$$\le \frac{C}{\Phi\left(\frac{1}{(c-a)^\alpha}\int_b^c \frac{1}{(c-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)ds\right)}\int_b^c \Phi\left(\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)\right)\varepsilon v(s)ds.$$

Since $\Phi \in \Delta_2$ there exists $\beta_1 > 1$ such that $s\varphi(s) < \beta_1\Phi(s)$ for all $s > 0$. Then the right hand side of the previous inequality is dominated by

$$\frac{C\beta_1}{\frac{1}{(c-a)^\alpha}\int_b^c \frac{1}{(c-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)ds\,\varphi\left(\frac{1}{(c-a)^\alpha}\int_b^c \frac{1}{(c-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)ds\right)}$$

$$\times \int_b^c \Phi\left(\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)\right)\varepsilon v(s)ds.$$

We have obtained

$$\frac{\int_a^b \varepsilon u(s)ds}{(c-a)^\alpha}\varphi\left(\frac{1}{(c-a)^\alpha}\int_b^c \frac{1}{(c-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)ds\right)$$
$$\leq \frac{C\beta_1}{\int_b^c \frac{1}{(c-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)ds}\int_b^c \Phi\left(\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)\right)\varepsilon v(s)ds.$$

Since $\Phi \in \Delta_2$, $\Phi(\psi(s)) \leq Cs\psi(s)$ for all $s > 0$. Then the right hand side of the last inequality is dominated by

$$\frac{C\beta_1}{\int_b^c \frac{1}{(c-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)ds}\int_b^c \frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)\varepsilon v(s)ds = C\beta_1 = K.$$

We have found a constant K such that

$$\frac{\int_a^b \varepsilon u(s)ds}{(c-a)^\alpha}\varphi\left(\frac{\int_b^c \frac{1}{(c-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon v(s))^{-1}}{(c-s)^{1-\alpha}}\right)ds}{(c-a)^\alpha}\right) \leq K$$

for every $a, b, c \in \mathbf{R}$ with $a < b < c$ and all $\varepsilon > 0$.

□

3. Proof of Theorem 1.5.

In order to characterize the strong type inequality we will need the following lemma, which generalizes lemma 3.1. in [8]

Lemma 3.1. *Let Φ be an N-function with complementary N-function Ψ such that $\Phi, \Psi \in \Delta_2$. Let $0 < \alpha \leq 1$. For every $c \in \mathbf{R}$ let us consider the function $g_c(s) = \frac{1}{(c-s)^{1-\alpha}}\chi_{(-\infty,c)}(s)$. Suppose that $w \in A^+_{\Phi,\alpha}$. Then $w \in A^+_\Phi(g_c)$ uniformly in c, i.e., there exists a constant $C > 0$ independent of c such that*

$$\frac{\int_x^y \varepsilon w(s)ds}{\int_x^z g_c(s)ds}\varphi\left(\frac{\int_y^z g_c(s)\psi\left(\frac{g_c(s)}{\varepsilon w(s)}\right)ds}{\int_x^z g_c(s)ds}\right) \leq C$$

for all $x, y, z \in \mathbf{R}$ with $x < y < z < c$ and all $\varepsilon > 0$.

Proof. Let us fix $c \in \mathbf{R}$, let $x, y, z \in \mathbf{R}$ with $x < y < z < c$ and let $\varepsilon > 0$. Define the points x_i by $x_0 = z$ and $\int_x^{x_{i+1}} w = \int_{x_{i+1}}^{x_i} w$, if $i \geq 0$. The sequence $\{x_i\}$ decreases strictly and has limit x. There exists one and only one N such that $x_N \leq y < x_{N-1}$. This N satisfies

$$\int_{x_{N+1}}^{x_N} w = \frac{1}{2}\int_{x_N}^{x_{N-1}} w = \frac{1}{4}\int_x^{x_{N-1}} w > \frac{1}{4}\int_x^y w.$$

For $i < N-1$

$$\int_{x_{i+1}}^{x_i} w = \int_x^{x_{i+1}} w > \int_x^y w > \frac{1}{4}\int_x^y w.$$

Therefore for all $i \leq N-1$ we have

$$\int_{x_{i+2}}^{x_{i+1}} w > \frac{1}{4}\int_x^y w. \tag{3.1}$$

(Observe that the proof of $\int_{x_N}^{x_{N-1}} w > \frac{1}{4}\int_x^y w$ is included in the first case).
Then

$$\int_y^z g_c(s)\psi\left(\frac{g_c(s)}{\varepsilon w(s)}\right)ds \leq \sum_{i=0}^{N-1}\int_{x_{i+1}}^{x_i}\frac{1}{(c-s)^{1-\alpha}}\psi\left(\frac{(\varepsilon w(s))^{-1}}{(c-s)^{1-\alpha}}\right)ds$$

$$=\sum_{i=0}^{N-1}\int_{x_{i+1}}^{x_i}\left(\frac{x_i-s}{c-s}\right)^{1-\alpha}\psi\left(\left(\frac{x_i-s}{c-s}\right)^{1-\alpha}\frac{(\varepsilon w(s))^{-1}}{(x_i-s)^{1-\alpha}}\right)\frac{1}{(x_i-s)^{1-\alpha}}ds. \tag{3.2}$$

Since the function $\cdot \to \left(\frac{x_i-\cdot}{c-\cdot}\right)^{1-\alpha}$ is decreasing we have that (3.2) is dominated by

$$\sum_{i=0}^{N-1}\left(\frac{x_i-x_{i+2}}{c-x_{i+2}}\right)^{1-\alpha}\int_{x_{i+1}}^{x_i}\psi\left(\left(\frac{x_i-x_{i+2}}{c-x_{i+2}}\right)^{1-\alpha}\frac{(\varepsilon w(s))^{-1}}{(x_i-s)^{1-\alpha}}\right)\frac{1}{(x_i-s)^{1-\alpha}}ds$$

$$=\sum_{i=0}^{N-1}\left(\frac{x_i-x_{i+2}}{c-x_{i+2}}\right)^{1-\alpha}\int_{x_{i+1}}^{x_i}\psi\left(\left[\left(\frac{c-x_{i+2}}{x_i-x_{i+2}}\right)^{1-\alpha}\varepsilon\right]^{-1}\frac{(w(s))^{-1}}{(x_i-s)^{1-\alpha}}\right)\frac{1}{(x_i-s)^{1-\alpha}}ds.$$

Applying condition $A^+_{\Phi,\alpha}$ in each sumand of the last term and besides $\Phi,\Psi\in\Delta_2$ (and then, $t \leq (\psi\circ\varphi)(t)\leq Ct$), we obtain

$$\int_y^z g_c(s)\psi\left(\frac{g_c(s)}{\varepsilon w(s)}\right)ds$$

$$\leq \sum_{i=0}^{N-1}\left(\frac{x_i-x_{i+2}}{c-x_{i+2}}\right)^{1-\alpha}\psi\left(\frac{K(x_i-x_{i+2})^{\alpha}}{\int_{x_{i+2}}^{x_{i+1}}\left(\frac{c-x_{i+2}}{x_i-x_{i+2}}\right)^{1-\alpha}\varepsilon w(s)ds}\right)(x_i-x_{i+2})^{\alpha}$$

$$=\frac{1}{K}\sum_{i=0}^{N-1}\left(\frac{x_i-x_{i+2}}{c-x_{i+2}}\right)^{1-\alpha}\frac{K(x_i-x_{i+2})^{\alpha}}{\int_{x_{i+2}}^{x_{i+1}}\varepsilon w(s)ds}\int_{x_{i+2}}^{x_{i+1}}\varepsilon w(s)ds$$

$$\psi\left(\frac{K(x_i-x_{i+2})^{\alpha}}{\int_{x_{i+2}}^{x_{i+1}}\left(\frac{c-x_{i+2}}{x_i-x_{i+2}}\right)^{1-\alpha}\varepsilon w(s)ds}\right)=\frac{1}{K}\sum_{i=0}^{N-1}\int_{x_{i+2}}^{x_{i+1}}\varepsilon w(s)ds$$

$$\frac{K(x_i-x_{i+2})^{\alpha}}{\int_{x_{i+2}}^{x_{i+1}}\left(\frac{c-x_{i+2}}{x_i-x_{i+2}}\right)^{1-\alpha}\varepsilon w(s)ds}\psi\left(\frac{K(x_i-x_{i+2})^{\alpha}}{\int_{x_{i+2}}^{x_{i+1}}\left(\frac{c-x_{i+2}}{x_i-x_{i+2}}\right)^{1-\alpha}\varepsilon w(s)ds}\right). \tag{3.3}$$

Since $\Psi\in\Delta_2$, there exists $K_1>1$ such that $t\psi(t)<K_1\Psi(t)$, for all $t>0$. Then, (3.3) is dominated by

$$\frac{1}{K}\sum_{i=0}^{N-1}\left(\int_{x_{i+2}}^{x_{i+1}}\varepsilon w(s)ds\right)K_1\Psi\left(\frac{K(x_i-x_{i+2})^{\alpha}}{\int_{x_{i+2}}^{x_{i+1}}\left(\frac{c-x_{i+2}}{x_i-x_{i+2}}\right)^{1-\alpha}\varepsilon w(s)ds}\right)$$

$$= \frac{K_1}{K}\sum_{i=0}^{N-1}\left(\int_{x_{i+2}}^{x_{i+1}} \varepsilon w(s)ds\right)\Psi\left(\frac{K\left(\frac{x_i-x_{i+2}}{c-x_{i+2}}\right)^{1-\alpha}(x_i-x_{i+2})^{\alpha}}{\int_{x_{i+2}}^{x_{i+1}}\varepsilon w(s)ds}\right). \qquad (3.4)$$

If we apply now that the function $t \to t\Psi\left(\frac{\lambda}{t}\right)$ decreases together with (3.1), then the right hand side of (3.4) is less than

$$\frac{K_1}{K}\sum_{i=0}^{N-1}\frac{1}{4}\left(\int_x^y \varepsilon w(s)ds\right)\Psi\left(\frac{K\frac{x_i-x_{i+2}}{(c-x_{i+2})^{1-\alpha}}}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\right)$$
$$= \frac{K_1}{K}\frac{1}{4}\left(\int_x^y \varepsilon w(s)ds\right)\sum_{i=0}^{N-1}\Psi\left(\frac{K\frac{x_i-x_{i+2}}{(c-x_{i+2})^{1-\alpha}}}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\right)$$
$$\le \frac{K_1}{K}\frac{1}{4}\left(\int_x^y \varepsilon w(s)ds\right)\Psi\left(\frac{K}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\sum_{i=0}^{N-1}\frac{x_i-x_{i+2}}{(c-x_{i+2})^{1-\alpha}}\right)$$
$$= \frac{K_1}{K}\frac{1}{4}\left(\int_x^y \varepsilon w(s)ds\right)\Psi\left(\frac{K}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\sum_{i=0}^{N-1}\int_{x_{i+2}}^{x_i}\frac{1}{(c-x_{i+2})^{1-\alpha}}ds\right)$$
$$\le \frac{K_1}{K}\frac{1}{4}\left(\int_x^y \varepsilon w(s)ds\right)\Psi\left(\frac{K}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\sum_{i=0}^{N-1}\int_{x_{i+2}}^{x_i}\frac{1}{(c-s)^{1-\alpha}}ds\right)$$
$$\le \frac{K_1}{K}\frac{1}{4}\left(\int_x^y \varepsilon w(s)ds\right)\Psi\left(\frac{K}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\int_x^z\frac{1}{(c-s)^{1-\alpha}}ds\right)$$
$$= \frac{K_1}{K}\frac{1}{4}\left(\int_x^y \varepsilon w(s)ds\right)\Psi\left(\frac{K}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\int_x^z g_c(s)ds\right).$$

Since $\Psi(t) \le t\psi(t)$, for all $t > 0$, then the last term is less than

$$\frac{K_1}{K}\frac{1}{4}\left(\int_x^y \varepsilon w(s)ds\right)\frac{K}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\left(\int_x^z g_c(s)ds\right)\psi\left(\frac{K}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\int_x^z g_c(s)ds\right)$$
$$= K_1\left(\int_x^z g_c(s)ds\right)\psi\left(\frac{K}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\int_x^z g_c(s)ds\right).$$

We have obtained

$$\frac{\int_y^z g_c(s)\psi\left(\frac{g_c(s)}{\varepsilon w(s)}\right)ds}{\int_x^z g_c(s)ds} \le K_1\psi\left(\frac{K}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\int_x^z g_c(s)ds\right).$$

Then

$$\varphi\left(\frac{\int_y^z g_c(s)\psi\left(\frac{g_c(s)}{\varepsilon w(s)}\right)ds}{\int_x^z g_c(s)ds}\right) \le \varphi\left(K_1\psi\left(\frac{K}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\int_x^z g_c(s)ds\right)\right). \qquad (3.5)$$

Condition Δ_2 on Φ and Ψ implies that for every $C > 0$ there exists $C' > 0$, depending only on C, such that $\varphi(Cs) \le C'\varphi(s)$ for all $s > 0$. Then, from (3.5) we obtain

$$\varphi\left(\frac{\int_y^z g_c(s)\psi\left(\frac{g_c(s)}{\varepsilon w(s)}\right)ds}{\int_x^z g_c(s)ds}\right) \le C\varphi\left(\psi\left(\frac{K}{\frac{1}{4}\int_x^y\varepsilon w(s)ds}\int_x^z g_c(s)ds\right)\right)$$

$$\leq C\frac{K}{\frac{1}{4}\int_x^y \varepsilon w(s)ds}\int_x^z g_c(s)ds = \frac{K}{\int_x^y \varepsilon w(s)ds}\int_x^z g_c(s)ds.$$

We have proved

$$\frac{\int_x^y \varepsilon w(s)ds}{\int_x^z g_c(s)ds}\varphi\left(\frac{\int_y^z g_c(s)\psi\left(\frac{g_c(s)}{\varepsilon w(s)}\right)ds}{\int_x^z g_c(s)ds}\right) \leq K.$$

□

Let us prove Theorem 1.5. Implication *(i)* $\Rightarrow$ *(ii)* is trivial and *(ii)* $\Rightarrow$ *(iii)* was proved in theorem 1.4.

(iii) $\Rightarrow$ *(iv)*. According to lemma 3.1, $A^+_{\Phi,\alpha}$ implies $A^+_\Phi(g_c)$ for all $c \in \mathbf{R}$ with constant independent of c. On the other hand, $A^+_\Phi(g_c)$ implies $A^+_p(g_c)$ (Theorem 2 of [10]), which is equivalent to $A^+_{p,\alpha}$ (Lemma 3.1. of [8]).

(iv) $\Rightarrow$ *(i)*. $A^+_{p,\alpha}$ means:

$$\left(\int_a^b w(s)ds\right)^{\frac{1}{p}}\left(\int_b^c \frac{w^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}}ds\right)^{\frac{1}{p'}} \leq C_1(c-a)^\alpha$$

for all $a, b, c \in \mathbf{R}$ with $a < b < c$, where p' is the conjugate exponent of p. This is equivalent to say that there exists a constant C_2, which depends only on C_1, such that,

$$\left(\int_x^y w(s)ds\right)^{\frac{1}{p}}\left(\int_y^z \frac{w^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}}ds\right)^{\frac{1}{p'}} \leq C_2\int_x^z \frac{1}{(c-s)^{1-\alpha}}ds$$

for every $x, y, z, c \in \mathbf{R}$ with $x < y < z < c$, i.e., w verifies $A^+_p(g_c)$ for all $c \in \mathbf{R}$ uniformly. Then, we can ensure that there exists $\varepsilon > 0$, depending only on C_2, such that $w \in A^+_{p-\varepsilon}(g_c)$, or equivalently $w \in A^+_{p-\varepsilon,\alpha}$. This means that M^+_α is of weak type $(p-\varepsilon, p-\varepsilon)$.

On the other hand, it is well known that $w \in A^+_p(g_c)$ implies $w \in A^+_s(g_c)$ for all $s > p$. In particular, $w \in A^+_s(g_c)$ for all $s > q$ with constant independent of c, where q is the upper exponent of Φ. This gives that M^+_α is of weak type (s, s).

We have shown that M^+_α is of weak type $(p-\varepsilon, p-\varepsilon)$ and of weak type (s, s) for $1 < p-\varepsilon < p$ and $q < s < \infty$. Applying Theorem 1.3, we obtain that M^+_α maps $L_\Phi(w)$ into $L_\Phi(w)$, i.e.,

$$\int_{\mathbf{R}} \Phi(|M^+_\alpha f|)w \leq C\int_{\mathbf{R}} \Phi(|f|)w.$$

□

4. Proof of Theorem 1.6.

(i) $\Rightarrow$ *(ii)*. Let $N \in \mathbf{N}$ and let us consider the truncated Cesàro maximal operator

$$M^+_N f(x) \equiv M^+_{\alpha,T,N} f(x) = \sup_{0<n\leq N}\frac{1}{n^\alpha}\sum_{i=0}^{n-1}\frac{|f(T^i x)|}{(n-i)^{1-\alpha}}.$$

Let f be a positive function and let $\lambda > 0$. Let us fix $L \in \mathbf{N}$. Since T is a measure preserving transformation, we have

$$u(\{x \in \mathcal{X} : M_N^+ f(x) > \lambda\}) = \frac{1}{L}\sum_{i=0}^{L-1} u(\{x \in \mathcal{X} : M_N^+ f(x) > \lambda\})$$

$$= \frac{1}{L}\sum_{i=0}^{L-1}\int_{\mathcal{X}} \chi_{\{x\in\mathcal{X}:M_N^+ f(x)>\lambda\}}(y)u(y)d\mu(y)$$

$$= \frac{1}{L}\sum_{i=0}^{L-1}\int_{\mathcal{X}} \chi_{\{x\in\mathcal{X}:M_N^+ f(x)>\lambda\}}(T^i y)u(T^i y)d\mu(y)$$

$$= \frac{1}{L}\sum_{i=0}^{L-1}\int_{\mathcal{X}} \chi_{\{x\in\mathcal{X}:M_N^+ f(x)>\lambda\}}(T^i y)u^y(i)d\mu(y). \tag{4.1}$$

It is clear that

$$T^i y \in \{x \in \mathcal{X} : M_N^+ f(x) > \lambda\} \Leftrightarrow i \in \{j \in \mathbf{Z} : m_N^+(f^y \chi_{[0,N+L-1]})(j) > \lambda\},$$

where f^y, $y \in \mathcal{X}$, is the function on $\mathbf{Z}$ defined by $f^y(j) = f(T^j y)$ and m_N^+ is the truncated operator of m_α^+.

Then (4.1) equals

$$\frac{1}{L}\sum_{i=0}^{L-1}\int_{\mathcal{X}} \chi_{\{j\in\mathbf{Z}:m_N^+(f^y\chi_{[0,N+L-1]})(j)>\lambda\}}(i)u^y(i)d\mu(y)$$

$$= \frac{1}{L}\int_{\mathcal{X}}\sum_{i=0}^{L-1} \chi_{\{j\in\mathbf{Z}:m_N^+(f^y\chi_{[0,N+L-1]})(j)>\lambda\}}(i)u^y(i)d\mu(y). \tag{4.2}$$

Observe that the sum $\sum_{i=0}^{L-1} \chi_{\{j\in\mathbf{Z}:m_N^+(f^y\chi_{[0,N+L-1]})(j)>\lambda\}}(i)u^y(i)$ is nothing but the sum of the images by u^y of the integers between 0 and $L-1$ which are in the set $\{j \in \mathbf{Z} : m_N^+(f^y\chi_{[0,N+L-1]})(j) > \lambda\}$. Therefore,

$$\sum_{i=0}^{L-1} \chi_{\{j\in\mathbf{Z}:m_N^+(f^y\chi_{[0,N+L-1]})(j)>\lambda\}}(i)u^y(i)$$

$$= u^y(\{i \in [0, L-1] : m_N^+(f^y\chi_{[0,N+L-1]})(i) > \lambda\})$$

$$\leq u^y(\{i \in \mathbf{Z} : m_N^+(f^y\chi_{[0,N+L-1]})(i) > \lambda\}). \tag{4.3}$$

To say that the couple (u^y, v^y) verifies condition $A_{\Phi,\alpha}^+(\mathbf{Z})$ is equivalent to say that the operator m_α^+ verifies the weak type Φ-inequality. Then, the truncated operator is of weak type too. Therefore, the last term of (4.3) is less than or equal to

$$\frac{C}{\Phi(\lambda)}\sum_{i\in\mathbf{Z}}\Phi(f^y\chi_{[0,L+N-1]}(i))v^y(i) = \frac{C}{\Phi(\lambda)}\sum_{i=0}^{N+L-1}\Phi(f^y(i))v^y(i) =$$

$$\frac{C}{\Phi(\lambda)}\sum_{i=0}^{N+L-1}\Phi(f(T^i y))v(T^i y).$$

We have obtained

$$\sum_{i=0}^{L-1} \chi_{\{j\in\mathbf{Z}:m_N^+(f^y\chi_{[0,N+L-1]})(j)>\lambda\}}(i)u^y(i) \le \frac{C}{\Phi(\lambda)} \sum_{i=0}^{N+L-1} \Phi(f(T^iy))v(T^iy).$$

By this inequality and the fact that the operator T is a measure preserving transformation, we can affirm that (4.2) is dominated by

$$\frac{1}{L}\int_{\mathcal{X}} \frac{C}{\Phi(\lambda)} \sum_{i=0}^{N+L-1} \Phi(f(T^iy))v(T^iy)d\mu(y) = \frac{1}{L}\frac{C}{\Phi(\lambda)} \sum_{i=0}^{N+L-1} \int_{\mathcal{X}} \Phi(f(T^iy))v(T^iy)d\mu(y)$$

$$= \frac{1}{L}\frac{C}{\Phi(\lambda)} \sum_{i=0}^{N+L-1} \int_{\mathcal{X}} \Phi(f(y))v(y)d\mu(y) = \frac{N+L}{L}\frac{C}{\Phi(\lambda)} \int_{\mathcal{X}} \Phi(f(y))v(y)d\mu(y).$$

We have proved

$$u(\{x\in\mathcal{X} : M_N^+f(x) > \lambda\}) \le \frac{N+L}{L}\frac{C}{\Phi(\lambda)}\int_{\mathcal{X}} \Phi(f(y))v(y)d\mu(y).$$

Since this inequality holds for all $L, N \in \mathbf{N}$, letting L to ∞ and then N to ∞ we obtain

$$u(\{x\in\mathcal{X} : M_{\alpha,T}^+f(x) > \lambda\}) \le \frac{C}{\Phi(\lambda)}\int_{\mathcal{X}} \Phi(f(y))v(y)d\mu(y).$$

(ii) $\Rightarrow$ *(i).* We will use the following lemma due to F. J. Martín-Reyes:

Lemma 4.1 (Lemma 2.9 in [4])**.** *Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a σ-finite space and let $T : \mathcal{X} \to \mathcal{X}$ be an ergodic measure preserving transformation. Then, for every positive integer k there exists a countable family of measurable sets $\{B_n^{(k)} : n = 0, 1, \dots\}$, such that*

(i) *For every n, $T^iB_n^{(k)} \cap T^jB_n^{(k)} = \emptyset$ if $i \ne j$, $0 \le i, j < k$.*
(ii) *$\mathcal{X} = \cup_n B_n^{(k)}$.*

The sets $B_n^{(k)}$ are called basis of ergodic rectangles associated to k.

Let s, k be integers with $0 \le s < k$ and let $\varepsilon > 0$. Let $\{B_i\}$ be the partition of $\mathcal{X}$ in basis of rectangles associated to k provided by the lemma. Let us fix a nonnegative integer i and let $m \in \mathbf{Z}$. Let us define

$$H_{i,m} = \left\{ x \in B_i : 2^m < \frac{\sum_{i=s}^{k-1} \psi\left(\frac{\varepsilon^{-1}v^{-1}(T^ix)}{(k-i)^{1-\alpha}}\right)\frac{1}{(k-i)^{1-\alpha}}}{k^\alpha} \le 2^{m+1} \right\}.$$

Let A be a measurable subset of $H_{i,m}$. Then the sets $A, TA, T^2A, \dots, T^{k-1}A$ are pairwise disjoint. Let us consider the sets

$$R_1 = \bigcup_{j=0}^{s} T^jA \quad \text{and} \quad R_2 = \bigcup_{j=s}^{k-1} T^jA.$$

Let f be the function defined by

$$f(x) = \psi\left(\sum_{j=s}^{k-1} \chi_{T^jA}(x)\frac{1}{(k-j)^{1-\alpha}}\frac{1}{\varepsilon v(x)}\right).$$

Let us see that

$$R_1 \subset \{x : M^+_{\alpha,T} f(x) > 2^m\}.$$

If $x \in R_1$, there exists one and only one $h \in [0, s]$ such that $x = T^h y$ for some $y \in A$. For this x we have:

$$M^+_{\alpha,T} f(x) = \sup_{N>0} \frac{1}{N^\alpha} \sum_{j=0}^{N-1} \frac{f(T^j x)}{(N-j)^{1-\alpha}}$$

$$= \sup_{N>0} \frac{1}{N^\alpha} \sum_{j=0}^{N-1} \frac{\psi\left(\sum_{r=s}^{k-1} \chi_{T^r A}(T^j x) \frac{1}{(k-r)^{1-\alpha}} \frac{1}{\varepsilon v(T^j x)}\right)}{(N-j)^{1-\alpha}}$$

$$= \sup_{N>0} \frac{1}{N^\alpha} \sum_{j=0}^{N-1} \frac{\psi\left(\sum_{r=s}^{k-1} \chi_{T^r A}(T^{j+h} y) \frac{1}{(k-r)^{1-\alpha}} \frac{1}{\varepsilon v(T^{j+h} y)}\right)}{(N-j)^{1-\alpha}}$$

$$= \sup_{N>0} \frac{1}{N^\alpha} \sum_{j=h}^{N+h-1} \frac{\psi\left(\sum_{r=s}^{k-1} \chi_{T^r A}(T^j y) \frac{1}{(k-r)^{1-\alpha}} \frac{1}{\varepsilon v(T^j y)}\right)}{(N-j+h)^{1-\alpha}}$$

$$\geq \frac{1}{(k-h)^\alpha} \sum_{j=h}^{k-1} \frac{\psi\left(\sum_{r=s}^{k-1} \chi_{T^r A}(T^j y) \frac{1}{(k-r)^{1-\alpha}} \frac{1}{\varepsilon v(T^j y)}\right)}{(k-j)^{1-\alpha}}$$

$$\geq \frac{1}{k^\alpha} \sum_{j=s}^{k-1} \frac{\psi\left(\sum_{r=s}^{k-1} \chi_{T^r A}(T^j y) \frac{1}{(k-r)^{1-\alpha}} \frac{1}{\varepsilon v(T^j y)}\right)}{(k-j)^{1-\alpha}} = \frac{1}{k^\alpha} \sum_{j=s}^{k-1} \frac{\psi\left(\frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon v(T^j y)}\right)}{(k-j)^{1-\alpha}} > 2^m$$

where the last inequality follows from the fact that $y \in A$ and $A \subset H_{i,m}$.

Using this inclusion and the weak type inequality, we obtain:

$$\int_{R_1} u d\mu \leq \int_{\{x: M^+_{\alpha,T} f(x) > 2^m\}} u d\mu \leq \frac{C}{\Phi(2^m)} \int_{\mathcal{X}} \Phi(|f|) v d\mu.$$

Therefore,

$$\int_A \sum_{j=0}^{s} u(T^j x) d\mu \leq \frac{C}{\Phi(2^m)} \int_{\mathcal{X}} \Phi\left(\psi\left(\sum_{j=s}^{k-1} \chi_{T^j A}(x) \frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon v(x)}\right)\right) v(x) d\mu(x)$$

$$= \frac{C}{\Phi(2^m)} \int_{R_2} \Phi\left(\psi\left(\sum_{j=s}^{k-1} \chi_{T^j A}(x) \frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon v(x)}\right)\right) v(x) d\mu(x)$$

$$= \frac{C}{\Phi(2^m)} \int_{R_2} \sum_{j=s}^{k-1} \Phi\left(\psi\left(\chi_{T^j A}(x) \frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon v(x)}\right)\right) v(x) d\mu(x).$$

Applying that $\Phi(\psi(s)) \approx s\psi(s)$, the definition of $H_{i,m}$ and $\Phi(s) \approx s\varphi(s)$, we can ensure that the right hand side of the last inequality is less than

$$\frac{C}{\Phi(2^m)} \int_{R_2} \sum_{j=s}^{k-1} \chi_{T^j A}(x) \frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon v(x)} \psi\left(\frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon v(x)}\right) v(x) d\mu(x)$$

$$= \frac{C}{\Phi(2^m)} \sum_{j=s}^{k-1} \int_{T^jA} \frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon} \psi\left(\frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon v(x)} \right) d\mu(x)$$

$$= \frac{C}{\Phi(2^m)} \int_A \sum_{j=s}^{k-1} \frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon} \psi\left(\frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon v(T^jx)} \right) d\mu(x)$$

$$\le \frac{C}{\varepsilon} \int_A \frac{\sum_{j=s}^{k-1} \frac{1}{(k-j)^{1-\alpha}} \psi\left(\frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon v(T^jx)} \right)}{\Phi\left(\frac{\sum_{i=s}^{k-1} \psi\left(\frac{\varepsilon^{-1}v^{-1}(T^ix)}{(k-i)^{1-\alpha}} \right) \frac{1}{(k-i)^{1-\alpha}}}{k^\alpha} \right)} d\mu(x)$$

$$\le \frac{C}{\varepsilon} \int_A \frac{\sum_{j=s}^{k-1} \frac{1}{(k-j)^{1-\alpha}} \psi\left(\frac{1}{(k-j)^{1-\alpha}} \frac{1}{\varepsilon v(T^jx)} \right)}{\frac{\sum_{i=s}^{k-1} \psi\left(\frac{\varepsilon^{-1}v^{-1}(T^ix)}{(k-i)^{1-\alpha}} \right) \frac{1}{(k-i)^{1-\alpha}}}{k^\alpha} \varphi\left(\frac{\sum_{i=s}^{k-1} \psi\left(\frac{\varepsilon^{-1}v^{-1}(T^ix)}{(k-i)^{1-\alpha}} \right) \frac{1}{(k-i)^{1-\alpha}}}{k^\alpha} \right)} d\mu(x)$$

$$= \frac{Ck^\alpha}{\varepsilon} \int_A \frac{1}{\varphi\left(\frac{\sum_{i=s}^{k-1} \psi\left(\frac{\varepsilon^{-1}v^{-1}(T^ix)}{(k-i)^{1-\alpha}} \right) \frac{1}{(k-i)^{1-\alpha}}}{k^\alpha} \right)} d\mu(x).$$

Since this holds for every measurable subset $A \subset H_{i,m}$ we obtain:

$$\sum_{i=0}^{s} u(T^ix) \le \frac{Ck^\alpha}{\varepsilon} \frac{1}{\varphi\left(\frac{\sum_{i=s}^{k-1} \psi\left(\frac{\varepsilon^{-1}v^{-1}(T^ix)}{(k-i)^{1-\alpha}} \right) \frac{1}{(k-i)^{1-\alpha}}}{k^\alpha} \right)}$$

for almost every $x \in H_{i,m}$. Then, since the union of $H_{i,m}$ is B_i, the inequality holds for almost all $x \in B_i$. Then

$$\frac{1}{k^\alpha} \left(\sum_{i=0}^{s} \varepsilon u(T^ix) \right) \varphi\left(\frac{1}{k^\alpha} \sum_{i=s}^{k-1} \psi\left(\frac{\varepsilon^{-1}v^{-1}(T^ix)}{(k-i)^{1-\alpha}} \right) \frac{1}{(k-i)^{1-\alpha}} \right) \le C,$$

for almost every $x \in \mathcal{X}$.

□

5. Proof of Theorem 1.7.

Implication *(i)* ⇒ *(ii)* is trivial, *(ii)* ⇒ *(iii)* was proved in the previous theorem and *(iii)* ⇔ *(iv)* is an immediate consequence of the result in the integers.

(iii) ⇒ *(i)*. Let f be a positive and measurable function and let $L, N \in \mathbf{N}$. Since T is a measurable transformation which preserves the measure we have

$$\int_X \Phi(M_N^+ f(x)) w(x) d\mu(x) = \frac{1}{L} \sum_{i=0}^{L-1} \int_X \Phi(M_N^+ f(x)) w(x) d\mu(x)$$

$$= \frac{1}{L} \sum_{i=0}^{L-1} \int_X \Phi(M_N^+ f(T^ix)) w(T^ix) d\mu(x). \tag{5.1}$$

It is clear that $M_N^+ f(T^i x) \leq m_\alpha^+(f^x \chi_{[0,N+L-2]})(i)$. Therefore, since Φ is increasing, the last term of (5.1) is dominated by

$$\frac{1}{L}\int_X \sum_{i=0}^{\infty} \Phi(m_\alpha^+(f^x \chi_{[0,N+L-2]})(i))w^x(i)d\mu(x). \tag{5.2}$$

As we have pointed out previously, $w \in A_{\Phi,\alpha}^+(T)$ is equivalent to say that $w^x \in A_{\Phi,\alpha}^+(\mathbf{Z})$ uniformly in x. On the other hand, $w^x \in A_{\Phi,\alpha}^+(\mathbf{Z})$ is equivalent to say that m_α^+ verifies the strong type inequality with weight w^x. Applying this theorem to the sum in (5.2) we can ensure that this expression is less than

$$\frac{C}{L}\int_X \sum_{i=0}^{\infty} \Phi(f^x \chi_{[0,N+L-2]}(i))w^x(i)d\mu(x) = \frac{C}{L}\int_X \sum_{i=0}^{L+N-2} \Phi(f^x(i))w^x(i)d\mu(x)$$

$$= \frac{C}{L}\sum_{i=0}^{L+N-2}\int_X \Phi(f^x(i))w^x(i)d\mu(x) = \frac{C}{L}\sum_{i=0}^{L+N-2}\int_X \Phi(f(T^i x))w(T^i x)d\mu(x).$$

Using again that T is a measure preserving transformation we obtain that the right hand side of the last equality is

$$\frac{C}{L}\sum_{i=0}^{L+N-2}\int_X \Phi(f(x))w(x)d\mu(x) = \frac{L+N-1}{L} C \int_X \Phi(f(x))w(x)d\mu(x).$$

We have proved

$$\int_X \Phi(M_N^+ f(x))w(x)d\mu(x) \leq \frac{L+N-1}{L} C \int_X \Phi(f(x))w(x)d\mu(x).$$

Letting L to ∞ and then N to ∞ we obtain

$$\int_X \Phi(M_{\alpha,T}^+ f(x))w(x)d\mu(x) \leq \int_X \Phi(f(x))w(x)d\mu(x).$$

□

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Pedro Ortega Salvador, Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
Email address: ortega@anamat.cie.uma.es

Consuelo Ramírez Torreblanca, Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
Email address: ramirez@anamat.cie.uma.es