WEIGHTED BILINEAR HARDY INEQUALITIES

M. ISABEL AGUILAR CAÑESTRO, PEDRO ORTEGA SALVADOR, AND C. RAMÍREZ TORREBLANCA

Abstract. We characterize the weights w, w_1, w_2 such that the weighted bilinear Hardy inequality

$$\left(\int_a^b \left(\int_a^x f\right)^q \left(\int_a^x g\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C \left(\int_a^b f^{p_1} w_1\right)^{\frac{1}{p_1}} \left(\int_a^b g^{p_2} w_2\right)^{\frac{1}{p_2}}$$

holds for all nonnegative functions f and g, with a positive constant C independent of f and g, for all possible values of q, p_1 and p_2 with $1 < q, p_1, p_2 < \infty$.

We also characterize the good weights for the weighted bilinear n-dimensional Hardy inequality to hold.

1. Introduction and results.

G. H. Hardy established in [8] that if $p > 1$, then the inequality

$$\int_0^\infty \left(\frac{1}{t}\int_0^t f(x)dx\right)^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(t)dt \tag{1.1}$$

holds for all nonnegative functions f on $(0, \infty)$.

Shortly after, (1.1) led to prove (see [9], Theorem 330) that if $p > 1$ and $\varepsilon < p - 1$, then

$$\int_0^\infty \left(\int_0^t f(x)dx\right)^p t^{\varepsilon - p} dt \leq \left(\frac{p}{p-1-\varepsilon}\right)^p \int_0^\infty f^p(t)t^\varepsilon dt,$$

which can be seen as a weighted strong-type inequality for the operator $Pf(x) = \int_0^x f$, with weights $t^{\varepsilon-p}$ and t^ε. It is natural to consider the problem of characterizing the pairs (u, v) of nonnegative measurable functions on an interval (a, b), with $-\infty \leq a < b \leq \infty$, such that the inequality

$$\left(\int_a^b \left(\int_a^x f\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C \left(\int_a^b f^p v\right)^{\frac{1}{p}} \tag{1.2}$$

holds for all nonnegative f with a positive constant C independent of f, where $q, p > 1$. This problem was solved by Muckenhoupt [14] and Bradley [2] in the case $1 < p \leq q < \infty$ and by Mazja [13] in the case $1 < q < p < \infty$ (see [10] and [15] for more references and details). The next theorem collects the results mentioned above:

Theorem A ([2],[13],[14]). *Let $1 \leq q, p < \infty$ and let u, v be positive measurable functions on (a, b). Then the inequality (1.2) holds for all positive measurable functions f with a constant C independent of f if and only if:*

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(i) *in the case* $1 < p \le q < \infty$,

$$A = \sup_{a<x<b} \left(\int_x^b u \right)^{\frac{1}{q}} \left(\int_a^x v^{1-p'} \right)^{\frac{1}{p'}} < \infty,$$

and the best constant C *in* (1.2) *verifies* $C = A$;

(ii) *in the case* $1 < q < p < \infty$, *if* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$,

$$B = \left\{ \int_a^b \left(\int_x^b w(t)dt \right)^{\frac{r}{q}} \left(\int_a^x v^{1-p'}(t)dt \right)^{\frac{r}{q'}} v^{1-p'}(x)dx \right\}^{\frac{1}{r}}$$

$$= \left(\frac{p'}{q} \right)^{\frac{1}{r}} \left\{ \int_a^b \left(\int_x^b w(t)dt \right)^{\frac{r}{p}} \left(\int_a^x v^{1-p'}(t)dt \right)^{\frac{r}{p'}} w(x)dx \right\}^{\frac{1}{r}} < \infty,$$

and the best constant C *in* (1.2) *verifies* $q^{\frac{1}{q}} \left(\frac{p'q}{r} \right)^{\frac{1}{q'}} B \le C \le q^{\frac{1}{q}} (p')^{\frac{1}{q'}} B$.

The inequality (1.2) is closely related to the embeddings of weighted Sobolev spaces into weighted Lebesgue spaces. More precisely, if, following [10], we define the space $W_L^{1,p}(u)$ as the one consisting of the functions $f : (a,b) \to \mathbb{R}$ such that f is absolutely continuous, $\lim_{x \to a^+} f(x) = 0$ and $f' \in L^p(u)$, normed with $\|f\|_{W_L^{1,p}(u)} = \|f'\|_{L^p(u)}$, then the inequality (1.2) is equivalent to

$$\|f\|_{L^q(w)} \le C \|f\|_{W_L^{1,p}(v)},$$

i. e., the continuity of the identity map from $W_L^{1,p}(v)$ to $L^q(w)$.

This paper deals with the bilinear Hardy operator H defined for pairs (f,g) of nonnegative measurable functions on (a,b), $-\infty \le a < b \le \infty$, by

$$H(f,g)(x) = \int_a^x \int_a^x f(s)g(t)dsdt.$$

Our purpose is to characterize the positive measurable functions w, w_1, w_2 such that the inequality

$$\left(\int_a^b (H(f,g)(x))^q w(x)dx \right)^{\frac{1}{q}} \le C \left(\int_a^b f^{p_1} w_1 \right)^{\frac{1}{p_1}} \left(\int_a^b g^{p_2} w_2 \right)^{\frac{1}{p_2}} \tag{1.3}$$

holds for all pairs (f,g) of nonnegative functions with a positive constant C independent of f and g, where $q, p_1, p_2 > 1$.

As far as we know, this problem has not been considered previously in the literature, apart from the papers [4] and [7], which work with general bilinear operators and characterize their boundedness, in the case $\frac{1}{q} \ge \frac{1}{p_1} + \frac{1}{p_2}$, by means of a Schur-type criterion.

Our work has been motivated by the paper [11], where a weight theory has been developed for a new multi(sub)linear maximal function introduced in order to control the multilinear Calderón-Zygmund operators.

It is worth noting that the inequality (1.3) is equivalent to

$$\|fg\|_{L^q(w)} \le C \|f\|_{W_L^{1,p_1}(w_1)} \|g\|_{W_L^{1,p_2}(w_2)}, \tag{1.4}$$

which means that the characterization of the weights w, w_1, w_2 such that the inequality (1.3) holds is equivalent to the characterization of the weights w, w_1, w_2 for the product to be continuous from $W_L^{1,p_1}(w_1) \times W_L^{1,p_2}(w_2)$ to $L^q(w)$.

Our results are the following ones:

Theorem 1. *Let $q, p_1, p_2 > 1$ with $q \geq p_1$ and $q \geq p_2$. Let w, w_1, w_2 be positive measurable functions defined on (a,b). Then there exists a positive constant C such that the inequality (1.3) holds for all positive functions f and g if and only if*

$$A_L = \sup_{a<x<b} \left(\int_x^b w\right)^{\frac{1}{q}} \left(\int_a^x w_1^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\int_a^x w_2^{1-p_2'}\right)^{\frac{1}{p_2'}} < \infty.$$

Moreover, the best constant C in inequality (1.3) verifies $A_L \leq C \leq 8(1+4^q)^{\frac{1}{q}} A_L$.

Theorem 2. *Let $q, p_1, p_2 > 1$ with $q \geq p_1$ and $q < p_2$. Let w, w_1, w_2 be positive measurable functions defined on (a,b). Then there exists a positive constant C such that the inequality (1.3) holds for all positive functions f and g if and only if the following conditions hold:*

(i)

$$A_L = \sup_{a<x<b} \left(\int_x^b w\right)^{\frac{1}{q}} \left(\int_a^x w_1^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\int_a^x w_2^{1-p_2'}\right)^{\frac{1}{p_2'}} < \infty;$$

(ii)

$$B_L = \sup_{a<x<b} \left(\int_a^x w_1^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\int_x^b \left(\int_y^b w\right)^{\frac{r_2}{q}} \left(\int_a^y w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy\right)^{\frac{1}{r_2}} < \infty,$$

where $\dfrac{1}{r_2} = \dfrac{1}{q} - \dfrac{1}{p_2}$.

Moreover, the best constant C in inequality (1.3) verifies

$$\max\{A_L, q^{\frac{1}{q}} \left(\frac{qp_2'}{r_2}\right)^{\frac{1}{q'}} B_L\} \leq C \leq 8\left(A_L + q^{\frac{1}{q}} (p_2')^{\frac{1}{q'}} B_L\right).$$

Theorem 3. *Let $q, p_1, p_2 > 1$ with $q < p_1$, $q < p_2$ and $\dfrac{1}{p_1} + \dfrac{1}{p_2} \geq \dfrac{1}{q}$. Let w, w_1, w_2 be positive measurable functions defined on (a,b). Then there exists a positive constant C such that the inequality (1.3) holds for all positive functions f and g if and only if the following conditions hold:*

(i)

$$A_L = \sup_{a<x<b} \left(\int_x^b w\right)^{\frac{1}{q}} \left(\int_a^x w_1^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\int_a^x w_2^{1-p_2'}\right)^{\frac{1}{p_2'}} < \infty;$$

(ii)

$$B_L = \sup_{a<x<b} \left(\int_a^x w_1^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\int_x^b \left(\int_y^b w\right)^{\frac{r_2}{q}} \left(\int_a^y w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy\right)^{\frac{1}{r_2}} < \infty;$$

(iii)

$$C_L = \sup_{a<x<b} \left(\int_a^x w_2^{1-p_2'}\right)^{\frac{1}{p_2'}} \left(\int_x^b \left(\int_y^b w\right)^{\frac{r_1}{q}} \left(\int_a^y w_1^{1-p_1'}\right)^{\frac{r_1}{q'}} w_1^{1-p_1'}(y)dy\right)^{\frac{1}{r_1}} < \infty,$$

where $\dfrac{1}{r_1}=\dfrac{1}{q}-\dfrac{1}{p_1}$ *and* $\dfrac{1}{r_2}=\dfrac{1}{q}-\dfrac{1}{p_2}$.

Moreover, the best constant C *in inequality (*1.3*) verifies*

$$\max\{A_L, q^{\frac{1}{q}}\left(\frac{qp_2'}{r_2}\right)^{\frac{1}{q'}}B_L, q^{\frac{1}{q}}\left(\frac{qp_1'}{r_1}\right)^{\frac{1}{q'}}C_L\}\le C\le 8\left(8C_L+4\left(\frac{p_1'}{r_1}\right)^{\frac{1}{r_1}}A_L+q^{\frac{1}{q}}(p_2')^{\frac{1}{q'}}B_L\right).$$

Theorem 4. *Let* $q,p_1,p_2>1$ *with* $q<p_1$*,* $q<p_2$ *and* $\dfrac{1}{p_1}+\dfrac{1}{p_2}<\dfrac{1}{q}$ *. Let* w,w_1,w_2 *be positive measurable functions defined on* (a,b)*. Then there exists a positive constant* C *such that the inequality (*1.3*) holds for all positive functions* f *and* g *if and only if the following conditions hold:*

(i)

$$D_L=\left\{\int_a^b\left(\int_x^b w\right)^{\frac{s}{p_1}+\frac{s}{p_2}}\left(\int_a^x w_1^{1-p_1'}\right)^{\frac{s}{p_1'}}\left(\int_a^x w_2^{1-p_2'}\right)^{\frac{s}{p_2'}}w(x)dx\right\}^{\frac{1}{s}}<\infty;$$

(ii)

$$E_L=\left\{\int_a^b\left(\int_x^b\left(\int_y^b w\right)^{\frac{r_2}{q}}\left(\int_a^y w_2^{1-p_2'}\right)^{\frac{r_2}{q'}}w_2^{1-p_2'}(y)dy\right)^{\frac{s}{r_2}}\left(\int_a^x w_1^{1-p_1'}\right)^{\frac{s}{r_2'}}w_1^{1-p_1'}(x)dx\right\}^{\frac{1}{s}}<\infty;$$

(iii)

$$F_L=\left\{\int_a^b\left(\int_x^b\left(\int_y^b w\right)^{\frac{r_1}{q}}\left(\int_a^y w_1^{1-p_1'}\right)^{\frac{r_1}{q'}}w_1^{1-p_1'}(y)dy\right)^{\frac{s}{r_1}}\left(\int_a^x w_2^{1-p_2'}\right)^{\frac{s}{r_1'}}w_2^{1-p_2'}(x)dx\right\}^{\frac{1}{s}}<\infty,$$

where $\dfrac{1}{r_1}=\dfrac{1}{q}-\dfrac{1}{p_1}$, $\dfrac{1}{r_2}=\dfrac{1}{q}-\dfrac{1}{p_2}$ *and* $\dfrac{1}{s}=\dfrac{1}{r_1}-\dfrac{1}{p_2}=\dfrac{1}{q}-\dfrac{1}{p_1}-\dfrac{1}{p_2}$.

Moreover, the best constant C *in inequality (*1.3*) verifies*

$$C\le 8\left(2r_1^{\frac{1}{r_1}}(p_2')^{\frac{1}{r_1'}}\left(\frac{s}{p_2'}+1\right)^{\frac{1}{s}}\left(\left(\frac{r_1}{p_1'}\right)^{\frac{1}{r_1}}F_L+\left(\frac{s}{q}\right)^{\frac{1}{s}}D_L\right)+q^{\frac{1}{q}}(p_2')^{\frac{1}{q'}}\left(\frac{s}{p_1'}\right)^{\frac{1}{s}}E_L\right)$$

and

$$C\ge\max\left\{\widetilde{D_L},\widetilde{E_L},\widetilde{F_L}\right\},$$

where

$$\widetilde{D_L}=D_L\left(\frac{q}{r_1}\right)^{\frac{1}{p_2}}r_1^{\frac{1}{r_1}}\left(\frac{p_2'r_1}{s}\right)^{\frac{1}{r_1'}}\left(\frac{p_2'}{r_1}\right)^{\frac{1}{s}}q^{\frac{1}{q}}\left(\frac{p_1'q}{r_1}\right)^{\frac{1}{q'}}\left(\frac{p_1'}{q}\right)^{\frac{1}{r_1}},$$

$$\widetilde{E_L}=E_Lq^{\frac{1}{q}}\left(\frac{p_2'q}{r_2}\right)^{\frac{1}{q'}}r_2^{\frac{1}{r_2}}\left(\frac{p_1'r_2}{s}\right)^{\frac{1}{r_2'}}$$

and

$$\widetilde{F_L}=F_Lq^{\frac{1}{q}}\left(\frac{p_1'q}{r_1}\right)^{\frac{1}{q'}}r_1^{\frac{1}{r_1}}\left(\frac{p_2'r_1}{s}\right)^{\frac{1}{r_1'}}.$$

The proofs of Theorems 1, 2, 3 and 4 are contained in sections 2, 3, 4 and 5, respectively. From the technical view point, it is worth noting that the proofs of the sufficiency of the conditions in Theorems 3 and 4 need an estimation which is performed by means of the known results about weighted inequalities for the discrete Hardy operator. This technique is inspired in [3] (see also [1] and [16]).

The theory of weighted Hardy inequalities was extended to the n-dimensional setting by P. Drábek, H. P. Heinig and A. Kufner, who characterized in [5] (see also [12] for more general results and applications) the good weights for the inequality

$$\left(\int_{\mathbb{R}^n}\left(\int_{B(0,|x|)} f\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n} f^p v\right)^{\frac{1}{p}} \tag{1.5}$$

to hold for all nonnegative f with a positive constant C independent of f, where $q,p>1$. They proved the following result.

Theorem B ([5]). *Let $1<q,p<\infty$ and let u,v be positive measurable functions on $\mathbb{R}^n$. Then the inequality (1.5) holds for all positive measurable functions f with a constant C independent of f if and only if:*

(i) *in the case $1<p\leq q<\infty$,*

$$\sup_{\rho>0}\left(\int_{|x|>\rho} u\right)^{\frac{1}{q}}\left(\int_{|x|<\rho} v^{1-p'}\right)^{\frac{1}{p'}}<\infty;$$

(ii) *in the case $1<q<p<\infty$, if $\dfrac{1}{r}=\dfrac{1}{q}-\dfrac{1}{p}$,*

$$\left\{\int_{\mathbb{R}^n}\left(\int_{|y|>|x|} w\right)^{\frac{r}{q}}\left(\int_{|y|<|x|} v^{1-p'}\right)^{\frac{r}{q'}} v^{1-p'}(x)dx\right\}^{\frac{1}{r}}$$

$$=\left(\frac{p'}{q}\right)^{\frac{1}{r}}\left\{\int_{\mathbb{R}^n}\left(\int_{|y|>|x|} w\right)^{\frac{r}{p}}\left(\int_{|y|<|x|} v^{1-p'}\right)^{\frac{r}{p'}} w(x)dx\right\}^{\frac{1}{r}}<\infty.$$

Arguing as in the proofs of Theorems 1, 2, 3 and 4, and applying Theorem B instead of Theorem A when necessary, we also characterize the weighted inequality

$$\left(\int_{\mathbb{R}^n}(T(f,g)(x))^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n} f^{p_1}w_1\right)^{\frac{1}{p_1}}\left(\int_{\mathbb{R}^n} g^{p_2}w_2\right)^{\frac{1}{p_2}}, \tag{1.6}$$

where T stands for the n-dimensional bilinear Hardy operator defined for pairs (f,g) of nonnegative measurable functions on $\mathbb{R}^n$ by

$$T(f,g)(x)=\int_{B(0,|x|)}\int_{B(0,|x|)} f(s)g(t)dsdt.$$

Some norm inequalities with power weights for a multilinear n-dimensional Hardy operator have been characterized in [6]. The operator T we have just defined is closely related to the operator $\mathcal{H}^2$ considered in [6], which is defined for pairs (f,g) of nonnegative functions on $\mathbb{R}^n$ and $x\in\mathbb{R}^n$ by

$$\mathcal{H}^2(f,g)(x)=\int_{|(y_1,y_2)|<|x|} f(y_1)g(y_2)dy_1dy_2.$$

It is clear that $\mathcal{H}^2(f,g)(x) \leq T(f,g)(x) \leq \mathcal{H}^2(f,g)(\sqrt{2}x)$, which allows to establish relationships between the good weights for T and $\mathcal{H}^2$. In particular, the good triplets (w, w_1, w_2) of power weights coincide.

The results for T are the following ones:

Theorem 5. *Let $q, p_1, p_2 > 1$ with $q \geq p_1$ and $q \geq p_2$. Let w, w_1, w_2 be positive measurable functions on $\mathbb{R}^n$. Then there exists a positive constant C such that the inequality (1.6) holds for all positive functions f and g if and only if*

$$A^n = \sup_{0<\rho<\infty} \left(\int_{|x|>\rho} w(x)dx \right)^{\frac{1}{q}} \left(\int_{|x|<\rho} w_1^{1-p_1'}(x)dx \right)^{\frac{1}{p_1'}} \left(\int_{|x|<\rho} w_2^{1-p_2'}(x)dx \right)^{\frac{1}{p_2'}} < \infty.$$

Theorem 6. *Let $q, p_1, p_2 > 1$ with $q \geq p_1$ and $q < p_2$. Let w, w_1, w_2 be positive measurable functions on $\mathbb{R}^n$. Then there exists a positive constant C such that the inequality (1.6) holds for all positive functions f and g if and only if the following conditions hold:*

(i)

$$A^n = \sup_{0<\rho<\infty} \left(\int_{|x|>\rho} w(x)dx \right)^{\frac{1}{q}} \left(\int_{|x|<\rho} w_1^{1-p_1'}(x)dx \right)^{\frac{1}{p_1'}} \left(\int_{|x|<\rho} w_2^{1-p_2'}(x)dx \right)^{\frac{1}{p_2'}} < \infty$$

(ii)

$$B^n = \sup_{0<\rho<\infty} \left(\int_{|x|<\rho} w_1^{1-p_1'}(x)dx \right)^{\frac{1}{p_1'}} \times \left(\int_{|y|>\rho} \left(\int_{|x|>|y|} w(x)dx \right)^{\frac{r_2}{q}} \left(\int_{|x|<|y|} w_2^{1-p_2'}(x)dx \right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy \right)^{\frac{1}{r_2}} < \infty,$$

where $\dfrac{1}{r_2} = \dfrac{1}{q} - \dfrac{1}{p_2}$.

Theorem 7. *Let $q, p_1, p_2 > 1$ with $q < p_1$, $q < p_2$ and $\dfrac{1}{p_1} + \dfrac{1}{p_2} \geq \dfrac{1}{q}$. Let w, w_1, w_2 be positive measurable functions on $\mathbb{R}^n$. Then there exists a positive constant C such that the inequality (1.6) holds for all positive functions f and g if and only if the following conditions hold:*

(i)

$$A^n = \sup_{0<\rho<\infty} \left(\int_{|x|>\rho} w(x)dx \right)^{\frac{1}{q}} \left(\int_{|x|<\rho} w_1^{1-p_1'}(x)dx \right)^{\frac{1}{p_1'}} \left(\int_{|x|<\rho} w_2^{1-p_2'}(x)dx \right)^{\frac{1}{p_2'}} < \infty;$$

(ii)

$$B^n = \sup_{0<\rho<\infty} \left(\int_{|x|<\rho} w_1^{1-p_1'}(x)dx \right)^{\frac{1}{p_1'}} \times \left(\int_{|y|>\rho} \left(\int_{|x|>|y|} w(x)dx \right)^{\frac{r_2}{q}} \left(\int_{|x|<|y|} w_2^{1-p_2'}(x)dx \right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy \right)^{\frac{1}{r_2}} < \infty;$$

(iii)

$$C^n = \sup_{0<\rho<\infty} \left(\int_{|x|<\rho} w_2^{1-p_2'}(x)dx \right)^{\frac{1}{p_2'}}$$
$$\times \left(\int_{|y|>\rho} \left(\int_{|x|>|y|} w(x)dx \right)^{\frac{r_1}{q}} \left(\int_{|x|<|y|} w_1^{1-p_1'}(x)dx \right)^{\frac{r_1}{q'}} w_1^{1-p_1'}(y)dy \right)^{\frac{1}{r_1}} < \infty,$$

where $\frac{1}{r_1} = \frac{1}{q} - \frac{1}{p_1}$ *and* $\frac{1}{r_2} = \frac{1}{q} - \frac{1}{p_2}$.

Theorem 8. *Let* $q, p_1, p_2 > 1$ *with* $q < p_1$, $q < p_2$ *and* $\frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{q}$. *Let* w, w_1, w_2 *be positive measurable functions on* $\mathbb{R}^n$. *Then there exists a positive constant* C *such that the inequality (*1.6*) holds for all positive functions* f *and* g *if and only if the following conditions hold:*

(i)

$$D^n = \left\{ \int_{\mathbb{R}^n} \left(\int_{|y|>|x|} w \right)^{\frac{s}{p_1}+\frac{s}{p_2}} \left(\int_{|y|<|x|} w_1^{1-p_1'} \right)^{\frac{s}{p_1'}} \left(\int_{|y|<|x|} w_2^{1-p_2'} \right)^{\frac{s}{p_2'}} w(x)dx \right\}^{\frac{1}{s}} < \infty;$$

(ii)

$$E^n = \left\{ \int_{\mathbb{R}^n} \left(\int_{|y|>|x|} \left(\int_{|z|>|y|} w \right)^{\frac{r_2}{q}} \left(\int_{|z|<|y|} w_2^{1-p_2'} \right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy \right)^{\frac{s}{r_2}} \right.$$
$$\left. \times \left(\int_{|z|<|x|} w_1^{1-p_1'} \right)^{\frac{s}{r_2'}} w_1^{1-p_1'}(x)dx \right\}^{\frac{1}{s}} < \infty;$$

(iii)

$$F^n = \left\{ \int_{\mathbb{R}^n} \left(\int_{|y|>|x|} \left(\int_{|z|>|y|} w \right)^{\frac{r_1}{q}} \left(\int_{|z|<|y|} w_1^{1-p_1'} \right)^{\frac{r_1}{q'}} w_1^{1-p_1'}(y)dy \right)^{\frac{s}{r_1}} \right.$$
$$\left. \times \left(\int_{|z|<|x|} w_2^{1-p_2'} \right)^{\frac{s}{r_1'}} w_2^{1-p_2'}(x)dx \right\}^{\frac{1}{s}} < \infty;$$

where $\frac{1}{r_1} = \frac{1}{q} - \frac{1}{p_1}$, $\frac{1}{r_2} = \frac{1}{q} - \frac{1}{p_2}$ *and* $\frac{1}{s} = \frac{1}{r_1} - \frac{1}{p_2} = \frac{1}{q} - \frac{1}{p_1} - \frac{1}{p_2}$.

2. Proof of Theorem 1.

Proof. For the necessity of $A_L < \infty$, we fix $x \in (a, b)$ and test the inequality (1.3) with the functions $f = \chi_{(a,x)} w_1^{1-p_1'}$ and $g = \chi_{(a,x)} w_2^{1-p_2'}$, obtaining immediately

$$\left(\int_x^b \left(\int_a^x w_1^{1-p_1'} \right)^q \left(\int_a^x w_2^{1-p_2'} \right)^q w(t)dt \right)^{\frac{1}{q}} \le C \left(\int_a^x w_1^{1-p_1'} \right)^{\frac{1}{p_1}} \left(\int_a^x w_2^{1-p_2'} \right)^{\frac{1}{p_2}},$$

which means that $A_L \le C$.

Conversely, let f,g be positive functions and let us define the sequence $\{x_i\}$ by $x_0=b$ and x_{i+1} is the unique number such that $\int_a^{x_{i+1}} f = \int_{x_{i+1}}^{x_i} f$. Then

$$\begin{aligned}
&\int_a^b \left(\int_a^x f\right)^q \left(\int_a^x g\right)^q w(x)dx = \sum_{i=0}^{\infty}\int_{x_{i+1}}^{x_i} \left(\int_a^x f\right)^q \left(\int_a^x g\right)^q w(x)dx \\
&\le \sum_{i=0}^{\infty}\int_{x_{i+1}}^{x_i} \left(\int_a^{x_i} f\right)^q \left(\int_a^x g\right)^q w(x)dx = 4^q \sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f\right)^q \int_{x_{i+1}}^{x_i} \left(\int_a^x g\right)^q w(x)dx \\
&\le 8^q \left(\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f\right)^q \int_{x_{i+1}}^{x_i} \left(\int_a^{x_{i+1}} g\right)^q w(x)dx\right. \\
&\left. + \sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f\right)^q \int_{x_{i+1}}^{x_i} \left(\int_{x_{i+1}}^x g\right)^q w(x)dx\right) = 8^q(I+II).
\end{aligned}$$

In order to estimate I, we apply Hölder inequality and the condition and then take into account that $\frac{q}{p_1}\ge 1$ as follows:

$$\begin{aligned}
I &= \sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f\right)^q \left(\int_{x_{i+1}}^{x_i} w\right)\left(\int_a^{x_{i+1}} g\right)^q \\
&\le \sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1}w_1\right)^{\frac{q}{p_1}} \left(\int_{x_{i+1}}^{x_i} w\right)\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{q}{p_1'}} \left(\int_a^{x_{i+1}} g^{p_2}w_2\right)^{\frac{q}{p_2}} \left(\int_a^{x_{i+1}} w_2^{1-p_2'}\right)^{\frac{q}{p_2'}} \\
&\le \sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1}w_1\right)^{\frac{q}{p_1}} \left(\int_a^{x_{i+1}} g^{p_2}w_2\right)^{\frac{q}{p_2}} \left(\int_{x_{i+1}}^{b} w\right)\left(\int_a^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{q}{p_1'}} \left(\int_a^{x_{i+1}} w_2^{1-p_2'}\right)^{\frac{q}{p_2'}} \\
&\le A_L^q \left(\int_a^b g^{p_2}w_2\right)^{\frac{q}{p_2}} \sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1}w_1\right)^{\frac{q}{p_1}} \le A_L^q \left(\int_a^b g^{p_2}w_2\right)^{\frac{q}{p_2}} \left(\int_a^b f^{p_1}w_1\right)^{\frac{q}{p_1}}.
\end{aligned}$$

For the estimation of II, let us split each interval (x_{i+1},x_i) by means of the sequence $\{z_j^i\}$ defined by $z_0^i=x_i$ and, once given z_j^i, z_{j+1}^i is the unique number such that $\int_{x_{i+1}}^{z_{j+1}^i} g = \int_{z_{j+1}^i}^{z_j^i} g$. Then

$$\begin{aligned}
II &= \sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f\right)^q \int_{x_{i+1}}^{x_i}\left(\int_{x_{i+1}}^x g\right)^q w(x)dx = \sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f\right)^q \sum_{j=0}^{\infty}\int_{z_{j+1}^i}^{z_j^i}\left(\int_{x_{i+1}}^x g\right)^q w(x)dx \\
&\le \sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f\right)^q \sum_{j=0}^{\infty}\left(\int_{x_{i+1}}^{z_j^i} g\right)^q \left(\int_{z_{j+1}^i}^{z_j^i} w\right) = 4^q \sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f\right)^q \sum_{j=0}^{\infty}\left(\int_{z_{j+2}^i}^{z_{j+1}^i} g\right)^q \left(\int_{z_{j+1}^i}^{z_j^i} w\right).
\end{aligned}$$

Applying Hölder inequality, the condition and the facts that $\frac{q}{p_1}\ge 1$ and $\frac{q}{p_2}\ge 1$, we obtain that II is less than

$$4^q\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1}w_1\right)^{\frac{q}{p_1}} \left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{q}{p_1'}} \sum_{j=0}^{\infty}\left(\int_{z_{j+2}^i}^{z_{j+1}^i} g^{p_2}w_2\right)^{\frac{q}{p_2}} \left(\int_{z_{j+2}^i}^{z_{j+1}^i} w_2^{1-p_2'}\right)^{\frac{q}{p_2}} \left(\int_{z_{j+1}^i}^{z_j^i} w\right)$$

$$\leq 4^q \sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1} w_1\right)^{\frac{q}{p_1}} \sum_{j=0}^{\infty} \left(\int_{z_{j+2}^i}^{z_{j+1}^i} g^{p_2} w_2\right)^{\frac{q}{p_2}} \left(\int_a^{z_{j+1}^i} w_1^{1-p_1'}\right)^{\frac{q}{p_1'}} \left(\int_a^{z_{j+1}^i} w_2^{1-p_2'}\right)^{\frac{q}{p_2}} \left(\int_{z_{j+1}^i}^b w\right)$$

$$\leq 4^q A_L^q \sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1} w_1\right)^{\frac{q}{p_1}} \sum_{j=0}^{\infty} \left(\int_{z_{j+2}^i}^{z_{j+1}^i} g^{p_2} w_2\right)^{\frac{q}{p_2}} \leq 4^q A_L^q \sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1} w_1\right)^{\frac{q}{p_1}} \left(\int_{x_{i+1}}^{x_i} g^{p_2} w_2\right)^{\frac{q}{p_2}}$$

$$\leq 4^q A_L^q \left(\int_a^b g^{p_2} w_2\right)^{\frac{q}{p_2}} \sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1} w_1\right)^{\frac{q}{p_1}} \leq 4^q A_L^q \left(\int_a^b f^{p_1} w_1\right)^{\frac{q}{p_1}} \left(\int_a^b g^{p_2} w_2\right)^{\frac{q}{p_2}}.$$

This shows that

$$\left(\int_a^b \left(\int_a^x f\right)^q \left(\int_a^x g\right)^q w(x)dx\right)^{\frac{1}{q}} \leq 8(1+4^q)^{\frac{1}{q}} A_L \left(\int_a^b f^{p_1} w_1\right)^{\frac{1}{p_1}} \left(\int_a^b g^{p_2} w_2\right)^{\frac{1}{p_2}},$$

and we are done.

□

3. Proof of Theorem 2.

Proof. The necessity of condition (i) follows as in Theorem 1, obtaining $A_L \leq C$. For the necessity of condition (ii), we observe that (1.3) is equivalent to

$$\left(\int_a^b \left(\int_a^x g\right)^q \left(\int_a^x \frac{f}{||f||_{p_1,w_1}}\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C \left(\int_a^b g^{p_2} w_2\right)^{\frac{1}{p_2}}. \tag{3.1}$$

This means that the Hardy operator is bounded from $L^{p_2}(w_2)$ to $L^q(u)$, where

$$u(x) = \left(\int_a^x \frac{f}{||f||_{p_1,w_1}}\right)^q w(x),$$

and the constant C does not depend on f.

Since $q < p_2$, by Theorem A, the inequality (3.1) implies

$$\left\{\int_a^b \left(\int_x^b \left(\int_a^t \frac{f}{||f||_{p_1,w_1}}\right)^q w(t)dt\right)^{\frac{r_2}{q}} \left(\int_a^x w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(x)dx\right\}^{\frac{1}{r_2}} \leq Cq^{-\frac{1}{q}} \left(\frac{p_2' q}{r_2}\right)^{-\frac{1}{q'}},$$

or, equivalently,

$$\left\{\int_a^b \left(\int_x^b \left(\int_a^t f\right)^q w(t)dt\right)^{\frac{r_2}{q}} \left(\int_a^x w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(x)dx\right\}^{\frac{1}{r_2}} \leq Cq^{-\frac{1}{q}} \left(\frac{p_2' q}{r_2}\right)^{-\frac{1}{q'}} ||f||_{p_1,w_1}.$$

This inequality implies immediately that

$$\left\{\int_a^b \left(\int_a^x f\right)^{r_2} \left(\int_x^b w\right)^{\frac{r_2}{q}} \left(\int_a^x w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(x)dx\right\}^{\frac{1}{r_2}} \leq Cq^{-\frac{1}{q}} \left(\frac{p_2' q}{r_2}\right)^{-\frac{1}{q'}} ||f||_{p_1,w_1},$$

which means that the Hardy operator $Pf(x) = \int_a^x f$ is bounded from $L^{p_1}(w_1)$ to $L^{r_2}(\tilde{u})$, where

$$\tilde{u}(x) = \left(\int_x^b w\right)^{\frac{r_2}{q}} \left(\int_a^x w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(x).$$

Since $r_2 \geq p_1$, applying Theorem A, this inequality holds if and only if $B_L < \infty$. Moreover $C \geq q^{\frac{1}{q}} \left(\frac{p_2' q}{r_2}\right)^{\frac{1}{q'}} B_L$.

For the sufficiency of the conditions, we work exactly as in the proof Theorem 1, obtaining

$$\int_a^b \left(\int_a^x f\right)^q \left(\int_a^x g\right)^q w(t)dt \leq 8^q(I + II), \tag{3.2}$$

where

$$I = \sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} f\right)^q \left(\int_a^{x_{i+1}} g\right)^q \left(\int_{x_{i+1}}^{x_i} w\right)$$

and

$$II = \sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} f\right)^q \int_{x_{i+1}}^{x_i} \left(\int_{x_{i+1}}^{x} g\right)^q w(x)dx.$$

Proceeding as in the proof of Theorem 1, we obtain, by means of condition (i),

$$I \leq A_L^q \left(\int_a^b f^{p_1} w_1\right)^{\frac{q}{p_1}} \left(\int_a^b g^{p_2} w_2\right)^{\frac{q}{p_2}}. \tag{3.3}$$

In order to estimate II, we work as in the proof of Mazja's Theorem (see Theorem 1.15 in [15]), obtaining, for every i,

$$\begin{aligned} &\int_{x_{i+1}}^{x_i} \left(\int_{x_{i+1}}^{x} g\right)^q w(x)dx \\ &\leq q(p_2')^{q-1} \left(\int_{x_{i+1}}^{x_i} g^{p_2} w_2\right)^{\frac{q}{p_2}} \left(\int_{x_{i+1}}^{x_i} \left(\int_y^b w\right)^{\frac{r_2}{q}} \left(\int_a^y w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy\right)^{\frac{q}{r_2}}. \end{aligned} \tag{3.4}$$

Then, applying (3.4), Hölder inequality and condition (ii), we have

$$\begin{aligned} II &\leq q(p_2')^{q-1} \sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} f\right)^q \left(\int_{x_{i+1}}^{x_i} g^{p_2} w_2\right)^{\frac{q}{p_2}} \times \\ &\quad \left(\int_{x_{i+1}}^{x_i} \left(\int_y^b w\right)^{\frac{r_2}{q}} \left(\int_a^y w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy\right)^{\frac{q}{r_2}} \\ &\leq q(p_2')^{q-1} \sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1} w_1\right)^{\frac{q}{p_1}} \left(\int_{x_{i+1}}^{x_i} g^{p_2} w_2\right)^{\frac{q}{p_2}} \times \\ &\quad \left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{q}{p_1'}} \left(\int_{x_{i+1}}^{x_i} \left(\int_y^b w\right)^{\frac{r_2}{q}} \left(\int_a^y w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy\right)^{\frac{q}{r_2}} \end{aligned}$$

$$\leq q(p_2')^{q-1}B_L^q\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}}f^{p_1}w_1\right)^{\frac{q}{p_1}}\left(\int_{x_{i+1}}^{x_i}g^{p_2}w_2\right)^{\frac{q}{p_2}}$$
$$\leq q(p_2')^{q-1}B_L^q\left(\int_a^b g^{p_2}w_2\right)^{\frac{q}{p_2}}\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}}f^{p_1}w_1\right)^{\frac{q}{p_1}}. \tag{3.5}$$

Since $\dfrac{q}{p_1}\geq 1$, from (3.5) we get

$$II\leq q(p_2')^{q-1}B_L^q\left(\int_a^b f^{p_1}w_1\right)^{\frac{q}{p_1}}\left(\int_a^b g^{p_2}w_2\right)^{\frac{q}{p_2}}. \tag{3.6}$$

Finally, from (3.2), (3.3) and (3.6) we obtain

$$\left(\int_a^b\left(\int_a^x f\right)^q\left(\int_a^x g\right)^q w(x)dx\right)^{\frac{1}{q}}\leq 8\left(A_L+q^{\frac{1}{q}}(p_2')^{\frac{1}{q'}}B_L\right)\left(\int_a^b f^{p_1}w_1\right)^{\frac{1}{p_1}}\left(\int_a^b g^{p_2}w_2\right)^{\frac{1}{p_2}},$$

and we are done. □

4. Proof of Theorem 3.

Proof. The necessity of the conditions follows as in Theorem 2. Observe that, by simmetry, the condition $C_L<\infty$ can be deduced from (1.3) as $B_L<\infty$ was deduced in the proof of Theorem 2.

For the sufficiency, we work again as in the proof of Theorem 1 and get

$$\int_a^b\left(\int_a^x f\right)^q\left(\int_a^x g\right)^q w(x)dx\leq 8^q(I+II).$$

Let us estimate I. Applying twice Hölder inequality, we obtain

$$\begin{aligned}
I &=\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}}f\right)^q\left(\int_{x_{i+1}}^{x_i}w\right)\left(\int_a^{x_{i+1}}g\right)^q\\
&\leq\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}}f^{p_1}w_1\right)^{\frac{q}{p_1}}\left(\int_{x_{i+2}}^{x_{i+1}}w_1^{1-p_1'}\right)^{\frac{q}{p_1'}}\left(\int_{x_{i+1}}^{x_i}w\right)\left(\int_a^{x_{i+1}}g\right)^q\\
&\leq\left(\sum_{i=0}^{\infty}\int_{x_{i+2}}^{x_{i+1}}f^{p_1}w_1\right)^{\frac{q}{p_1}}\left(\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}}w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i}w\right)^{\frac{r_1}{q}}\left(\int_a^{x_{i+1}}g\right)^{r_1}\right)^{\frac{q}{r_1}}\\
&\leq\left(\int_a^b f^{p_1}w_1\right)^{\frac{q}{p_1}}\left(\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}}w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i}w\right)^{\frac{r_1}{q}}\left(\sum_{j=i}^{\infty}\int_{x_{j+2}}^{x_{j+1}}g\right)^{r_1}\right)^{\frac{q}{r_1}}\\
&=\left(\int_a^b f^{p_1}w_1\right)^{\frac{q}{p_1}}\left(\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}}w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i}w\right)^{\frac{r_1}{q}}\left(h^+(\{a_n\})(i)\right)^{r_1}\right)^{\frac{q}{r_1}},
\end{aligned} \tag{4.1}$$

where $h^+(\{a_n\})(i)=\sum_{j=i}^{\infty}a_n$ and $a_n=\int_{x_{n+2}}^{x_{n+1}}g$ for all n.

Observe that h^+ is nothing but the discrete Hardy operator. Applying the well-known results about the weighted inequalities for h^+ (see [17]) and taking into account that $p_2\leq r_1$, we have that

$$\left(\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}}\left(h^+(\{a_n\})(i)\right)^{r_1}\right)^{\frac{q}{r_1}} \le C\left(\sum_{i=0}^{\infty} a_i^{p_2} v_i\right)^{\frac{q}{p_2}}, \tag{4.2}$$

whenever $\{v_i\}$ to be a sequence verifying

$$K=\sup_{m\ge 0}\left(\sum_{i=0}^{m}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}}\right)^{\frac{1}{r_1}}\left(\sum_{i=m}^{\infty} v_i^{1-p_2'}\right)^{\frac{1}{p_2'}}<\infty. \tag{4.3}$$

In such a case, the constant C in (4.2) verifies $C \le 4^q K^q$. Let $v_i=\left(\int_{x_{i+2}}^{x_{i+1}} w_2^{1-p_2'}\right)^{\frac{-p_2}{p_2'}}$. Let us see that for this sequence $\{v_i\}$, the condition (4.3) holds. Let $m \ge 0$. Then

$$\begin{aligned}
&\left(\sum_{i=0}^{m}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}}\right)^{\frac{1}{r_1}}\left(\sum_{i=m}^{\infty} v_i^{1-p_2'}\right)^{\frac{1}{p_2'}}\\
&=\left(\sum_{i=0}^{m}\int_{x_{i+2}}^{x_{i+1}}\left(\int_{x_{i+2}}^{x} w_1^{1-p_1'}\right)^{\frac{r_1}{q'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}} w_1^{1-p_1'}(x)dx\right)^{\frac{1}{r_1}}\left(\int_a^{x_{m+1}} w_2^{1-p_2'}\right)^{\frac{1}{p_2'}}\\
&\le\left(\sum_{i=0}^{m}\int_{x_{i+2}}^{x_{i+1}}\left(\int_{x_{i+2}}^{x} w_1^{1-p_1'}\right)^{\frac{r_1}{q'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}} w_1^{1-p_1'}(x)dx\right)^{\frac{1}{r_1}}\left(\int_a^{x_{m+2}} w_2^{1-p_2'}\right)^{\frac{1}{p_2'}}\\
&+\left(\sum_{i=0}^{m-1}\int_{x_{i+2}}^{x_{i+1}}\left(\int_{x_{i+2}}^{x} w_1^{1-p_1'}\right)^{\frac{r_1}{q'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}} w_1^{1-p_1'}(x)dx\right)^{\frac{1}{r_1}}\left(\int_{x_{m+2}}^{x_{m+1}} w_2^{1-p_2'}\right)^{\frac{1}{p_2'}}\\
&+\left(\int_{x_{m+2}}^{x_{m+1}}\left(\int_{x_{m+2}}^{x} w_1^{1-p_1'}\right)^{\frac{r_1}{q'}}\left(\int_{x_{m+1}}^{x_m} w\right)^{\frac{r_1}{q}} w_1^{1-p_1'}(x)dx\right)^{\frac{1}{r_1}}\left(\int_{x_{m+2}}^{x_{m+1}} w_2^{1-p_2'}\right)^{\frac{1}{p_2'}}\\
&\le\left(\int_{x_{m+2}}^{x_1}\left(\int_{x_{m+2}}^{x} w_1^{1-p_1'}\right)^{\frac{r_1}{q'}}\left(\int_x^b w\right)^{\frac{r_1}{q}} w_1^{1-p_1'}(x)dx\right)^{\frac{1}{r_1}}\left(\int_a^{x_{m+2}} w_2^{1-p_2'}\right)^{\frac{1}{p_2'}}\\
&+\left(\int_{x_{m+1}}^{x_1}\left(\int_{x_{m+1}}^{x} w_1^{1-p_1'}\right)^{\frac{r_1}{q'}}\left(\int_x^b w\right)^{\frac{r_1}{q}} w_1^{1-p_1'}(x)dx\right)^{\frac{1}{r_1}}\left(\int_a^{x_{m+1}} w_2^{1-p_2'}\right)^{\frac{1}{p_2'}}\\
&+\left(\frac{p_1'}{r_1}\right)^{\frac{1}{r_1}}\left(\int_{x_{m+1}}^{b} w\right)^{\frac{1}{q}}\left(\int_a^{x_{m+1}} w_1^{1-p_1'}\right)^{\frac{1}{p_1'}}\left(\int_a^{x_{m+1}} w_2^{1-p_2'}\right)^{\frac{1}{p_2'}} \le 2C_L+\left(\frac{p_1'}{r_1}\right)^{\frac{1}{r_1}} A_L.
\end{aligned}$$

Therefore, (4.2) holds. Transporting (4.2) to (4.1) and applying Hölder inequality, we obtain

$$I \le 4^q\left(2C_L+\left(\frac{p_1'}{r_1}\right)^{\frac{1}{r_1}} A_L\right)^q\left(\int_a^b f^{p_1} w_1\right)^{\frac{q}{p_1}}\left(\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} g\right)^{p_2}\left(\int_{x_{i+2}}^{x_{i+1}} w_2^{1-p_2'}\right)^{-\frac{p_2}{p_2'}}\right)^{\frac{q}{p_2}}$$

$$
\begin{aligned}
&\leq 4^q \left(2C_L + \left(\frac{p_1'}{r_1}\right)^{\frac{1}{r_1}} A_L\right)^q \left(\int_a^b f^{p_1} w_1\right)^{\frac{q}{p_1}} \times \\
&\qquad \left(\sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} g^{p_2} w_2\right) \left(\int_{x_{i+2}}^{x_{i+1}} w_2^{1-p_2'}\right)^{\frac{p_2}{p_2'}} \left(\int_{x_{i+2}}^{x_{i+1}} w_2^{1-p_2'}\right)^{-\frac{p_2}{p_2'}}\right)^{\frac{q}{p_2}} \\
&= 4^q \left(2C_L + \left(\frac{p_1'}{r_1}\right)^{\frac{1}{r_1}} A_L\right)^q \left(\int_a^b f^{p_1} w_1\right)^{\frac{q}{p_1}} \left(\sum_{i=0}^{\infty} \int_{x_{i+2}}^{x_{i+1}} g^{p_2} w_2\right)^{\frac{q}{p_2}} \\
&\leq 4^q \left(2C_L + \left(\frac{p_1'}{r_1}\right)^{\frac{1}{r_1}} A_L\right)^q \left(\int_a^b f^{p_1} w_1\right)^{\frac{q}{p_1}} \left(\int_a^b g^{p_2} w_2\right)^{\frac{q}{p_2}}.
\end{aligned} \tag{4.4}
$$

For the estimation of II, we work as in the estimation of II in Theorem 2, obtaining

$$
\begin{aligned}
II &\leq q(p_2')^{q-1} \sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1} w_1\right)^{\frac{q}{p_1}} \left(\int_{x_{i+1}}^{x_i} g^{p_2} w_2\right)^{\frac{q}{p_2}} \\
&\times \left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{q}{p_1'}} \left(\int_{x_{i+1}}^{x_i} \left(\int_y^{x_i} w\right)^{\frac{r_2}{q}} \left(\int_{x_{i+1}}^{y} w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy\right)^{\frac{q}{r_2}}.
\end{aligned}
$$

Now, applying condition (ii) and Hölder inequality and taking into account that $\dfrac{r_1}{p_2} \geq 1$, we get

$$
\begin{aligned}
II &\leq q(p_2')^{q-1} B_L^q \sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1} w_1\right)^{\frac{q}{p_1}} \left(\int_{x_{i+1}}^{x_i} g^{p_2} w_2\right)^{\frac{q}{p_2}} \\
&\leq q(p_2')^{q-1} B_L^q \left(\sum_{i=0}^{\infty} \int_{x_{i+2}}^{x_{i+1}} f^{p_1} w_1\right)^{\frac{q}{p_1}} \left(\sum_{i=0}^{\infty} \left(\int_{x_{i+1}}^{x_i} g^{p_2} w_2\right)^{\frac{r_1}{p_2}}\right)^{\frac{q}{r_1}} \\
&\leq q(p_2')^{q-1} B_L^q \left(\int_a^b f^{p_1} w_1\right)^{\frac{q}{p_1}} \left(\int_a^b g^{p_2} w_2\right)^{\frac{q}{p_2}}.
\end{aligned}
$$

From the above estimates, we finally get

$$
\|H(f,g)\|_{q,w} \leq 8 \left(8C_L + 4\left(\frac{p_1'}{r_1}\right)^{\frac{1}{r_1}} A_L + q^{\frac{1}{q}} (p_2')^{\frac{1}{q'}} B_L\right) \|f\|_{p_1,w_1} \|g\|_{p_2,w_2},
$$

as we wished to prove.

□

5. Proof of theorem 4

Proof. For the necessity, we begin by showing that (1.3) implies $D_L < \infty$.

The inequality (1.3) is equivalent to

$$
\left(\int_a^b \left(\int_a^x f\right)^q \left(\int_a^x \frac{g}{||g||_{p_2,w_2}}\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C \left(\int_a^b f^{p_1} w_1\right)^{\frac{1}{p_1}}. \tag{5.1}
$$

This means that the Hardy operator is bounded from $L^{p_1}(w_1)$ to $L^q(u)$, where

$$
u(x) = \left(\int_a^x \frac{g}{||g||_{p_2,w_2}}\right)^q w(x),
$$

and the constant C does not depend on g.

Since $q < p_1$, applying Theorem A, the inequality (5.1) implies

$$\left\{\int_a^b \left(\int_x^b \left(\int_a^t \frac{g}{||g||_{p_2,w_2}}\right)^q w(t)dt\right)^{\frac{r_1}{p_1}} \left(\int_a^x w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}} \left(\int_a^x \frac{g}{||g||_{p_2,w_2}}\right)^q w(x)dx\right\}^{\frac{1}{r_1}}$$
$$\leq Cq^{-\frac{1}{q}} \left(\frac{p_1'q}{r_1}\right)^{-\frac{1}{q'}} \left(\frac{q}{p_1'}\right)^{\frac{1}{r_1}},$$

or, equivalently,

$$\left\{\int_a^b \left(\int_x^b \left(\int_a^t g\right)^q w(t)dt\right)^{\frac{r_1}{p_1}} \left(\int_a^x w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}} \left(\int_a^x g\right)^q w(x)dx\right\}^{\frac{1}{r_1}}$$
$$\leq Cq^{-\frac{1}{q}} \left(\frac{p_1'q}{r_1}\right)^{-\frac{1}{q'}} \left(\frac{q}{p_1'}\right)^{\frac{1}{r_1}} ||g||_{p_2,w_2}.$$

This inequality implies immediately that

$$\left\{\int_a^b \left(\int_a^x g\right)^{r_1} \left(\int_x^b w\right)^{\frac{r_1}{p_1}} \left(\int_a^x w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}} w(x)dx\right\}^{\frac{1}{r_1}} \leq Cq^{-\frac{1}{q}} \left(\frac{p_1'q}{r_1}\right)^{-\frac{1}{q'}} \left(\frac{q}{p_1'}\right)^{\frac{1}{r_1}} ||g||_{p_2,w_2},$$

which means that the Hardy operator $Pg(x) = \int_a^x g$ is bounded from $L^{p_2}(w_2)$ to $L^{r_1}(\overline{u})$, where

$$\overline{u}(x) = \left(\int_x^b w\right)^{\frac{r_1}{p_1}} \left(\int_a^x w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}} w(x).$$

Since $p_2 > r_1$, applying Theorem A, this inequality holds if and only if

$$\overline{D_L} = \left\{\int_a^b \left(\int_x^b \overline{u}\right)^{\frac{s}{p_2}} \left(\int_a^x w_2^{1-p_2'}\right)^{\frac{s}{p_2'}} \overline{u}(x)dx\right\}^{\frac{1}{s}} < \infty,$$

where $\frac{1}{s} = \frac{1}{r_1} - \frac{1}{p_2}$, and

$$Cq^{-\frac{1}{q}} \left(\frac{p_1'q}{r_1}\right)^{-\frac{1}{q'}} \left(\frac{q}{p_1'}\right)^{\frac{1}{r_1}} \geq \overline{D_L} r_1^{\frac{1}{r_1}} \left(\frac{p_2'r_1}{s}\right)^{\frac{1}{r_1'}} \left(\frac{p_2'}{r_1}\right)^{\frac{1}{s}}.$$

Taking into account that $\int_x^b \overline{u} \geq \frac{q}{r_1} \left(\int_a^x w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}} \left(\int_x^b w\right)^{\frac{r_1}{p_1}+1}$ and the identities $\frac{r_1}{p_1'}\frac{s}{p_2} + \frac{r_1}{p_1'} = \frac{s}{p_1'}$ and $\left(\frac{r_1}{p_1}+1\right)\frac{s}{p_2} + \frac{r_1}{p_1} = \frac{s}{p_1} + \frac{s}{p_2}$, we obtain $D_L < \infty$ and

$$Cq^{-\frac{1}{q}} \left(\frac{p_1'q}{r_1}\right)^{-\frac{1}{q'}} \left(\frac{q}{p_1'}\right)^{\frac{1}{r_1}} \geq D_L \left(\frac{q}{r_1}\right)^{\frac{1}{p_2}} r_1^{\frac{1}{r_1}} \left(\frac{p_2'r_1}{s}\right)^{\frac{1}{r_1'}} \left(\frac{p_2'}{r_1}\right)^{\frac{1}{s}}, \text{ i.e.,}$$

$$C \geq D_L \left(\frac{q}{r_1}\right)^{\frac{1}{p_2}} r_1^{\frac{1}{r_1}} \left(\frac{p_2'r_1}{s}\right)^{\frac{1}{r_1'}} \left(\frac{p_2'}{r_1}\right)^{\frac{1}{s}} q^{\frac{1}{q}} \left(\frac{p_1'q}{r_1}\right)^{\frac{1}{q'}} \left(\frac{p_1'}{q}\right)^{\frac{1}{r_1}}.$$

In order to prove $F_L < \infty$ we begin as in the proof of $D_L < \infty$. We have proved that (1.3) is equivalent to the Hardy operator to be bounded from $L^{p_1}(w_1)$ to $L^q(u)$, where

$$u(x) = \left(\int_a^x \frac{g}{||g||_{p_2,w_2}} \right)^q w(x),$$

and the constant C does not depend on g.

Since $q < p_1$, by Theorem A, the inequality (5.1) implies

$$\left\{ \int_a^b \left(\int_x^b \left(\int_a^t \frac{g}{||g||_{p_2,w_2}} \right)^q w(t)dt \right)^{\frac{r_1}{q}} \left(\int_a^x w_1^{1-p_1'} \right)^{\frac{r_1}{q'}} w_1^{1-p_1'}(x)dx \right\}^{\frac{1}{r_1}} \le Cq^{-\frac{1}{q}} \left(\frac{p_1' q}{r_1} \right)^{-\frac{1}{q'}},$$

or, equivalently,

$$\left\{ \int_a^b \left(\int_x^b \left(\int_a^t g \right)^q w(t)dt \right)^{\frac{r_1}{q}} \left(\int_a^x w_1^{1-p_1'} \right)^{\frac{r_1}{q'}} w_1^{1-p_1'}(x)dx \right\}^{\frac{1}{r_1}} \le Cq^{-\frac{1}{q}} \left(\frac{p_1' q}{r_1} \right)^{-\frac{1}{q'}} ||g||_{p_2,w_2}.$$

This inequality implies immediately that

$$\left\{ \int_a^b \left(\int_a^x g \right)^{r_1} \left(\int_x^b w \right)^{\frac{r_1}{q}} \left(\int_a^x w_1^{1-p_1'} \right)^{\frac{r_1}{q'}} w_1^{1-p_1'}(x)dx \right\}^{\frac{1}{r_1}} \le Cq^{-\frac{1}{q}} \left(\frac{p_1' q}{r_1} \right)^{-\frac{1}{q'}} ||g||_{p_2,w_2},$$

which means that the Hardy operator P is bounded from $L^{p_2}(w_2)$ to $L^{r_1}(\tilde{u})$, where

$$\tilde{u}(x) = \left(\int_x^b w \right)^{\frac{r_1}{q}} \left(\int_a^x w_1^{1-p_1'} \right)^{\frac{r_1}{q'}} w_1^{1-p_1'}(x).$$

Since $p_2 > r_1$, applying Theorem A, this inequality holds if and only if $F_L < \infty$. Moreover

$$Cq^{-\frac{1}{q}} \left(\frac{p_1' q}{r_1} \right)^{-\frac{1}{q'}} \ge F_L r_1^{\frac{1}{r_1}} \left(\frac{p_2' r_1}{s} \right)^{\frac{1}{r_1'}} \quad \text{, i.e.,} \quad C \ge F_L r_1^{\frac{1}{r_1}} \left(\frac{p_2' r_1}{s} \right)^{\frac{1}{r_1'}} q^{\frac{1}{q}} \left(\frac{p_1' q}{r_1} \right)^{\frac{1}{q'}}.$$

Observe that, by simmetry, the condition $E_L < \infty$ can be deduced from (1.3) as $F_L < \infty$ was deduced in the above proof. Moreover, $C \ge E_L q^{\frac{1}{q}} \left(\frac{p_2' q}{r_2} \right)^{\frac{1}{q'}} r_2^{\frac{1}{r_2}} \left(\frac{p_1' r_2}{s} \right)^{\frac{1}{r_2'}}$.

For the sufficiency, we work as in Theorem 3, obtaining

$$\int_a^b \left(\int_a^x f \right)^q \left(\int_a^x g \right)^q w(x)dx \le 8^q (I + II)$$

and

$$I \le \left(\int_a^b f^{p_1} w_1 \right)^{\frac{q}{p_1}} \left(\sum_{i=0}^{\infty} \left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'} \right)^{\frac{r_1}{p_1'}} \left(\int_{x_{i+1}}^{x_i} w \right)^{\frac{r_1}{q}} \left(h^+(\{a_n\})(i) \right)^{r_1} \right)^{\frac{q}{r_1}}, \tag{5.2}$$

where $h^+(\{a_n\})(i) = \sum_{j=i}^{\infty} a_n$ and $a_n = \int_{x_{n+2}}^{x_{n+1}} g$ for all n.

Applying the well-known results about the weighted inequalities for h^+ (see [17]) and taking into account that $r_1 < p_2$, we have that the inequality

$$\left(\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}}\left(h^+(\{a_n\})(i)\right)^{r_1}\right)^{\frac{q}{r_1}} \le C\left(\sum_{i=0}^{\infty} a_i^{p_2} v_i\right)^{\frac{q}{p_2}}, \tag{5.3}$$

holds whenever $\{v_i\}$ to be a sequence verifying

$$K=\left\{\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}}\right)^{\frac{s}{r_1}}\left(\sum_{i=n}^{\infty} v_i^{1-p_2'}\right)^{\frac{s}{r_1'}} v_n^{1-p_2'}\right\}^{\frac{1}{s}} < \infty. \tag{5.4}$$

In such a case, the constant C in (5.3) verifies $C^{\frac{1}{q}} \le r_1^{\frac{1}{r_1}}(p_2')^{\frac{1}{r_1'}}K$. Again as in the proof of Theorem 3, let $v_i=\left(\int_{x_{i+2}}^{x_{i+1}} w_2^{1-p_2'}\right)^{\frac{-p_2}{p_2'}}$. Let us see that for this sequence $\{v_i\}$, the condition (5.4) holds:

$$\begin{aligned}
K^s &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}}\right)^{\frac{s}{r_1}}\left(\int_a^{x_{n+1}} w_2^{1-p_2'}\right)^{\frac{s}{r_1'}}\int_{x_{n+2}}^{x_{n+1}} w_2^{1-p_2'}\\
&\le 2^{\frac{s}{r_1'}}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}}\right)^{\frac{s}{r_1}}\left(\int_{x_{n+2}}^{x_{n+1}} w_2^{1-p_2'}\right)^{\frac{s}{r_1'}+1}\\
&+2^{\frac{s}{r_1'}}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}}\right)^{\frac{s}{r_1}}\left(\int_a^{x_{n+2}} w_2^{1-p_2'}\right)^{\frac{s}{r_1'}}\int_{x_{n+2}}^{x_{n+1}} w_2^{1-p_2'}\\
&= J_1+J_2.
\end{aligned}$$

Let us estimate J_1:

$$\begin{aligned}
J_1 &= 2^{\frac{s}{r_1'}}\frac{s}{p_2'}\sum_{n=0}^{\infty}\int_{x_{n+2}}^{x_{n+1}}\left(\sum_{i=0}^{n}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}}\right)^{\frac{s}{r_1}}\left(\int_{x_{n+2}}^{x} w_2^{1-p_2'}\right)^{\frac{s}{r_1'}} w_2^{1-p_2'}(x)dx\\
&\le 2^s\frac{s}{p_2'}\sum_{n=0}^{\infty}\int_{x_{n+2}}^{x_{n+1}}\left(\sum_{i=0}^{n-1}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}}\right)^{\frac{s}{r_1}}\left(\int_{x_{n+2}}^{x} w_2^{1-p_2'}\right)^{\frac{s}{r_1'}} w_2^{1-p_2'}(x)dx\\
&+2^s\frac{s}{p_2'}\sum_{n=0}^{\infty}\left(\int_{x_{n+2}}^{x_{n+1}} w_1^{1-p_1'}\right)^{\frac{s}{p_1'}}\left(\int_{x_{n+1}}^{x_n} w\right)^{\frac{s}{q}}\left(\int_{x_{n+2}}^{x_{n+1}} w_2^{1-p_2'}\right)^{\frac{s}{p_2'}}\\
&\le 2^s\frac{s}{p_2'}\left(\frac{r_1}{p_1'}\right)^{\frac{s}{r_1}}\sum_{n=0}^{\infty}\int_{x_{n+2}}^{x_{n+1}}\left(\int_{x_{n+1}}^{x_1}\left(\int_a^y w_1^{1-p_1'}\right)^{\frac{r_1}{q'}} w_1^{1-p_1'}(y)\left(\int_y^b w\right)^{\frac{r_1}{q}}dy\right)^{\frac{s}{r_1}}\\
&\qquad\times\left(\int_a^x w_2^{1-p_2'}\right)^{\frac{s}{r_1'}} w_2^{1-p_2'}(x)dx\\
&+2^s\frac{s}{p_2'}\frac{s}{q}\sum_{n=0}^{\infty}\int_{x_{n+1}}^{x_n}\left(\int_{x_{n+2}}^{x_{n+1}} w_1^{1-p_1'}\right)^{\frac{s}{p_1'}}\left(\int_x^{x_{n+2}} w\right)^{\frac{s}{q}-1}\left(\int_a^{x_{n+1}} w_2^{1-p_2'}\right)^{\frac{s}{p_2'}} w(x)dx
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^s \frac{s}{p_2'}\left(\frac{r_1}{p_1'}\right)^{\frac{s}{r_1}} \sum_{n=0}^{\infty}\int_{x_{n+2}}^{x_{n+1}} \left(\int_x^b \left(\int_a^y w_1^{1-p_1'}\right)^{\frac{r_1}{q'}} w_1^{1-p_1'}(y)\left(\int_y^b w\right)^{\frac{r_1}{q}} dy\right)^{\frac{s}{r_1}} \\
&\qquad\qquad \times \left(\int_a^x w_2^{1-p_2'}\right)^{\frac{s}{r_1'}} w_2^{1-p_2'}(x)dx \\
&+2^s \frac{s}{p_2'}\frac{s}{q}\sum_{n=0}^{\infty}\int_{x_{n+1}}^{x_n}\left(\int_a^x w_1^{1-p_1'}\right)^{\frac{s}{p_1'}}\left(\int_x^b w\right)^{\frac{s}{p_1}+\frac{s}{p_2}}\left(\int_a^x w_2^{1-p_2'}\right)^{\frac{s}{p_2'}} w(x)dx \\
&\leq 2^s \frac{s}{p_2'}\left(\frac{r_1}{p_1'}\right)^{\frac{s}{r_1}}\int_a^b \left(\int_x^b \left(\int_a^y w_1^{1-p_1'}\right)^{\frac{r_1}{q'}} w_1^{1-p_1'}(y)\left(\int_y^b w\right)^{\frac{r_1}{q}} dy\right)^{\frac{s}{r_1}}\left(\int_a^x w_2^{1-p_2'}\right)^{\frac{s}{r_1'}} w_2^{1-p_2'}(x)dx \\
&\qquad +2^s \frac{s}{p_2'}\frac{s}{q}\int_a^b \left(\int_a^x w_1^{1-p_1'}\right)^{\frac{s}{p_1'}}\left(\int_x^b w\right)^{\frac{s}{p_1}+\frac{s}{p_2}}\left(\int_a^x w_2^{1-p_2'}\right)^{\frac{s}{p_2'}} w(x)dx \\
&\qquad = 2^s \frac{s}{p_2'}\left(\frac{r_1}{p_1'}\right)^{\frac{s}{r_1}} F_L^s + 2^s\frac{s}{p_2'}\frac{s}{q}D_L^s < \infty.
\end{aligned}
$$

Now, we estimate J_2 in the same way:

$$
\begin{aligned}
J_2 &\leq 2^s \sum_{n=0}^{\infty}\int_{x_{n+2}}^{x_{n+1}}\left(\sum_{i=0}^{n-1}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\left(\int_{x_{i+1}}^{x_i} w\right)^{\frac{r_1}{q}}\right)^{\frac{s}{r_1}}\left(\int_a^x w_2^{1-p_2'}\right)^{\frac{s}{r_1'}} w_2^{1-p_2'}(x)dx \\
&+2^s\sum_{n=0}^{\infty}\left(\int_{x_{n+2}}^{x_{n+1}} w_1^{1-p_1'}\right)^{\frac{s}{p_1'}}\left(\int_{x_{n+1}}^{x_n} w\right)^{\frac{s}{q}}\left(\int_a^{x_{n+1}} w_2^{1-p_2'}\right)^{\frac{s}{p_2'}} \\
&\leq 2^s\left(\frac{r_1}{p_1'}\right)^{\frac{s}{r_1}} F_L^s + 2^s\frac{s}{q}D_L^s < \infty.
\end{aligned}
$$

Therefore, (5.4) holds, which implies (5.3). Transporting (5.3) to (5.2) and working exactly as in (4.4), we obtain

$$
I \leq r_1^{\frac{q}{r_1}}(p_2')^{\frac{q}{r_1'}}2^q\left(\frac{s}{p_2'}+1\right)^{\frac{q}{s}}\left(\left(\frac{r_1}{p_1'}\right)^{\frac{s}{r_1}} F_L^s + \frac{s}{q}D_L^s\right)^{\frac{q}{s}}\left(\int_a^b f^{p_1}w_1\right)^{\frac{q}{p_1}}\left(\int_a^b g^{p_2}w_2\right)^{\frac{q}{p_2}}.
$$

For the estimation of II, we begin as in the estimation of II in Theorems 2 and 3, obtaining

$$
\begin{aligned}
II &\leq q(p_2')^{q-1}\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} f^{p_1}w_1\right)^{\frac{q}{p_1}}\left(\int_{x_{i+1}}^{x_i} g^{p_2}w_2\right)^{\frac{q}{p_2}} \\
&\times\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{q}{p_1'}}\left(\int_{x_{i+1}}^{x_i}\left(\int_y^{x_i} w\right)^{\frac{r_2}{q}}\left(\int_{x_{i+1}}^{y} w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy\right)^{\frac{q}{r_2}}.
\end{aligned}
$$

Applying now the triple Hölder inequality with exponents $\frac{p_1}{q}$, $\frac{p_2}{q}$ and $\frac{s}{q}$, we have

$$\begin{aligned} II &\le q(p_2')^{q-1}\left(\int_a^b f^{p_1}w_1\right)^{\frac{q}{p_1}}\left(\int_a^b g^{p_2}w_2\right)^{\frac{q}{p_2}} \\ &\times\left(\sum_{i=0}^{\infty}\left(\int_{x_{i+2}}^{x_{i+1}} w_1^{1-p_1'}\right)^{\frac{s}{p_1'}}\left(\int_{x_{i+1}}^{x_i}\left(\int_y^{x_i} w\right)^{\frac{r_2}{q}}\left(\int_{x_{i+1}}^{y} w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy\right)^{\frac{s}{r_2}}\right)^{\frac{q}{s}}. \end{aligned}$$

If we call S the sum in brackets in the above inequality, it is simple to see that $S \le \frac{s}{p_1'}E_L^s$:

$$\begin{aligned} S &= \frac{s}{p_1'}\sum_{i=0}^{\infty}\int_{x_{i+2}}^{x_{i+1}}\left(\int_{x_{i+2}}^{x} w_1^{1-p_1'}\right)^{\frac{s}{r_2'}} w_1^{1-p_1'}(x)dx\left(\int_{x_{i+1}}^{x_i}\left(\int_y^{x_i} w\right)^{\frac{r_2}{q}}\left(\int_{x_{i+1}}^{y} w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy\right)^{\frac{s}{r_2}} \\ &\le \frac{s}{p_1'}\sum_{i=0}^{\infty}\int_{x_{i+2}}^{x_{i+1}}\left(\int_x^b\left(\int_y^b w\right)^{\frac{r_2}{q}}\left(\int_a^y w_2^{1-p_2'}\right)^{\frac{r_2}{q'}} w_2^{1-p_2'}(y)dy\right)^{\frac{s}{r_2}}\left(\int_a^x w_1^{1-p_1'}\right)^{\frac{s}{r_2'}} w_1^{1-p_1'}(x)dx \\ &= \frac{s}{p_1'}E_L^s. \end{aligned}$$

Then

$$II \le q(p_2')^{q-1}\left(\frac{s}{p_1'}\right)^{\frac{q}{s}} E_L^q\left(\int_a^b f^{p_1}w_1\right)^{\frac{q}{p_1}}\left(\int_a^b g^{p_2}w_2\right)^{\frac{q}{p_2}}$$

and we are done.

□

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M. Isabel Aguilar Cañestro, Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
E-mail address: mi_aguilar@hotmail.com

Pedro Ortega Salvador, Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
E-mail address: portega@uma.es

C. Ramírez Torreblanca, Departamento de Matemáticas, Escuela Politécnica Superior, Universidad de Córdoba, Córdoba, Spain
E-mail address: ma1ratoc@uco.es