Ann. Funct. Anal. (2024) 15:26
https://doi.org/10.1007/s43034-024-00327-y

Tusi Mathematical Research Group

ORIGINAL PAPER

Some new weighted weak-type iterated and bilinear modified Hardy inequalities

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Received: 6 September 2023 / Accepted: 29 January 2024

Abstract
We characterize the good weights for some weighted weak-type iterated and bilinear modified Hardy inequalities to hold.

Keywords Bilinear modified Hardy inequalities · Iterated modified Hardy inequalities · Weighted weak-type inequalities · Weighted inequalities

Mathematics Subject Classification 26D10 · 26D15

1 Introduction and results

The initial problem in the theory of weighted Hardy inequalities was the one of characterizing the positive functions w, v, the weights, such that

$$\left(\int_a^b \left(\int_a^x f\right)^q w(x)\mathrm{d}x\right)^{\frac{1}{q}} \leq C \left(\int_a^b f^p v\right)^{\frac{1}{p}} \tag{1.1}$$

holds for all positive measurable function f with a positive constant C independent of f, which means that the Hardy operator $Tf(x) = \int_a^x f$ is bounded from $L^p(v)$ to $L^q(w)$.

This problem was solved by Talenti [31], Muckenhoupt [23] and Bradley [4] in the case $p \leq q$, by Mazja [22] when $1 \leq q < p$, Sinnamon [27, 28] for $0 < q < 1 < p$

Communicated by Feng Dai.

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Published online: 02 March 2024

and Sinnamon and Stepanov [29] for $0 < q < 1 = p$. Their results are the following ones.

Theorem A ([4, 22, 23, 29, 31]) *Let $1 < q < \infty$, $1 \le p < \infty$ and let w, v be positive measurable functions on (a, b), where $-\infty \le a < b \le \infty$. Then there exists a positive constant C such that inequality (1.1) holds for all nonnegative functions f if and only if*

(i) *in the case $p \le q$,*

$$B_1 \equiv \sup_{s\in(a,b)} \left(\int_s^b w \right)^{\frac{1}{q}} \|\chi_{(a,s)} v^{-\frac{1}{p}}\|_{p'} < \infty,$$

and the best constant C in inequality (1.1) verifies $B_1 \le C \le K(q, p)B_1$, where $K(q, p) = \left(1 + \frac{q}{p'}\right)^{\frac{1}{q}} \left(1 + \frac{p'}{q}\right)^{\frac{1}{p'}}$ if $p > 1$ and $K(q, 1) = 1$;

(ii) *in the case $q < p$,*

$$B_2 \equiv \left(\int_a^b \left(\int_t^b w \right)^{\frac{r}{q}} \|\chi_{(a,t)} v^{-\frac{1}{p}}\|_{p'}^{\frac{rp'}{q'}} v^{1-p'}(t)dt \right)^{\frac{1}{r}} < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, and the best constant C in inequality (1.1) verifies $q \left(\frac{p'}{r}\right)^{\frac{1}{q'}} B_2 \le C \le q^{\frac{1}{q}} (p')^{\frac{1}{q'}} B_2$.

Weighted weak-type inequalities for T were also studied. By a weighted weak-type (p, q) inequality for T we mean the boundedness of T from $L^p(v)$ to $L^{q,\infty}(w)$, where

$$L^{q,\infty}(w) = \left\{ f : \|f\|_{q,\infty;w} = \sup_{\lambda>0} \lambda \left(\int_{\{x:|f(x)|>\lambda\}} w \right)^{\frac{1}{q}} < \infty \right\}.$$

Really, weighted weak-type inequalities have been studied for the modified Hardy operators $T_\beta f(x) = \beta(x) \int_a^x f$. This kind of inequalities are technically more difficult than the strong-type ones. In fact, the problem of characterizing the boundedness of T_β from $L^p(v)$ to $L^{q,\infty}(w)$ in the case $q < p$ is not completely solved yet.

The first results on weighted weak-type inequalities for modified Hardy operators are due to Andersen and Muckenhoupt [2], who worked with $\beta(x) = x^\alpha$, $\alpha \in \mathbb{R}$, on $(0, \infty)$. The weighted weak-type inequalities with more general functions β were characterized in [6, 20, 21]. The following two theorems contain such characterizations.

Theorem B ([6, 21]) *Let $1 \le p \le q < \infty$ and β, v and w be positive measurable functions on (a, b), where $-\infty \le a < b \le \infty$. Then there exists a positive constant C such that inequality*

$$\left\| \beta(x) \left(\int_a^x f \right) \right\|_{q,\infty;w} \le C \| f \|_{p,v} \tag{1.2}$$

holds for all nonnegative functions f if and only if

$$B_3 \equiv \sup_{a<s<b} \|\beta \chi_{(s,b)}\|_{q,\infty;w} \|\chi_{(a,s)} v^{-\frac{1}{p}}\|_{p'} < \infty, \tag{1.3}$$

and the best constant C in inequality (1.2) verifies $B_3 \le C \le 4B_3$.

Theorem C ([20]) *Let $0 < q < p < \infty$ with $p \ge 1$ and β, v and w be positive measurable functions on (a, b), where $-\infty \le a < b \le \infty$ and β is a monotone function. Then there exists a positive constant C such that inequality (1.2) holds for all nonnegative functions f if and only if the function Ψ defined on (a, b) by*

$$\Psi(x) = \sup_{b>c>x} \left[\left(\inf_{y\in(x,c)} \beta(y) \right) \left(\int_x^c w \right)^{\frac{1}{p}} \right] \|\chi_{(a,x)} v^{-\frac{1}{p}}\|_{p'}$$

belongs to $L^{r,\infty}(w)$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. In this case, the best constant C in inequality (1.2) verifies $2^{-\frac{1}{p}} \|\Psi\|_{r,\infty;w} \le C \le (1 + 4^p)^{\frac{1}{q}} \|\Psi\|_{r,\infty;w}$.

It is worth noting that weighted weak-type inequalities for modified linear or sublinear operators are included in the topic of weighted mixed weak-type inequalities, which goes back to the work of Andersen and Muckenhoupt [2] and have been studied by several authors (see [5, 16–19, 21, 26]).

Two new kinds of Hardy inequalities are the weighted iterated and bilinear Hardy inequalities. On one hand, weighted iterated Hardy inequalities are of the form

$$\left\| \left(\int_a^x \left(\int_a^t f \right)^r u(t)\mathrm{d}t \right)^{\frac{1}{r}} \right\|_{q,w} \le C \| f \|_{p,v} \tag{1.4}$$

or

$$\left\| \left(\int_a^x \left(\int_t^x f \right)^r u(t)\mathrm{d}t \right)^{\frac{1}{r}} \right\|_{q,w} \le C \| f \|_{p,v}, \tag{1.5}$$

and have been studied by many authors [3, 8–11, 24, 25, 30].

On the other hand, weighted strong-type bilinear Hardy inequalities

$$\left\| \left(\int_a^x f \right) \left(\int_a^x g \right) \right\|_{q,w} \leq C \|f\|_{p_1,v_1} \|g\|_{p_2,v_2} \tag{1.6}$$

were characterized in [1] and some of their generalizations and variants have also been studied later (see, for instance, [12–14, 30]).

Recently, the authors have characterized in [7] the weights w, v_1, v_2 for which the weighted weak-type bilinear modified Hardy inequality

$$\left\| \beta(x) \left(\int_a^x f \right) \left(\int_a^x g \right) \right\|_{q,\infty;w} \leq C \|f\|_{p_1,v_1} \|g\|_{p_2,v_2} \tag{1.7}$$

holds in the cases $0 < q < \infty$, $1 \leq p_1, p_2 < \infty$, $q < p_1$, $q < p_2$ and $\frac{1}{q} \leq \frac{1}{p_1} + \frac{1}{p_2}$. In this paper, we will complete the characterization of inequality (1.7) solving the problem for the case $\frac{1}{q} > \frac{1}{p_1} + \frac{1}{p_2}$.

As we showed in [7], inequality (1.7) is equivalent to two weighted weak-type iterated modified Hardy inequalities of the form

$$\left\| \alpha(x) \left\| u(t) \chi_{(a,x)}(t) \int_a^t f \right\|_r \right\|_{q,\infty;w} \leq C \|f\|_{p,v}, \tag{1.8}$$

where $q < p$. Therefore, we will solve the problem of the characterization of (1.8) in the case $q < p$ and then we will get immediately the characterization of (1.7). It is worth noting that the good weights for (1.8) to hold in the case $p \leq q$ were characterized by the authors in [7].

In order to state the results for the iterated inequality (1.8), we define two functions Φ, Ψ on (a, b) by

$$\Phi(x) = \sup_{a<e<c<x<d<b} \left(\inf_{t\in(c,d)} \alpha(t) \|\chi_{(e,t)} u\|_r \right) \left(\int_c^d w \right)^{\frac{1}{p}} \|\chi_{(a,e)} v^{-\frac{1}{p}}\|_{p'}$$

and

$$\Psi(x) = \sup_{a<c<x<d<b} \left(\inf_{t\in(c,d)} \alpha(t) \right) \left(\int_c^d w \right)^{\frac{1}{p}} \times \left(\int_a^c \left(\int_t^c u^r \right)^{\frac{\theta}{r}} \left(\int_a^t v^{1-p'} \right)^{\frac{\theta}{r'}} v^{1-p'}(t) \mathrm{d}t \right)^{\frac{1}{\theta}},$$

where $\frac{1}{\theta} = \frac{1}{r} - \frac{1}{p}$.

The results are the following ones.

Theorem 1 *Let p, q, r with $0 < q < p$, $1 \leq p < \infty$ and $p \leq r \leq \infty$. Let α be a positive function in (a, b) such that*

$$\inf_{t \in (\rho, \nu)} \alpha(t) > 0 \tag{1.9}$$

for all ρ, ν with $a < \rho < \nu < b$. Let us suppose that for all $e \in (a, b)$ and all measurable sets $\Omega \subset (e, b)$, the function $\alpha(t)\|\chi_{(e,t)}u\|_r$ verifies

$$\inf_{t \in \Omega}\{\alpha(t)\|\chi_{(e,t)}u\|_r\} = \inf_{t \in (\rho_1, \rho_2)}\{\alpha(t)\|\chi_{(e,t)}u\|_r\}, \tag{1.10}$$

where $\rho_1 = \inf \Omega$ and $\rho_2 = \sup \Omega$. Then, (1.8) holds for all nonnegative functions f if and only if $\Phi \in L^{\eta,\infty}(w)$, where $\frac{1}{\eta} = \frac{1}{q} - \frac{1}{p}$. Moreover, the best constant C in inequality (1.8) verifies

$$2^{\frac{-1}{p}}\|\Phi\|_{\eta,\infty;w} \leq C \leq (\|\Phi\|^{\eta}_{\eta,\infty;w} + 2^p 4^{(1+\frac{1}{r})p} + 2^p 4^{\frac{p}{r}} K(r, p)^p)^{\frac{1}{q}}$$

if $r < \infty$ and

$$2^{\frac{-1}{p}}\|\Phi\|_{\eta,\infty;w} \leq C \leq (2^{\eta}\|\Phi\|^{\eta}_{\eta,\infty;w} + 8^p + 2^p)^{\frac{1}{q}}$$

if $r = \infty$.

Theorem 2 *Let p, q, r with $0 < q < p$ and $1 < r < p < \infty$. Let α be a positive monotone function in (a, b) and let us suppose that (1.10) holds. Then, the weighted iterated weak-type modified Hardy inequality (1.8) holds for all nonnegative functions f if and only if $\Phi, \Psi \in L^{\eta,\infty}(w)$, where $\frac{1}{\eta} = \frac{1}{q} - \frac{1}{p}$. Moreover, the best constant C in (1.8) verifies*

$$\max\{2^{\frac{-1}{p}}\|\Phi\|_{\eta,\infty;w}, 2^{\frac{-1}{p}} r\left(\frac{p'}{\theta}\right)^{\frac{1}{r'}}\|\Psi\|_{\eta,\infty;w}\} \leq C$$
$$\leq (\|\Phi\|^{\eta}_{\eta,\infty,w} + \|\Psi\|^{\eta}_{\eta,\infty;w} + 2^p 4^{(1+\frac{1}{r})p} + 2^p 4^{\frac{p}{r}} C^p_{r,p})^{\frac{1}{q}},$$

where $C_{r,p} = r^{\frac{1}{r}}(p')^{\frac{1}{r'}}$.

Observe that condition (1.10) holds if the function $\alpha(t)\|\chi_{(e,t)}u\|_r$ is monotone or increases in an interval (e, x_0) and decreases in (x_0, b). In the same way, condition (1.9) holds, for instance, if α is a positive monotone function.

As consequences of Theorems 1 and 2 we get the results for the weighted weak-type bilinear modified Hardy inequalities. In order to state them, we define the next functions on (a, b):

$$\alpha_i(x) = \sup_{c>x} \left(\inf_{(x,c)} \beta \right) \left(\int_x^c w \right)^{\frac{1}{p_i}}, \quad i = 1, 2,$$

$$\Phi_1(x) = \sup_{a<e<c<x<d<b} \left(\inf_{t\in(c,d)} \alpha_1(t) \| \chi_{(e,t)} v_1^{\frac{-1}{p_1}} \|_{p_1'} \right) \times \left(\int_c^d w \right)^{\frac{1}{p_2}} \| \chi_{(a,e)} v_2^{-\frac{1}{p_2}} \|_{p_2'},$$

$$\Phi_2(x) = \sup_{a<e<c<x<d<b} \left(\inf_{t\in(c,d)} \alpha_2(t) \| \chi_{(e,t)} v_2^{\frac{-1}{p_2}} \|_{p_2'} \right) \times \left(\int_c^d w \right)^{\frac{1}{p_1}} \| \chi_{(a,e)} v_1^{-\frac{1}{p_1}} \|_{p_1'},$$

$$\Psi_1(x) = \sup_{a<c<x<d<b} \left(\inf_{t\in(c,d)} \alpha_1(t) \right) \left(\int_c^d w \right)^{\frac{1}{p_2}} \times \left(\int_a^c \left(\int_t^c v_1^{1-p_1'} \right)^{\frac{\theta}{p_1'}} \left(\int_a^t v_2^{1-p_2'} \right)^{\frac{\theta}{p_1}} v_2^{1-p_2'}(t) \mathrm{d}t \right)^{\frac{1}{\theta}},$$

$$\Psi_2(x) = \sup_{a<c<x<d<b} \left(\inf_{t\in(c,d)} \alpha_2(t) \right) \left(\int_c^d w \right)^{\frac{1}{p_1}} \times \left(\int_a^c \left(\int_t^c v_2^{1-p_2'} \right)^{\frac{\theta}{p_2'}} \left(\int_a^t v_1^{1-p_1'} \right)^{\frac{\theta}{p_2}} v_1^{1-p_1'}(t) \mathrm{d}t \right)^{\frac{1}{\theta}},$$

where $\frac{1}{\theta} = \frac{1}{p_2'} - \frac{1}{p_1} = \frac{1}{p_1'} - \frac{1}{p_2} = 1 - \frac{1}{p_2} - \frac{1}{p_1}$.

The theorems read as follows.

Theorem 3 *Let* $0 < q < p_1, p_2 < \infty$, $p_1, p_2 \geq 1$, $\frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{q}$ *and* $p_2 \leq p_1'$. *Let* w, v_1, v_2, β *be positive measurable functions on* (a, b) *with* β *monotone. Assume that* *the functions* α_i *verify* (1.9) *and that for all* $e \in (a, b)$ *and all measurable sets* $\Omega \subset (e, b)$*, the functions* $\alpha_i(t)\|\chi_{(e,t)} v_i^{\frac{-1}{p_i}}\|_{p_i'}$*, i=1,2, verify* (1.10)*. Then the weighted weak-type bilinear modified Hardy inequality* (1.7) *holds if and only if* $\Phi_1, \Phi_2 \in L^{\eta,\infty}(w)$*, where* $\frac{1}{\eta} = \frac{1}{q} - \frac{1}{p_1} - \frac{1}{p_2}$.

Theorem 4 *Let* $0 < q < p_1, p_2 < \infty$*,* $p_1, p_2 \geq 1$*,* $\frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{q}$ *and* $p_2 > p_1'$*. Let* w, v_1, v_2, β *be positive measurable functions on* (a, b) *with* β *monotone. Assume that the functions* α_i *are monotone and that for all* $e \in (a, b)$ *and all measurable sets* $\Omega \subset (e, b)$*, the functions* $\alpha_i(t)\|\chi_{(e,t)} v_i^{\frac{-1}{p_i}}\|_{p_i'}$*, i=1,2, verify* (1.10)*. Then the weighted weak-type bilinear modified Hardy inequality* (1.7) *holds if and only if* $\Phi_1, \Phi_2, \Psi_1, \Psi_2 \in L^{\eta,\infty}(w)$*, where* $\frac{1}{\eta} = \frac{1}{q} - \frac{1}{p_1} - \frac{1}{p_2}$.

The proofs of Theorems 1 and 2 are included in Sects. 2 and 3, respectively, while Sect. 4 contains the proofs of Theorems 2 and 4.

2 Proof of Theorem 1

First of all, let us prove the necessity of the condition. Assume that the weak-type inequality (1.8) holds and let us see that $\Phi \in L^{\eta,\infty}(w)$. Let $\lambda > 0$ and $S_\lambda = \{x \in (a, b) : \Phi(x) > \lambda\}$. We will prove that

$$\lambda \left(\int_{S_\lambda} w \right)^{\frac{1}{\eta}} \leq 2^{\frac{1}{p}} C,$$

where C is the constant in (1.8). Let K be a compact subset of S_λ. For all $z \in K$, $z \in S_\lambda$ and then $\Phi(z) > \lambda$. This implies the existence of c_z, d_z, e_z with $e_z < c_z < d_z$ such that $z \in (c_z, d_z)$ and

$$\inf_{t \in (c_z, d_z)} \left[\alpha(t) \|\chi_{(e_z,t)} u\|_r \right] \left(\int_{c_z}^{d_z} w \right)^{\frac{1}{p}} \left(\int_a^{e_z} v^{1-p'} \right)^{\frac{1}{p'}} > \lambda. \tag{2.1}$$

Then, $K \subset \bigcup_{z \in K} (c_z, d_z)$. Since K is compact, there are $(c_{z_1}, d_{z_1}), (c_{z_2}, d_{z_2}) \ldots (c_{z_N}, d_{z_N})$ such that $K \subset \bigcup_{j=1}^N (c_{z_j}, d_{z_j})$. We can also suppose that

$$\sum_{j=1}^N \chi_{(c_{z_j}, d_{z_j})} \leq 2\chi_{\bigcup_{j=1}^N (c_{z_j}, d_{z_j})}.$$

Let, for all $j \in \{1, 2, \ldots, N\}$ and $x \in (a, b)$,

$$f_j(x) = \left(\inf_{y \in (c_{z_j}, d_{z_j})} \left[\alpha(y) \| \chi_{(e_{z_j}, y)} u \|_r \right] \left(\int_a^{e_{z_j}} v^{1-p'} \right) \right)^{-p} v(x)^{-p'} \chi_{(a, e_{z_j})}(x)$$

and

$$f = \left(\sum_{j=1}^{N} f_j \right)^{\frac{1}{p}}.$$

Let ε with $0 < \varepsilon < 1$. Let us see that

$$\bigcup_{j=1}^{N} (c_{z_j}, d_{z_j}) \subset \left\{ x \in (a, b) : \alpha(x) \left\| \chi_{(a,x)}(t) u(t) \int_a^t f \right\|_r > \varepsilon \right\}. \tag{2.2}$$

Indeed, if $z \in (c_{z_j}, d_{z_j})$, then

$$\begin{aligned}
\alpha(z) \left\| \chi_{(a,z)}(t) u(t) \int_a^t f \right\|_r &\geq \alpha(z) \left\| \chi_{(e_{z_j}, z)}(t) u(t) \int_a^t f \right\|_r \\
&\geq \alpha(z) \| \chi_{(e_{z_j}, z)}(t) u(t) \|_r \int_a^{e_{z_j}} f \\
&\geq \alpha(z) \| \chi_{(e_{z_j}, z)}(t) u(t) \|_r \int_a^{e_{z_j}} f_j^{\frac{1}{p}} \\
&= \frac{\alpha(z) \| \chi_{(e_{z_j}, z)}(t) u(t) \|_r \int_a^{e_{z_j}} v^{-\frac{p'}{p}}}{\inf_{y \in (c_{z_j}, d_{z_j})} [\alpha(y) \| \chi_{(e_{z_j}, y)} u \|_r] \int_a^{e_{z_j}} v^{1-p'}} \\
&\geq 1 > \varepsilon.
\end{aligned}$$

This proves (2.2). Applying the weak-type inequality,

$$\int_{\cup_{j=1}^{N}(c_{z_j},d_{z_j})} w \leq \int_{\left\{x\in(a,b):\alpha(x)\left\|\chi_{(a,x)}(t)u(t)\int_a^t f\right\|_r>\varepsilon\right\}} w \leq \frac{C^q}{\varepsilon^q}\|f\|_{p,v}^q.$$

Since the last inequality holds for all ε with $0<\varepsilon<1$, letting $\varepsilon\to 1^-$ we get

$$\int_{\cup_{j=1}^{N}(c_{z_j},d_{z_j})} w \leq C^q\|f\|_{p,v}^q.$$

Let $\gamma_j=\inf_{y\in(c_{z_j},d_{z_j})}[\alpha(y)\|\chi_{(e_{z_j},y)}u\|_r]$. Then the inequality above and (2.1) yield

$$\begin{aligned}
\int_{\cup_{j=1}^{N}(c_{z_j},d_{z_j})} w &\leq C^q\left(\int_a^b\left(\sum_{j=1}^N f_j(x)\right)v(x)\mathrm{d}x\right)^{\frac{q}{p}}\\
&= C^q\left(\int_a^b\sum_{j=1}^N\frac{v^{-p'}(x)\chi_{(a,e_{z_j})}(x)}{\gamma_j^p\left(\int_a^{e_{z_j}}v^{1-p'}\right)^p}v(x)\mathrm{d}x\right)^{\frac{q}{p}}\\
&= C^q\left(\sum_{j=1}^N\frac{1}{\gamma_j^p\left(\int_a^{e_{z_j}}v^{1-p'}\right)^p}\int_a^{e_{z_j}}v^{1-p'}\right)^{\frac{q}{p}}\\
&= C^q\left(\sum_{j=1}^N\frac{1}{\gamma_j^p\left(\int_a^{e_{z_j}}v^{1-p'}\right)^{p-1}}\right)^{\frac{q}{p}} \leq C^q\left(\sum_{j=1}^N\frac{1}{\lambda^p}\int_{c_{z_j}}^{d_{z_j}}w\right)^{\frac{q}{p}}\\
&= \frac{C^q}{\lambda^q}\left(\sum_{j=1}^N\int_{c_{z_j}}^{d_{z_j}}w\right)^{\frac{q}{p}} \leq \frac{2^{\frac{q}{p}}C^q}{\lambda^q}\left(\int_{\cup_{j=1}^{N}(c_{z_j},d_{z_j})} w\right)^{\frac{q}{p}}.
\end{aligned}$$

Then, we have

$$\int_{\cup_{j=1}^{N}(c_{z_j},d_{z_j})} w \leq \frac{2^{\frac{q}{p}}C^q}{\lambda^q}\left(\int_{\cup_{j=1}^{N}(c_{z_j},d_{z_j})} w\right)^{\frac{q}{p}},$$

i.e.,

$$\lambda \left(\int_{\cup_{j=1}^{N}(c_{z_j}, d_{z_j})} w \right)^{\frac{1}{\eta}} \leq 2^{\frac{1}{p}} C.$$

The last inequality implies

$$\lambda \left(\int_K w \right)^{\frac{1}{\eta}} \leq 2^{\frac{1}{p}} C.$$

Since the inequality above holds for all compact $K \subset S_\lambda$, the regularity of the measure $w(x)\mathrm{d}x$ gives

$$\lambda \left(\int_{S_\lambda} w \right)^{\frac{1}{\eta}} \leq 2^{\frac{1}{p}} C,$$

what proves that $\|\Phi\|_{\eta,\infty;w} \leq 2^{\frac{1}{p}} C$, as we wished to show.

Now, let us prove the sufficiency of the condition. Let f be a positive function such that $\int_a^b f^p v = 1$. Let $\lambda > 0$ and $O_\lambda = \{x \in (a,b) : \alpha(x) \|\chi_{(a,x)}(t) u(t) \int_a^t f\|_r > \lambda\}$. Then,

$$\int_{O_\lambda} w = \int_{O_\lambda \cap \{x \in (a,b) : \Phi(x) > \lambda^{\frac{q}{\eta}}\}} w + \int_{O_\lambda \cap \{x \in (a,b) : \Phi(x) \leq \lambda^{\frac{q}{\eta}}\}} w = I + II.$$

The estimation of I is as follows:

$$\lambda^q \int_{O_\lambda \cap \{x \in (a,b) : \Phi(x) > \lambda^{\frac{q}{\eta}}\}} w \leq \sup_{z>0} z^\eta \int_{O_\lambda \cap \{x \in (a,b) : \Phi(x) > z\}} w = \|\Phi\|_{\eta,\infty;w}^\eta.$$

Now, we will estimate II. Assume first that $r < \infty$. Let us suppose that $\int_a^b f < \infty$ and $\int_a^b \left(\int_a^t f\right)^r u^r(t)\mathrm{d}t < \infty$, too. Let $\{x_k\}$ be the sequence defined by $x_0 = b$ and

$$\int_a^{x_{k+1}} \left(\int_a^t f \right)^r u^r(t)\mathrm{d}t = \int_{x_{k+1}}^{x_k} \left(\int_a^t f \right)^r u^r(t)\mathrm{d}t.$$

The sequence $\{x_k\}$ decreases to a and verifies

$$\int_a^{x_k} \left(\int_a^t f \right)^r u^r(t)\mathrm{d}t = 4 \int_{x_{k+2}}^{x_{k+1}} \left(\int_a^t f \right)^r u^r(t)\mathrm{d}t \tag{2.3}$$

for all k. Let $E_k = O_\lambda \cap \{x \in (a,b) : \Phi(x) \le \lambda^{\frac{q}{\eta}}\} \cap (x_{k+1}, x_k)$. If $x \in E_k$, we have

$$\begin{aligned}
\lambda &< \alpha(x) \left(\int_a^x \left(\int_a^t f \right)^r u^r(t)\mathrm{d}t \right)^{\frac{1}{r}} \\
&\le \alpha(x) \left(\int_a^{x_k} \left(\int_a^t f \right)^r u^r(t)\mathrm{d}t \right)^{\frac{1}{r}} \\
&= 4^{\frac{1}{r}} \alpha(x) \left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_a^t f \right)^r u^r(t)\mathrm{d}t \right)^{\frac{1}{r}} \\
&\le 4^{\frac{1}{r}} \alpha(x) \left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_{x_{k+2}}^t f \right)^r u^r(t)\mathrm{d}t \right)^{\frac{1}{r}} \\
&\quad + 4^{\frac{1}{r}} \alpha(x) \left(\int_{x_{k+2}}^{x_{k+1}} u^r \right)^{\frac{1}{r}} \int_a^{x_{k+2}} f.
\end{aligned} \tag{2.4}$$

It is clear that, for each k, $E_k = E_{k,1} \cup E_{k,2}$, where

$$E_{k,1} = \left\{ x \in E_k : \alpha(x) \left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_{x_{k+2}}^t f \right)^r u^r(t)\mathrm{d}t \right)^{\frac{1}{r}} > \frac{\lambda}{2 \cdot 4^{\frac{1}{r}}} \right\}$$

and

$$E_{k,2} = \left\{ x \in E_k : \alpha(x) \left(\int_{x_{k+2}}^{x_{k+1}} u^r \right)^{\frac{1}{r}} \int_a^{x_{k+2}} f > \frac{\lambda}{2 \cdot 4^{\frac{1}{r}}} \right\}.$$

Since $p \leq r$, by Theorem A (i), we have that

$$\left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_{x_{k+2}}^{t} f \right)^{r} u^{r}(t) \mathrm{d}t \right)^{\frac{1}{r}} \leq K(r,p) \sup_{x_{k+2} < \gamma < x_{k+1}} \left(\int_{\gamma}^{x_{k+1}} u^{r} \right)^{\frac{1}{r}} \| \chi_{(x_{k+2},\gamma)} v^{-\frac{1}{p}} \|_{p'} \left(\int_{x_{k+2}}^{x_{k+1}} f^{p} v \right)^{\frac{1}{p}}. \tag{2.5}$$

Let us see that the supremum in (2.5) is finite. Let $\gamma \in (x_{k+2}, x_{k+1})$. As $\Phi \in L^{\eta,\infty}(w)$, Φ is finite almost everywhere. Let ρ, ν with $x_{k+1} < \rho < \nu < b$ and let $t \in (\rho, \nu)$ such that $\Phi(t) < \infty$. Then,

$$1 + \Phi(t) > \left(\inf_{x \in (\rho,\nu)} \alpha(x) \left(\int_{\gamma}^{x} u^{r} \right)^{\frac{1}{r}} \right) \left(\int_{\rho}^{\nu} w \right)^{\frac{1}{p}} \| \chi_{(a,\gamma)} v^{-\frac{1}{p}} \|_{p'}.$$

Thus, there is $\tilde{x} \in (\rho, \nu)$, which depends on γ, such that

$$\begin{aligned} 1 + \Phi(t) &> \alpha(\tilde{x}) \left(\int_{\gamma}^{\tilde{x}} u^{r} \right)^{\frac{1}{r}} \left(\int_{\rho}^{\nu} w \right)^{\frac{1}{p}} \| \chi_{(a,\gamma)} v^{-\frac{1}{p}} \|_{p'} \\ &\geq \alpha(\tilde{x}) \left(\int_{\gamma}^{x_{k+1}} u^{r} \right)^{\frac{1}{r}} \left(\int_{\rho}^{\nu} w \right)^{\frac{1}{p}} \| \chi_{(x_{k+2},\gamma)} v^{-\frac{1}{p}} \|_{p'}. \end{aligned}$$

Then, applying (1.9), we get

$$\left(\int_{\gamma}^{x_{k+1}} u^{r} \right)^{\frac{1}{r}} \| \chi_{(x_{k+2},\gamma)} v^{-\frac{1}{p}} \|_{p'} < \frac{1 + \Phi(t)}{\left(\inf_{(\rho,\nu)} \alpha \right) \left(\int_{\rho}^{\nu} w \right)^{\frac{1}{p}}} < \infty.$$

Therefore, the supremum in (2.5) is finite. Then, for all $x \in E_{k,1}$ we have

$$\begin{aligned} &\frac{\lambda}{2 \cdot 4^{\frac{1}{r}} K(r,p)} \\ &< \alpha(x) \sup_{x_{k+2} < \gamma < x_{k+1}} \left(\int_{\gamma}^{x_{k+1}} u^{r} \right)^{\frac{1}{r}} \| \chi_{(x_{k+2},\gamma)} v^{-\frac{1}{p}} \|_{p'} \left(\int_{x_{k+2}}^{x_{k+1}} f^{p} v \right)^{\frac{1}{p}}. \end{aligned}$$

Let $\varepsilon > 1$. For every k, there is $\gamma_k \in (x_{k+2}, x_{k+1})$ such that

$$\sup_{x_{k+2}<\gamma<x_{k+1}} \left(\int_{\gamma}^{x_{k+1}} u^r \right)^{\frac{1}{r}} \|\chi_{(x_{k+2},\gamma)} v^{-\frac{1}{p}}\|_{p'} < \varepsilon \left(\int_{\gamma_k}^{x_{k+1}} u^r \right)^{\frac{1}{r}} \|\chi_{(x_{k+2},\gamma_k)} v^{-\frac{1}{p}}\|_{p'}.$$

Therefore, for all $x \in E_{k,1}$ the following inequality holds:

$$\frac{\lambda}{2 \cdot 4^{\frac{1}{r}} K(r,p)} < \varepsilon\alpha(x) \left(\int_{\gamma_k}^{x_{k+1}} u^r \right)^{\frac{1}{r}} \|\chi_{(x_{k+2},\gamma_k)} v^{-\frac{1}{p}}\|_{p'} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}}.$$

Since $x_{k+1} < x$, we get

$$\frac{\lambda}{2 \cdot 4^{\frac{1}{r}} K(r,p)} < \varepsilon\alpha(x) \left(\int_{\gamma_k}^{x} u^r \right)^{\frac{1}{r}} \|\chi_{(a,\gamma_k)} v^{-\frac{1}{p}}\|_{p'} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}}.$$

The last inequality holds for all $x \in E_{k,1}$. Then,

$$\frac{\lambda}{2 \cdot 4^{\frac{1}{r}} K(r,p)} \le \varepsilon \inf_{x\in E_{k,1}} \left(\alpha(x) \left(\int_{\gamma_k}^{x} u^r \right)^{\frac{1}{r}} \right) \|\chi_{(a,\gamma_k)} v^{-\frac{1}{p}}\|_{p'} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}}.$$

Now, if we multiply both sides of this inequality by $\left(\int_{E_{k,1}} w\right)^{\frac{1}{p}}$ and apply condition (1.10), we have

$$\frac{\lambda}{2\cdot 4^{\frac{1}{r}} K(r,p)\varepsilon}\left(\int\limits_{E_{k,1}} w\right)^{\frac{1}{p}} \leq \inf_{x\in(\rho_k^1,\rho_k^2)}\left\{\alpha(x)\left(\int\limits_{\gamma_k}^{x} u^r\right)^{\frac{1}{r}}\right\}$$

$$\times\left(\int\limits_{\rho_k^1}^{\rho_k^2} w\right)^{\frac{1}{p}} \|\chi_{(a,\gamma_k)} v^{-\frac{1}{p}}\|_{p'}\left(\int\limits_{x_{k+2}}^{x_{k+1}} f^p v\right)^{\frac{1}{p}}$$

$$\leq \lambda^{\frac{q}{\eta}}\left(\int\limits_{x_{k+2}}^{x_{k+1}} f^p v\right)^{\frac{1}{p}},$$

where $\rho_k^1 = \inf E_{k,1}$, $\rho_k^2 = \sup E_{k,1}$ and the last inequality holds since

$$\inf_{x\in(\rho_k^1,\rho_k^2)}\left\{\alpha(x)\left(\int\limits_{\gamma_k}^{x} u^r\right)^{\frac{1}{r}}\right\}\left(\int\limits_{\rho_k^1}^{\rho_k^2} w\right)^{\frac{1}{p}} \|\chi_{(a,\gamma_k)} v^{-\frac{1}{p}}\|_{p'} \leq \Phi(t) \leq \lambda^{\frac{q}{\eta}}$$

for all $t \in E_{k,1}$. Thus,

$$\lambda\left(\int\limits_{E_{k,1}} w\right)^{\frac{1}{p}} \leq \varepsilon 2\cdot 4^{\frac{1}{r}} K(r,p)\lambda^{\frac{q}{\eta}}\left(\int\limits_{x_{k+2}}^{x_{k+1}} f^p v\right)^{\frac{1}{p}}.$$

Since this inequality holds for every $\varepsilon > 1$, letting $\varepsilon \to 1^+$ and raising to p we get

$$\int\limits_{E_{k,1}} w \leq 2^p 4^{\frac{p}{r}} K(r,p)^p \lambda^{\frac{qp}{\eta}-p} \int\limits_{x_{k+2}}^{x_{k+1}} f^p v.$$

Now, summing up in k, we have

$$\int\limits_{\cup_k E_{k,1}} w \leq \frac{2^p\cdot 4^{\frac{p}{r}} K(r,p)^p}{\lambda^q}\int\limits_a^b f^p v = \frac{2^p\cdot 4^{\frac{p}{r}} K(r,p)^p}{\lambda^q},$$

what finishes the estimation of $\int_{\cup_k E_{k,1}} w$. In order to estimate $\int_{\cup_k E_{k,2}} w$, we will use the technique, due to Lai [15], that we have already used in [7]. Let us define the sequence $\{y'_m\}$ as $y'_0 = b$ and $\int_a^{y'_{m+1}} f = \int_{y'_{m+1}}^{y'_m} f$. Let $\{y_n\}$ be the subsequence of $\{y'_m\}$ defined by $y_0 = y'_0$ and by deleting y'_{m+1} if $[y'_{m+1}, y'_m) \cap \{x_k\} = \emptyset$. In this way, if $y'_{m+1} = y_{n+1} \le x_{k+2} < y_n$, then $x_{k+2} \le y'_m$ and $y_{n+2} \le y'_{m+2}$, which yields

$$\int_a^{x_{k+2}} f \le \int_a^{y'_m} f = 4\int_{y'_{m+2}}^{y'_{m+1}} f \le 4\int_{y_{n+2}}^{y_{n+1}} f. \tag{2.6}$$

Let $E_2^n = \bigcup_{\{k: y_{n+1} \le x_{k+2} < y_n\}} E_{k,2}$. If $x \in E_2^n$, there exists k with $y_{n+1} \le x_{k+2} < y_n$ such that $x \in E_{k,2}$ and then, by (2.6),

$$\frac{\lambda}{2 \cdot 4^{\frac{1}{r}}} < \alpha(x) \left(\int_{x_{k+2}}^{x_{k+1}} u^r \right)^{\frac{1}{r}} \int_a^{x_{k+2}} f \le 4\alpha(x) \left(\int_{y_{n+1}}^{x} u^r \right)^{\frac{1}{r}} \int_{y_{n+2}}^{y_{n+1}} f. \tag{2.7}$$

Since (2.7) holds for all $x \in E_2^n$, we have

$$\frac{\lambda}{2 \cdot 4^{1+\frac{1}{r}}} \le \inf_{x \in E_2^n} \left[\alpha(x) \left(\int_{y_{n+1}}^{x} u^r \right)^{\frac{1}{r}} \right] \int_{y_{n+2}}^{y_{n+1}} f. \tag{2.8}$$

Multiplying both sides of (2.8) by $\left(\int_{E_2^n} w\right)^{\frac{1}{p}}$, applying Holder's inequality and (1.10), we get

$$\begin{aligned} \frac{\lambda}{2 \cdot 4^{1+\frac{1}{r}}} \left(\int_{E_2^n} w \right)^{\frac{1}{p}} &\le \inf_{x \in (\rho_1^n, \rho_2^n)} \left[\alpha(x) \left(\int_{y_{n+1}}^{x} u^r \right)^{\frac{1}{r}} \right] \\ &\quad \times \left(\int_{\rho_1^n}^{\rho_2^n} w \right)^{\frac{1}{p}} \| \chi_{(a, y_{n+1})} v^{-\frac{1}{p}} \|_{p'} \left(\int_{y_{n+2}}^{y_{n+1}} f^p v \right)^{\frac{1}{p}} \\ &\le \lambda^{\frac{q}{\eta}} \left(\int_{y_{n+2}}^{y_{n+1}} f^p v \right)^{\frac{1}{p}}, \end{aligned} \tag{2.9}$$

where we have used that

$$\inf_{x\in(\rho_1^n,\rho_2^n)}\left[\alpha(x)\left(\int_{y_{n+1}}^{x} u^r\right)^{\frac{1}{r}}\right]\left(\int_{\rho_1^n}^{\rho_2^n} w\right)^{\frac{1}{p}}\|\chi_{(a,y_{n+1})}v^{-\frac{1}{p}}\|_{p'}\leq\Phi(t)$$

for all $t\in E_2^n$.

Then, raising to the p in (2.9) and summing, we get

$$\begin{aligned}\int_{\cup_k E_{k,2}} w &= \sum_{k=0}^{\infty}\int_{E_{k,2}} w = \sum_{n=0}^{\infty}\sum_{\{k:y_{n+1}\leq x_{k+2}<y_n\}}\int_{E_{k,2}} w = \sum_{n=0}^{\infty}\int_{E_2^n} w \\ &\leq \frac{2^p\cdot 4^{(1+\frac{1}{r})p}}{\lambda^q}\left(\int_a^b f^p v\right) = \frac{2^p\cdot 4^{(1+\frac{1}{r})p}}{\lambda^q},\end{aligned} \tag{2.10}$$

which implies

$$II\leq\int_{\cup_k E_{k,1}} w+\int_{\cup_k E_{k,2}} w\leq\frac{1}{\lambda^q}(2^p\cdot 4^{(1+\frac{1}{r})p}+2^p\cdot 4^{\frac{p}{r}}K(r,p)^p).$$

This finishes the proof of the sufficiency in the case $r<\infty$. Now, we will deal with the case $r=\infty$. Let us consider two sequences $\{a_n\}$ and $\{b_n\}$, with $\{a_n\}$ decreasing to a and $\{b_n\}$ increasing to b. Then, $\Phi\in L^{\eta,\infty}(w)$ implies that $\Phi_n\in L^{\eta,\infty}(w,(a_n,b_n))$, where

$$\Phi_n(x)=\sup_{a_n<e<c<x<d<b_n}\left(\inf_{t\in(c,d)}\left(\alpha(t)\|\chi_{(e,t)}u\|_\infty\right)\right)\times\left(\int_c^d w\right)^{\frac{1}{p}}\|\chi_{(a_n,e)}v^{-\frac{1}{p}}\|_{p'}.$$

For fixed n, there is $r_0>p$ such that $(b_n-a_n)^{\frac{1}{r}}\leq 2$ for all $r\geq r_0$. Then, if $r\geq r_0$ and $a_n<e<x<b_n$, we have

$$\|\chi_{(e,x)}u\|_r=\left(\int_e^x |u|^r\right)^{\frac{1}{r}}\leq\|\chi_{(e,x)}u\|_\infty(b_n-a_n)^{\frac{1}{r}}\leq 2\|\chi_{(e,x)}u\|_\infty.$$

Therefore, if we define $\Phi_{n,r}(x)$ as

$$\Phi_{n,r}(x) = \sup_{a_n<e<c<x<d<b_n} \left(\inf_{t\in(c,d)} \left(\alpha(t)\|\chi_{(e,t)}u\|_r\right)\right) \times \left(\int_c^d w\right)^{\frac{1}{p}} \|\chi_{(a_n,e)}v^{-\frac{1}{p}}\|_{p'},$$

we have that $\Phi_{n,r} \in L^{\eta,\infty}(w,(a_n,b_n))$ for all $r \geq r_0$ and their norms are bounded by $2\|\Phi\|_{\eta,\infty,w}$. Now, applying the Theorem in the case which we have already proved, we have that the weak-type inequality

$$\left\| \chi_{(a_n,b_n)}(x)\alpha(x) \left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^t f \right\|_r \right\|_{q,\infty;w} \leq C_{r,p,q}\|\chi_{(a_n,b_n)}f\|_{p,v} \quad (2.11)$$

holds, where $C_{r,p,q} = (2^\eta\|\Phi\|^\eta_{\eta,\infty,w} + 2^p 4^{\frac{p}{r}}(4^p + K(r,p)^p))^{\frac{1}{q}}$. Since

$$\left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^t f \right\|_\infty = \lim_{r\to\infty} \left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^t f \right\|_r$$

for every x, by Fatou's lemma we have

$$\begin{aligned} &\left\| \chi_{(a_n,b_n)}(x)\alpha(x) \left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^t f \right\|_\infty \right\|_{q,\infty;w} \\ &\leq \liminf_{r\to\infty} \left\| \chi_{(a_n,b_n)}(x)\alpha(x) \left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^t f \right\|_r \right\|_{q,\infty;w}. \end{aligned} \quad (2.12)$$

Now, from (2.11) and (2.12) we get

$$\left\| \chi_{(a_n,b_n)}(x)\alpha(x) \left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^t f \right\|_\infty \right\|_{q,\infty;w} \leq C_{p,q}\|\chi_{(a_n,b_n)}f\|_{p,v}, \quad (2.13)$$

where $C_{p,q} = (2^\eta\|\Phi\|^\eta_{\eta,\infty,w} + 8^p + 2^p)^{\frac{1}{q}}$. Finally, since (2.13) holds for all n with a constant independent of n, letting n tend to infinity and applying the monotone convergence theorem, we get (1.8) in the case $r = \infty$.

3 Proof of Theorem 2

The necessity of condition $\Phi \in L^{\eta,\infty}(w)$ follows as in the proof of Theorem 1. Therefore, the best constant C in (1.8) verifies $C \geq 2^{\frac{-1}{p}} \|\Phi\|_{\eta,\infty,w}$. Let us prove now that (1.8) implies $\Psi \in L^{\eta,\infty}(w)$. Let $\lambda > 0$ and $S_\lambda = \{x \in (a,b) : \Psi(x) > \lambda\}$. Let K be a compact subset of S_λ. If $z \in K$, there exist c_z, d_z with $c_z < z < d_z$ such that

$$\left(\inf_{(c_z,d_z)} \alpha\right) \left(\int_{c_z}^{d_z} w\right)^{\frac{1}{p}} \left(\int_a^{c_z} \left(\int_t^{c_z} u^r\right)^{\frac{\theta}{r}} \left(\int_a^t v^{1-p'}\right)^{\frac{\theta}{r'}} v^{1-p'}(t)\mathrm{d}t\right)^{\frac{1}{\theta}} > \lambda. \tag{3.1}$$

Since K is compact, there exist $z_1, z_2, \ldots, z_N \in K$ such that $K \subset \bigcup_{j=1}^{N} (c_{z_j}, d_{z_j})$ and

$$\sum_{j=1}^{N} \chi_{(c_{z_j},d_{z_j})} \leq 2\chi_{\cup_{j=1}^{N}(c_{z_j},d_{z_j})}. \tag{3.2}$$

Let, for each $j \in \{1, 2, \ldots, N\}$,

$$f_j(x) = \left(\inf_{(c_{z_j},d_{z_j})} \alpha\right)^{-p} \chi_{(a,c_{z_j})}(x) \left(\int_x^{c_{z_j}} u^r\right)^{\frac{\theta}{r}} \left(\int_a^x v^{1-p'}\right)^{\frac{\theta}{r'}} v^{-p'}(x)$$
$$\times \left(\int_a^{c_{z_j}} \left(\int_t^{c_{z_j}} u^r\right)^{\frac{\theta}{r}} \left(\int_a^t v^{1-p'}\right)^{\frac{\theta}{r'}} v^{1-p'}(t)\mathrm{d}t\right)^{-\frac{p}{r}}$$

and

$$f = \left(\sum_{j=1}^{N} f_j\right)^{\frac{1}{p}}.$$

If $z \in (c_{z_j}, d_{z_j})$ and $\gamma_j = \inf_{(c_{z_j},d_{z_j})} \alpha$, we have

$$\alpha(z) \left(\int_a^z \left(\int_a^t f\right)^r u^r(t)\mathrm{d}t\right)^{\frac{1}{r}} \geq \alpha(z) \left(\int_a^{c_{z_j}} \left(\int_a^t f_j^{\frac{1}{p}}\right)^r u^r(t)\mathrm{d}t\right)^{\frac{1}{r}}$$

$$= \frac{\alpha(z)}{\gamma_j \left(\int_a^{c_{z_j}} \left(\int_t^{c_{z_j}} u^r \right)^{\frac{\theta}{r}} \left(\int_a^t v^{1-p'} \right)^{\frac{\theta}{r'}} v^{1-p'}(t) \mathrm{d}t \right)^{\frac{1}{r}}}$$
$$\times \left(\int_a^{c_{z_j}} \left(\int_a^t \left(\int_x^{c_{z_j}} u^r \right)^{\frac{\theta}{rp}} \left(\int_a^x v^{1-p'} \right)^{\frac{\theta}{r'p}} v^{1-p'}(x) \mathrm{d}x \right)^r u^r(t) \mathrm{d}t \right)^{\frac{1}{r}}. \tag{3.3}$$

If $h(x) = \left(\int_x^{c_{z_j}} u^r \right)^{\frac{\theta}{rp}} \left(\int_a^x v^{1-p'} \right)^{\frac{\theta}{r'p}} v^{1-p'}(x)$, the last factor in (3.3) can be written as follows

$$\begin{aligned} \left(\int_a^{c_{z_j}} \left(\int_a^t h(x) \mathrm{d}x \right)^r u^r(t) \mathrm{d}t \right)^{\frac{1}{r}} &= \left(\int_a^{c_{z_j}} \left(\int_a^t \left[\left(\int_a^s h \right)^r \right]' (s) \mathrm{d}s \right) u^r(t) \mathrm{d}t \right)^{\frac{1}{r}} \\ &= r^{\frac{1}{r}} \left(\int_a^{c_{z_j}} \left(\int_a^t \left(\int_a^s h \right)^{r-1} h(s) \mathrm{d}s \right) u^r(t) \mathrm{d}t \right)^{\frac{1}{r}} \\ &= r^{\frac{1}{r}} \left(\int_a^{c_{z_j}} \left(\int_s^{c_{z_j}} u^r(t) \mathrm{d}t \right) \left(\int_a^s h \right)^{r-1} h(s) \mathrm{d}s \right)^{\frac{1}{r}}. \end{aligned} \tag{3.4}$$

Let us estimate now $\int_a^s h$:

$$\begin{aligned} \int_a^s h(x) \mathrm{d}x &= \int_a^s \left(\int_x^{c_{z_j}} u^r \right)^{\frac{\theta}{rp}} \left(\int_a^x v^{1-p'} \right)^{\frac{\theta}{r'p}} v^{1-p'}(x) \mathrm{d}x \\ &\geq \left(\int_s^{c_{z_j}} u^r \right)^{\frac{\theta}{rp}} \int_a^s \left(\int_a^x v^{1-p'} \right)^{\frac{\theta}{r'p}} v^{1-p'}(x) \mathrm{d}x \\ &= \frac{rp'}{\theta} \left(\int_s^{c_{z_j}} u^r \right)^{\frac{\theta}{rp}} \left(\int_a^x v^{1-p'} \right)^{\frac{\theta}{rp'}}. \end{aligned}$$

Taking this estimate to (3.4), we get

$$\left(\int_a^{c_{z_j}}\left(\int_a^t h(x)\mathrm{d}x\right)^r u^r(t)\mathrm{d}t\right)^{\frac{1}{r}} \geq r^{\frac{1}{r}}\left(\frac{rp'}{\theta}\right)^{\frac{1}{r'}}$$

$$\times\left(\int_a^{c_{z_j}}\left(\int_s^{c_{z_j}} u^r\right)\left(\left(\int_s^{c_{z_j}} u^r\right)^{\frac{\theta}{rp}}\left(\int_a^x v^{1-p'}\right)^{\frac{\theta}{rp'}}\right)^{r-1} h(s)\mathrm{d}s\right)^{\frac{1}{r}}$$

$$= r\left(\frac{p'}{\theta}\right)^{\frac{1}{r'}}\left(\int_a^{c_{z_j}}\left(\int_s^{c_{z_j}} u^r\right)^{\frac{\theta}{r}}\left(\int_a^s v^{1-p'}\right)^{\frac{\theta}{r'}} v^{1-p'}(s)\mathrm{d}s\right)^{\frac{1}{r}}.$$

Going back to (3.3) we get that for all $z \in (c_{z_j}, d_{z_j})$,

$$\alpha(z)\left(\int_a^z\left(\int_a^t f\right)^r u^r(t)\mathrm{d}t\right)^{\frac{1}{r}} \geq r\left(\frac{p'}{\theta}\right)^{\frac{1}{r'}}\frac{\alpha(z)}{\gamma_j} \geq r\left(\frac{p'}{\theta}\right)^{\frac{1}{r'}}.$$

Therefore, by (1.8), we have

$$\int\limits_{\cup_{j=1}^N (c_{z_j}, d_{z_j})} w \leq \frac{C^q}{\left(r\left(\frac{p'}{\theta}\right)^{\frac{1}{r'}}\right)^q}\|f\|_{p,v}^q. \tag{3.5}$$

Let us estimate $\|f\|_{p,v}^q$. By definition of f, (3.1) and (3.2),

$$\begin{aligned}\|f\|_{p,v}^q &= \left[\int_a^b \sum_{j=1}^N \frac{\left(\int_x^{c_{z_j}} u^r\right)^{\frac{\theta}{r}}\left(\int_a^x v^{1-p'}\right)^{\frac{\theta}{r'}} v^{1-p'}(x)\chi_{(a,c_{z_j})}(x)}{\gamma_j^p\left(\int_a^{c_{z_j}}\left(\int_s^{c_{z_j}} u^r\right)^{\frac{\theta}{r}}\left(\int_a^s v^{1-p'}\right)^{\frac{\theta}{r'}} v^{1-p'}(s)\mathrm{d}s\right)^{\frac{p}{r}}}\mathrm{d}x\right]^{\frac{q}{p}} \\ &= \left[\sum_{j=1}^N \gamma_j^{-p}\left(\int_a^{c_{z_j}}\left(\int_s^{c_{z_j}} u^r\right)^{\frac{\theta}{r}}\left(\int_a^s v^{1-p'}\right)^{\frac{\theta}{r'}} v^{1-p'}(s)\mathrm{d}s\right)^{-\frac{p}{\theta}}\right]^{\frac{q}{p}} \\ &\leq \left(\sum_{j=1}^N \frac{1}{\lambda^p}\int_{c_{z_j}}^{d_{z_j}} w\right)^{\frac{q}{p}} \leq \frac{2^{\frac{q}{p}}}{\lambda^q}\left(\int\limits_{\cup_{j=1}^N (c_{z_j}, d_{z_j})} w\right)^{\frac{q}{p}}.\end{aligned} \tag{3.6}$$

Finally, from (3.5) and (3.6) we get

$$\lambda^q \left(\int\limits_{\cup_{j=1}^N (c_{z_j}, d_{z_j})} w \right)^{\frac{q}{\eta}} \leq \frac{2^{\frac{q}{p}} C^q}{\left(r \left(\frac{p'}{\theta} \right)^{\frac{1}{r'}} \right)^q},$$

which implies

$$\lambda \left(\int\limits_K w \right)^{\frac{1}{\eta}} \leq \frac{2^{\frac{1}{p}} C}{r \left(\frac{p'}{\theta} \right)^{\frac{1}{r'}}}.$$

Since the inequality above holds for all compact set $K \subset S_\lambda$, we have that $\Psi \in L^{\eta,\infty}(w)$ and $C \geq 2^{\frac{-1}{p}} r \left(\frac{p'}{\theta} \right)^{\frac{1}{r'}} \|\Psi\|_{\eta,\infty;w}$.

Let us prove now the sufficiency. Let f be a nonnegative function with $f \in L^1$ and $\int_a^b f^p v = 1$. Let $\lambda > 0$ and

$$O_\lambda = \left\{ x \in (a,b) : \alpha(x) \|\chi_{(a,x)}(t) u(t) \int\limits_a^t f \|_r > \lambda \right\}.$$

Then, as in the proof of Theorem 1,

$$\int\limits_{O_\lambda} w = \int\limits_{O_\lambda \cap \{x \in (a,b): \Phi(x) > \lambda^{\frac{q}{\eta}}\}} w + \int\limits_{O_\lambda \cap \{x \in (a,b): \Phi(x) \leq \lambda^{\frac{q}{\eta}}\}} w = I + II.$$

The estimation of I can be done as in the case $p \leq r$. For the estimation of II, we work as follows:

$$\begin{aligned} II &= \int\limits_{O_\lambda \cap \{x \in (a,b): \Phi(x) \leq \lambda^{\frac{q}{\eta}}, \Psi(x) > \lambda^{\frac{q}{\eta}}\}} w + \int\limits_{O_\lambda \cap \{x \in (a,b): \Phi(x) \leq \lambda^{\frac{q}{\eta}}, \Psi(x) \leq \lambda^{\frac{q}{\eta}}\}} w \\ &= III + IV. \end{aligned}$$

Firstly,

$$III \leq \int\limits_{O_\lambda \cap \{x \in (a,b): \Psi(x) > \lambda^{\frac{q}{\eta}}\}} w$$

and then

$$\int\limits_{O_\lambda \cap \{x\in(a,b):\Psi(x)>\lambda^{\frac{q}{\eta}}\}} w \le \frac{\|\Psi\|^{\eta}_{\eta,\infty;w}}{\lambda^q} = \frac{\|\Psi\|^{\eta}_{\eta,\infty;w}}{\lambda^q}\|f\|^q_{p,v}.$$

Now, we will work on IV. Let $\{x_k\}$ be the sequence defined as in the proof of Theorem 1 and

$$E_k = O_\lambda \cap (x_{k+1}, x_k) \cap \{x \in (a,b) : \Phi(x) \le \lambda^{\frac{q}{\eta}}, \Psi(x) \le \lambda^{\frac{q}{\eta}}\}.$$

If $x \in E_k$,

$$\lambda < 4^{\frac{1}{r}}\alpha(x)\left(\int\limits_{x_{k+2}}^{x_{k+1}}\left(\int\limits_{x_{k+2}}^{t} f\right)^r u^r(t)\mathrm{d}t\right)^{\frac{1}{r}} + 4^{\frac{1}{r}}\alpha(x)\left(\int\limits_{x_{k+2}}^{x_{k+1}} u^r\right)^{\frac{1}{r}}\int_a^{x_{k+2}} f.$$

It is clear that, for each k, $E_k = E_{k,1} \cup E_{k,2}$, where

$$E_{k,1} = \left\{x \in E_k : \alpha(x)\left(\int\limits_{x_{k+2}}^{x_{k+1}}\left(\int\limits_{x_{k+2}}^{t} f\right)^r u^r(t)\mathrm{d}t\right)^{\frac{1}{r}} > \frac{\lambda}{2\cdot 4^{\frac{1}{r}}}\right\}$$

and

$$E_{k,2} = \left\{x \in E_k : \alpha(x)\left(\int\limits_{x_{k+2}}^{x_{k+1}} u^r\right)^{\frac{1}{r}}\int\limits_{a}^{x_{k+2}} f > \frac{\lambda}{2\cdot 4^{\frac{1}{r}}}\right\}.$$

Since $r < p$, by Theorem A (ii) we have

$$\begin{aligned}&\left(\int\limits_{x_{k+2}}^{x_{k+1}}\left(\int\limits_{x_{k+2}}^{t} f\right)^r u^r(t)\mathrm{d}t\right)^{\frac{1}{r}}\\ &\le C_{r,p}\left(\int\limits_{x_{k+2}}^{x_{k+1}}\left(\int\limits_{t}^{x_{k+1}} u^r\right)^{\frac{\theta}{r}}\left(\int\limits_{x_{k+2}}^{t} v^{1-p'}\right)^{\frac{\theta}{r'}} v^{1-p'}(t)\mathrm{d}t\right)^{\frac{1}{\theta}} \\ &\times\left(\int\limits_{x_{k+2}}^{x_{k+1}} f^p v\right)^{\frac{1}{p}},\end{aligned} \tag{3.7}$$

where $C_{r,p} = r^{\frac{1}{r}}(p')^{\frac{1}{r'}}$. The first integral in the right-hand side of (3.7) is finite due to the monotonicity of α and the proof of this fact follows the pattern of the one in Theorem 1.

If $x \in E_{k,1}$,

$$\lambda < 2 \cdot 4^{\frac{1}{r}} C_{r,p}\alpha(x) \left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_{t}^{x_{k+1}} u^r \right)^{\frac{\theta}{r}} \left(\int_{x_{k+2}}^{t} v^{1-p'} \right)^{\frac{\theta}{r'}} v^{1-p'}(t)\mathrm{d}t \right)^{\frac{1}{\theta}} \times \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}},$$

which implies, due to the monotonicity of α,

$$\lambda \leq 2 \cdot 4^{\frac{1}{r}} C_{r,p} \left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_{t}^{x_{k+1}} u^r \right)^{\frac{\theta}{r}} \left(\int_{x_{k+2}}^{t} v^{1-p'} \right)^{\frac{\theta}{r'}} v^{1-p'}(t)\mathrm{d}t \right)^{\frac{1}{\theta}} \times \left(\inf_{(\rho_k^1, \rho_k^2)} \alpha \right) \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}}.$$

If we multiply both terms of the last inequality by $\left(\int_{E_{k,1}} w \right)^{\frac{1}{p}}$, we get

$$\left(\int_{E_{k,1}} w \right)^{\frac{1}{p}} \leq \frac{2 \cdot 4^{\frac{1}{r}} C_{r,p}}{\lambda} \left(\inf_{(\rho_k^1, \rho_k^2)} \alpha \right) \left(\int_{\rho_k^1}^{\rho_k^2} w \right)^{\frac{1}{p}} \times \left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_{t}^{x_{k+1}} u^r \right)^{\frac{\theta}{r}} \left(\int_{x_{k+2}}^{t} v^{1-p'} \right)^{\frac{\theta}{r'}} v^{1-p'}(t)\mathrm{d}t \right)^{\frac{1}{\theta}} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}}$$

$$\leq \frac{2 \cdot 4^{\frac{1}{r}} C_{r,p}}{\lambda} \left(\inf_{(\rho_k^1, \rho_k^2)} \alpha \right) \left(\int_{\rho_k^1}^{\rho_k^2} w \right)^{\frac{1}{p}}$$

$$\times \left(\int_a^{\rho_k^1} \left(\int_t^{\rho_k^1} u^r \right)^{\frac{\theta}{r}} \left(\int_a^t v^{1-p'} \right)^{\frac{\theta}{r'}} v^{1-p'}(t) \mathrm{d}t \right)^{\frac{1}{\theta}} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}}$$

$$\leq \frac{2 \cdot 4^{\frac{1}{r}} C_{r,p}}{\lambda} \lambda^{\frac{q}{\eta}} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}},$$

where the last inequality holds since

$$\left(\inf_{(\rho_k^1, \rho_k^2)} \alpha \right) \left(\int_{\rho_k^1}^{\rho_k^2} w \right)^{\frac{1}{p}} \left(\int_a^{\rho_k^1} \left(\int_t^{\rho_k^1} u^r \right)^{\frac{\theta}{r}} \left(\int_a^t v^{1-p'} \right)^{\frac{\theta}{r'}} v^{1-p'}(t) \mathrm{d}t \right)^{\frac{1}{\theta}}$$

$\leq \Psi(t)$ for all $t \in E_{k,1}$. Raising to p, we have that

$$\int_{E_{k,1}} w \leq \frac{2^p \cdot 4^{\frac{p}{r}} C_{r,p}^p}{\lambda^{p-\frac{pq}{\eta}}} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right) = \frac{2^p \cdot 4^{\frac{p}{r}} C_{r,p}^p}{\lambda^q} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right).$$

Now, summing up in k,

$$\int_{\cup_k E_{k,1}} w \leq \frac{2^p \cdot 4^{\frac{p}{r}} C_{r,p}^p}{\lambda^q} \left(\int_a^b f^p v \right) = \frac{2^p \cdot 4^{\frac{p}{r}} C_{r,p}^p}{\lambda^q}.$$

The estimation of $\int_{\cup_k E_{k,2}} w$ is the same as the one in Theorem 1, because the relationship between r and p is not taken into account. Therefore, the proof is complete.

4 Proofs of Theorems 3 and 4

Working as in ([7], proof of Theorem 3), we have that (1.8) is equivalent to the two weighted weak-type bilinear inequalities

$$\left\| \beta(x) \int_a^x f(t) \left(\int_a^t g \right) \mathrm{d}t \right\|_{q,\infty;w} \leq C \|f\|_{p_1,v_1} \|g\|_{p_2,v_2} \tag{4.1}$$

and

$$\left\| \beta(x) \int_a^x g(t) \left(\int_a^t f \right) \mathrm{d}t \right\|_{q,\infty;w} \leq C \|f\|_{p_1,v_1} \|g\|_{p_2,v_2}. \tag{4.2}$$

Inequality (4.1) is equivalent to

$$\left\| \beta(x) \int_a^x h \right\|_{q,\infty;w} \leq C \|h\|_{p_1,\tilde{v}_1^g}, \tag{4.3}$$

where $\tilde{v}_1^g(x) = v_1(x) \left(\int_a^x \frac{g}{\|g\|_{p_2,v_2}} \right)^{-p_1}$ and the constant C does not depend on g.

Since $q < p_1$ and β is a monotone function, by Theorem C inequality (4.3) holds if and only if there exists $C > 0$ such that

$$\|\Psi_g\|_{r_1,\infty;w} \leq C \tag{4.4}$$

for all g, where $\frac{1}{r_1} = \frac{1}{q} - \frac{1}{p_1}$ and

$$\Psi_g(x) = \sup_{c>x} \left((\inf_{y\in(x,c)} \beta(y)) \left(\int_x^c w \right)^{\frac{1}{p_1}} \right) \|\chi_{(a,x)} (\tilde{v}_1^g)^{-\frac{1}{p_1}} \|_{p_1'}$$

$$= \alpha_1(x) \|\chi_{(a,x)} (\tilde{v}_1^g)^{-\frac{1}{p_1}} \|_{p_1'}.$$

Then (4.4) can be written as

$$\left\| \alpha_1(x) \left\| \chi_{(a,x)}(t) \left(v_1^{-\frac{1}{p_1}}(t) \int_a^t g \right) \right\|_{p_1'} \right\|_{r_1,\infty;w} \leq C \|g\|_{p_2,v_2}. \tag{4.5}$$

Therefore, inequality (4.1) holds if and only if inequality (4.5) holds. Since $p_2 > r_1$, by Theorems 1 and 2, (4.5) holds if and only $\Phi_1 \in L^{\eta,\infty}(w)$ in the case $p_2 \leq p_1'$ and $\Phi_1, \Psi_1 \in L^{\eta,\infty}(w)$ in the case $p_1' < p_2$.

In the same way, we see that (4.2) holds if and only if $\Phi_2 \in L^{\eta,\infty}(w)$ in the case $p_2 \leq p_1'$ and $\Phi_2, \Psi_2 \in L^{\eta,\infty}(w)$ in the case $p_1' < p_2$.

Acknowledgements This research has been supported in part by Ministerio de Economía y Competitividad (Grant PGC2018-096166-B-100) and Junta de Andalucía (Grants FQM354 and UMA18-FEDERJA-002).

Funding Funding for open access publishing: Universidad Málaga/CBUA.

References

1. Aguilar Cañestro, M.I., Ortega Salvador, P., Ramírez Torreblanca, C.: Weighted bilinear Hardy inequalities. J. Math. Anal. Appl. **387**(1), 320–334 (2012)
2. Andersen, K., Muckenhoupt, B.: Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions. Stud. Math. **72**(1), 9–26 (1982)
3. Bernardis, A.L., Ortega Salvador, P.: Some new iterated Hardy-type inequalities and applications. J. Math. Inequal. **11**(2), 577–594 (2017)
4. Bradley, J.: Hardy inequalities with mixed norms. Can. Math. Bull. **21**(4), 405–408 (1978)
5. Cruz-Uribe, D., Martell, J.M., Pérez, C.: Weighted weak-type inequalities and a conjecture of Sawyer. Int. Math. Res. Notices **30**, 1849–1871 (2005)
6. Ferreyra, E.V.: Weighted Lorentz norm inequalities for integral operators. Stud. Math. **96**(2), 125–134 (1990)
7. García García, V., Ortega Salvador, P.: Weighted weak-type iterated and bilinear Hardy inequalities. J. Math. Anal. Appl. **525**, 127284 (2023)
8. Gogatishvili, A., Mustafayev, R.C.: Weighted iterated Hardy-type inequalities. Math. Inequal. Appl. **20**(3), 683–728 (2017)
9. Gogatishvili, A., Mustafayev, R.C., Persson, L.-E.: Some new iterated Hardy-type inequalities. J. Funct. Spaces Appl. **2012**, 734194 (2012)
10. Gogatishvili, A., Mustafayev, R.C., Persson, L.-E.: Some new iterated Hardy-type inequalities: the case $\theta = 1$. J. Inequal. Appl. **2013**, 515 (2013)
11. Gogatishvili, A., Mihula, Z., Pick, L., Turcinovà, H., Unver, T.: Weighted inequalities for a superposition of the Copson operator and the Hardy operator. J. Fourier Anal. Appl. **28**, 24 (2022)
12. Gogatishvili, A., Jain, P., Kanjilal, S.: On bilinear Hardy inequality and corresponding geometric mean inequality. Ric. Mat. **71**(2), 581–608 (2022)
13. Krepela, M.: Iterating bilinear Hardy inequalities. Proc. Edinb. Math. Soc. **60**(4), 955–971 (2017)
14. Krepela, M.: Bilinear weighted Hardy inequality for nonincreasing functions. Publ. Mat. **61**, 3–50 (2017)
15. Lai, Q.: Weighted modular inequalities for Hardy type operators. Proc. Lond. Math. Soc. **79**, 649–672 (1999)
16. Li, K., Ombrosi, S., Pérez, C.: Proof of an extension of E. Sawyer's conjecture about weighted mixed weak-type estimates. Math. Ann. **374**(1), 907–929 (2019)
17. Li, K., Ombrosi, S., Picardi, M.B.: Weighted mixed weak-type inequalities for multilinear operators. Stud. Math. **244**(2), 203–215 (2019)
18. Lorente, M., Martín-Reyes, F.J.: Some mixed weak-type inequalities. J. Math. Inequal. **15**(2), 811–826 (2021)
19. Martín-Reyes, F.J., Ombrosi, S.: Mixed weak-type inequalities for one-sided operators. Quart. J. Math. **60**(1), 63–73 (2009)
20. Martín-Reyes, F.J., Ortega Salvador, P.: On weighted weak type inequalities for modified Hardy operators. Proc. Am. Math. Soc. **126**(6), 1739–1746 (1998)
21. Martín-Reyes, F.J., Ortega Salvador, P., Sarrión, M.D.: Boundedness of operators of Hardy type in $\Lambda^{p,q}$-spaces and weighted mixed inequalities for singular integral operators. Proc. R. Soc. Edinb. **127A**, 157–170 (1997)
22. Maz'ja, V.G.: Sobolev Spaces. Springer-Verlag, Berlin (1985)
23. Muckenhoupt, B.: Hardy's inequality with weights. Stud. Math. **44**, 31–38 (1972)

24. Oinarov, R., Kalybay, A.: Three-parameter weighted Hardy type inequalities. Banach J. Math. Anal. **2**(2), 85–93 (2008)
25. Oinarov, R., Kalybay, A.: Weighted inequalities for a class of semiadditive operators. Ann. Funct. Anal. **6**(4), 155–171 (2015)
26. Sawyer, E.: A weighted weak-type inequality for the maximal function. Proc. Am. Math. Soc. **93**(4), 610–614 (1985)
27. Sinnamon, G.: Operators on Lebesgue Spaces with General Measures. Doctoral thesis, McMaster University (1987)
28. Sinnamon, G.: A weighted gradient inequality. Proc. R. Soc. Edinb. **111A**(3–4), 329–335 (1989)
29. Sinnamon, G., Stepanov, V.D.: The weighted Hardy inequality: new proofs and the case $p = 1$. J. Lond. Math. Soc. **54**(1), 89–101 (1996)
30. Stepanov, V.D., Shambilova, G.E.: On iterated and bilinear integral Hardy-type operators. Math. Inequal. Appl. **22**(4), 1505–1533 (2019)
31. Talenti, G.: Osservazioni sopra una classe di disuguaglianze. Rend. Sem. Mat. Fis. Milano **39**, 171–185 (1969)