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Fuzzy relational Galois connections between fuzzy transitive digraphs

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Abstract

We present a fuzzy version of the notion of relational Galois connection between fuzzy transitive directed graphs (fuzzy T-digraphs) on the specific setting in which the underlying algebra of truth values is a complete Heyting algebra. The components of such fuzzy Galois connection are fuzzy relations satisfying certain reasonable properties expressed in terms of the so-called full powering. Moreover, we provide a necessary and sufficient condition under which it is possible to construct a right adjoint for a given fuzzy relation between a fuzzy T-digraph and an unstructured set.

Keywords: Galois connection; Fuzzy transitive digraph

1. Introduction

Since its introduction in 1944 by Ore [1], the abstract mathematical notion of Galois connection has become an established concept, and plays a key role in a broad range of directions in mathematics and computer science [2]. To illustrate the contemporary interest in Galois connections, we mention but a few very recent studies illustrating various new application areas. The purpose of our sample is to illustrate the broadness of this interest, and by no means has the intention to be comprehensive or complete. Al-Sihabi et al. [3] introduce a framework for the modular design of abstract domains for recursive types and higher-order functions, based on Galois connections and the theory of recursive domain equations. Oh and Kim [4] consider the notions of Galois and dual Galois connections to develop a topological view of concept lattices in complete residuated lattices. Fernández-Alonso and Magaña [5] study Galois connections between the lattices of preradicals of two rings A and B induced by an adjoint pair of functors between the categories A-Mod and B-Mod. Madrid et al. [6] propose an alternative definition of approximation operator based on closure and interior operators obtained from an isotone Galois connection. Horváth et al. [7] characterize invariance groups of sets of Boolean functions as Galois closures of a suitable Galois connection. Ararat and Hamel [8] show that

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a recently introduced lower cone distribution function, together with the set-valued multivariate quantile, generates a Galois connection between a complete lattice of closed convex sets and the real unit interval $[0, 1]$. Alexandru and Ciobanu [9] introduce Galois connections between finitely supported ordered structures, and show properties of finitely supported Galois connections between invariant complete lattices.

The above studies have in common that they consider Galois connections between sets with different levels of structuration. One recent research topic along this line is the following problem of constructing Galois connections [10]: given a mapping $f : A \to B$ between different structures (for instance, the domain A being a lattice and the codomain B a plain set), one wants to establish necessary and sufficient conditions under which it is possible to equip B with a desired structure and construct a mapping $g : B \to A$ such that the couple (f, g) is a Galois connection. It is important to note the fact that A and B need not have the same structure rules out the application of Freyd's adjoint functor theorem. In the first paper in this research direction [11], the domain A was considered to be a poset or preposet.

Being closely linked to set theory, it is not surprising that Galois connections have also been introduced in the setting of fuzzy set theory. The degrees of freedom that come along with such effort usually result in different levels of generality, and this is no different for the notion of fuzzy Galois connection [12–14]. In previous works [15,16] we explored the above-mentioned construction problem in various fuzzy settings, satisfactorily extending the problem to Galois connections between a fuzzy domain A and fuzzy range B, although still having crisp functions as components, hence not obtaining a truly fuzzy notion of Galois connection.

More recently, the approach presented in [17] drew us back to the crisp case by considering Galois connections of which the domain and range are just sets endowed with arbitrary relations and whose components are (proper) relations, resulting in what we called *relational Galois connections*. Subsequent studies focused on the cases of the domain A having the structure of a transitive digraph [18] or fuzzy transitive digraph [19], studying Galois connections whose left and right components are crisp relations satisfying certain reasonable properties expressed in terms of the so-called full powering.

In this work, we focus on the specific setting in which the underlying algebra of truth values is a complete Heyting algebra, and expound for the first time an adequate notion of *fuzzy relational Galois connection* between fuzzy transitive digraphs, with both components now being fuzzy relations. This notion of fuzzy relational Galois connection will be shown to inherit most of the interesting equivalent characterizations of the notion of crisp Galois connection discussed in detail in [19].

This paper is organized as follows. In Section 2, we both recall and extend the mathematical apparatus adopted in this paper. We pay particular attention to the different fuzzy powerings and relationships between them, and their properties, in particular in the context of singletons and (normal) cliques, thereby considerably extending the knowledge on the topic. In Section 3, we introduce the protagonists of this paper: fuzzy relational Galois connections between fuzzy transitive digraphs. We provide a characterization in terms of a natural fuzzy Galois condition in the presence of an appropriate clique condition. In Section 4, we explore links with other notions, as is commonly done in papers on the topic: fuzzy closure relations and fuzzy closure systems. We report on the main endeavour of this research in Section 5: we provide necessary and sufficient conditions under which it is possible to build a right adjoint for a given fuzzy relation, based on the notion of compatibility of a fuzzy closure system with a fuzzy relation. The approach is constructive and is amply illustrated with examples. The final section discusses further paths of research to explore along the lines of this paper.

2. Preliminary notions

The framework considered in this work is that of $\mathbb{L}$-fuzzy set theory, where $\mathbb{L}$ is a complete Heyting algebra. To keep the paper self-contained, we recall the necessary notions and related properties required in the core of this paper.

A complete Heyting algebra is an algebra $\mathbb{L} = (L, \leq, \bot, \top, \to)$, where $(L, \leq)$ is a complete lattice, $\bot$ is the bottom element, $\top$ is the top element, and the following adjointness property holds for all $p, q, r \in L$:

$$p \wedge q \leq r \qquad \Longleftrightarrow \qquad p \leq q \to r \,. \tag{1}$$

Basic consequences of this property, such as $q \to \top = \top$ and ($p \leq q$ if and only if $p \to q = \top$), will be used throughout this paper without explicit mentioning. We will also use the following properties, which hold for all $p, q, r, s \in L$:

$$(p \to q) \wedge (r \to s) \leq (p \wedge r) \to (q \wedge s) \tag{2}$$
$$p \wedge (p \to q) \leq p \wedge q \tag{3}$$
$$p \leq q \implies p \to r \leq p \to (r \wedge q) \tag{4}$$
$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \tag{5}$$
$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r) \tag{6}$$

Complete Heyting algebras are complete residuated lattices with the meet operation as product. As a consequence, the following properties hold as well:

$$(p \wedge q) \to r = p \to (q \to r) \tag{7}$$
$$(p \to q) \wedge (q \to r) \leq (p \to r) \tag{8}$$
$$(p \to q) \leq (p \wedge r) \to (q \wedge r) \tag{9}$$
$$p \to \bigwedge_{i \in I} q_i = \bigwedge_{i \in I} (p \to q_i) \tag{10}$$
$$p \leq q \implies r \wedge p \leq r \wedge q \tag{11}$$
$$p \leq q \implies r \to p \leq r \to q \tag{12}$$
$$p \leq q \implies q \to r \leq p \to r \tag{13}$$

for all $p, q, r \in L$ and $\{q_i : i \in I\} \subseteq L$.

An $\mathbb{L}$-*fuzzy set* on a universe A (also called $\mathbb{L}$-*fuzzy subset* of A) is a mapping $X\colon A \to \mathbb{L}$ from A to the algebra $\mathbb{L}$ of membership degrees, where $X(a)$ denotes the degree to which element a belongs to X.[1] A fuzzy set X is said to be *normal* if $\mathrm{Core}(X) := \{a \in A : X(a) = \top\} \neq \emptyset$, i.e., $X(a) = \top$ for some $a \in A$. Any element $a \in A$ induces a *singleton*, i.e., a fuzzy set $a\colon A \to \mathbb{L}$ defined by $a(x) = \top$ if $x = a$ and $a(x) = \bot$ otherwise.

An $\mathbb{L}$-*fuzzy relation* between two universes A and B is a mapping $\mu\colon A \times B \to \mathbb{L}$, where $\mu(a, b)$ denotes the degree of relationship between the elements a and b. Given a fuzzy relation μ and an element $a \in A$, the *afterset* a^μ of a is the fuzzy set $a^\mu\colon B \to \mathbb{L}$ defined by $a^\mu(b) = \mu(a, b)$. A fuzzy relation μ is said to be *total* if the aftersets a^μ are normal, for all $a \in A$. The *range* of μ is defined as $\mathrm{rng}(\mu) := \bigcup_{a \in A} \mathrm{Core}(a^\mu)$. The composition of an $\mathbb{L}$-fuzzy relation μ between two universes A and B and an $\mathbb{L}$-fuzzy relation ν between B and a universe C is the $\mathbb{L}$-fuzzy relation $\mu \circ \nu$ between A and C defined by

$$\mu \circ \nu(a, c) = \bigvee_{b \in B} (\mu(a, b) \wedge \nu(b, c)).$$

An $\mathbb{L}$-fuzzy relation on a universe A is a mapping $\rho\colon A \times A \to \mathbb{L}$, and is said to be:

- *reflexive* if $\rho(a, a) = \top$, for all $a \in A$.
- *transitive* if $\rho(a, b) \wedge \rho(b, c) \leq \rho(a, c)$, for all $a, b, c \in A$; or, equivalently, $\rho \circ \rho(a, c) \leq \rho(a, c)$, for all $a, c \in A$.

Definition 1. $\mathbb{A} = \langle A, \rho \rangle$ is said to be a *fuzzy T-digraph* if ρ is a transitive fuzzy relation on A.

Given a relation R on a set A, it is possible to lift R to the powerset 2^A by defining the following *powerings*, for all $X, Y \in 2^A$, which correspond to the construction of the Hoare, Smyth and Plotkin powersets, respectively:

$$\begin{aligned}
X\, R_H\, Y &\iff (\forall x \in X)(\exists y \in Y)(x R y) \\
X\, R_S\, Y &\iff (\forall y \in Y)(\exists x \in X)(x R y) \\
X\, R_\propto\, Y &\iff (\forall x \in X)(\forall y \in Y)(x R y).
\end{aligned}$$

The adaptation of these powerings to the present fuzzy framework is explained next [20].

[1] For convenience, hereafter, we will always omit the prefix $\mathbb{L}$.

Definition 2. Consider a fuzzy T-digraph $\langle A, \rho \rangle$. We define the Hoare, Smyth and full fuzzy powerings as follows, for any $X, Y : A \to \mathbb{L}$:

(i) $\rho_H(X, Y) = \bigwedge_{x \in A} \left(X(x) \to \bigvee_{y \in A} (Y(y) \wedge \rho(x, y)) \right)$;

(ii) $\rho_S(X, Y) = \bigwedge_{y \in A} \left(Y(y) \to \bigvee_{x \in A} (X(x) \wedge \rho(x, y)) \right)$;

(iii) $\rho_\propto(X, Y) = \bigwedge_{x \in A} \bigwedge_{y \in A} (X(x) \wedge Y(y) \to \rho(x, y))$.

In the particular case of *singletons* in the first argument of the fuzzy powerings, the expressions in the above definitions are greatly simplified. Indeed, for all $a \in A$, it holds:

(i) $\rho_H(a, Y) = \bigvee_{y \in A} (Y(y) \wedge \rho(a, y))$;

(ii) $\rho_S(a, Y) = \rho_\propto(a, Y) = \bigwedge_{y \in A} (Y(y) \to \rho(a, y))$.

We recall that in the characterisation of relational Galois connections in the crisp case, a key role was played by the notion of clique [18]. Not surprisingly, a fuzzy version of this notion will play a similar role here.

Definition 3. Let $\langle A, \rho \rangle$ be a fuzzy T-digraph. A fuzzy set $X : A \to \mathbb{L}$ is called a clique if, for all $x, y \in A$, it holds that

$$X(x) \wedge X(y) \leq \rho(x, y)\,,$$

or, equivalently, $\rho_\propto(X, X) = \top$.

The following lemmas list a number of technical results related to the fuzzy powerings. They will be extensively used throughout this paper.

Lemma 1. *Consider a fuzzy T-digraph $\langle A, \rho \rangle$, a fuzzy set $X : A \to \mathbb{L}$ and $a \in A$.*

(i) *If X is a normal fuzzy set, then*

$$\rho_S(a, X) = \rho_\propto(a, X) \leq \rho_H(a, X)\,.$$

(ii) *If X is a clique, then*

$$\rho_H(a, X) \leq \rho_S(a, X) = \rho_\propto(a, X)\,.$$

(iii) *If X is a normal clique and $x_0 \in \mathrm{Core}(X)$, then*

$$\rho_S(a, X) = \rho_\propto(a, X) = \rho_H(a, X) = \rho(a, x_0)\,.$$

(iv) *If X is a normal clique, then $\rho_\propto(X, Y) \leq \rho_S(X, Y)$, for all $Y : A \to \mathbb{L}$.*

Proof. Let us prove (i). Since X is normal, we can choose $x_0 \in A$ such that $X(x_0) = \top$. Hence, using (3), it follows that

$$\begin{aligned} X(x_0) \to \rho(a, x_0) &= \top \to \rho(a, x_0) = \top \wedge (\top \wedge \rho(a, x_0)) \\ &\leq \top \wedge \rho(a, x_0) = X(x_0) \wedge \rho(a, x_0)\,. \end{aligned}$$

This clearly implies

$$\rho_S(a,X)=\rho_\propto(a,X)=\bigwedge_{x\in A}(X(x)\to\rho(a,x))$$
$$\leq\bigvee_{x\in A}(X(x)\wedge\rho(a,x))=\rho_H(a,X)\,.$$

Next, we prove (ii). Since X is a clique and ρ is transitive, it holds for all $x, y\in A$ that

$$X(x)\wedge\rho(a,x)\wedge X(y)\leq\rho(a,x)\wedge\rho(x,y)\leq\rho(a,y)\,.$$

Using the adjointness property, the latter is equivalent to $X(x)\wedge\rho(a,x)\leq X(y)\to\rho(a,y)$. Hence,

$$\rho_H(a,X)=\bigvee_{x\in A}(X(x)\wedge\rho(a,x))$$
$$\leq\bigwedge_{y\in A}(X(y)\to\rho(a,y))=\rho_S(a,X)=\rho_\propto(a,X)\,.$$

To prove (iii), observe on the one hand that

$$\rho_S(a,X)\leq X(x_0)\to\rho(a,x_0)=\top\to\rho(a,x_0)=\rho(a,x_0)\,.$$

On the other hand, $\rho_H(a,X)\geq X(x_0)\wedge\rho(a,x_0)=\rho(a,x_0)$. Using (i) and (ii), it then follows that

$$\rho_H(a,X)=\rho_S(a,X)=\rho_\propto(a,X)=\rho(a,x_0)\,.$$

Finally, let us prove (iv). Consider again x_0 such that $X(x_0)=\top$. We then have

$$\rho_\propto(X,Y)\leq\rho_\propto(x_0,Y)=\rho_S(x_0,Y)=\bigwedge_{w\in A}\big(Y(w)\to\rho(x_0,w)\big)$$
$$\overset{(12)}{\leq}\bigwedge_{w\in A}\left(Y(w)\to\bigvee_{z\in A}(X(z)\wedge\rho(z,w))\right)=\rho_S(X,Y)\,.\quad\square$$

The following lemma expresses that the fuzzy powering $\rho_\propto$ is close to being a transitive fuzzy relation.

Lemma 2. *Consider a fuzzy T-digraph $\langle A,\rho\rangle$ and fuzzy sets $X,Y,Z\colon A\to\mathbb{L}$. If Y is normal, then $\rho_\propto(X,Y)\wedge\rho_\propto(Y,Z)\leq\rho_\propto(X,Z)$.*

Proof. The claim follows from the following chain of (in-)equalities:

$$\rho_\propto(X,Y)\wedge\rho_\propto(Y,Z)=$$
$$=\bigwedge_{a,b\in A}\big(X(a)\wedge Y(b)\to\rho(a,b)\big)\wedge\bigwedge_{c,d\in A}\big(Y(c)\wedge Z(d)\to\rho(c,d)\big)$$
$$=\bigwedge_{a,b,c,d\in A}\Big(\big(X(a)\wedge Y(b)\to\rho(a,b)\big)\wedge\big(Y(c)\wedge Z(d)\to\rho(c,d)\big)\Big)$$
$$\overset{(*)}{\leq}\bigwedge_{a,b,d\in A}\Big(\big(X(a)\wedge Y(b)\to\rho(a,b)\big)\wedge\big(Y(b)\wedge Z(d)\to\rho(b,d)\big)\Big)$$
$$\overset{(2)}{\leq}\bigwedge_{a,b,d\in A}(X(a)\wedge Y(b)\wedge Z(d)\to(\rho(a,b)\wedge\rho(b,d)))$$
$$\overset{(\triangle)}{\leq}\bigwedge_{a,d\in A}(X(a)\wedge Z(d)\to\rho(a,d))=\rho_\propto(X,Z)\,,$$

where $(*)$ follows considering $c=b$ and $(\triangle)$ by choosing b such that $Y(b)=\top$ (given that Y is normal). $\square$

Lemma 3. *Consider a fuzzy T-digraph $\langle A,\rho\rangle$ and fuzzy sets $X,Y\colon A\to\mathbb{L}$. For all $a\in A$, we have:*

(i) $\rho_\propto(X, Y) \wedge X(a) \le \rho_\propto(a, Y)$;
(ii) $\rho_\propto(X, Y) \wedge Y(a) \le \rho_\propto(X, a)$;
(iii) *If X is a normal clique, then* $\rho_\propto(a, Y) \wedge X(a) \le \rho_\propto(X, Y)$;
(iv) *If Y is a normal clique, then* $\rho_\propto(X, a) \wedge Y(a) \le \rho_\propto(X, Y)$.

Proof. (i) We will prove that $\rho_\propto(X, Y) \le X(a) \to \rho_\propto(a, Y)$, which is equivalent to $\rho_\propto(X, Y) \wedge X(a) \le \rho_\propto(a, Y)$. It holds that

$$\begin{aligned}\rho_\propto(X, Y) &= \bigwedge_{x\in A} \bigwedge_{y\in A} (X(x) \wedge Y(y) \to \rho(x, y)) \\ &\le \bigwedge_{y\in A} (X(a) \wedge Y(y) \to \rho(a, y)) \\ &\overset{(7)}{=} \bigwedge_{y\in A} (X(a) \to (Y(y) \to \rho(a, y))) \\ &\overset{(10)}{=} X(a) \to \bigwedge_{y\in A} (Y(y) \to \rho(a, y)) \\ &= X(a) \to \rho_\propto(a, Y)\,.\end{aligned}$$

(ii) The proof is similar to that of the previous item.
(iii) Since X is normal, we can choose x_0 such that $X(x_0) = \top$ and, hence, $X(z) \le \rho(x_0, z)$ and $X(z) \le \rho(z, x_0)$ for all $z \in A$. It follows, due to Lemma 2 and the equality, $\rho_\propto(x_0, a) = \rho(x_0, a)$ that

$$\begin{aligned}\rho_\propto(a, Y) \wedge X(a) \le \rho_\propto(a, Y) \wedge \rho(x_0, a) &\le \rho_\propto(x_0, Y) \\ &= \bigwedge_{y\in A} \big(Y(y) \to \rho(x_0, y)\big) \\ &\overset{(9)}{\le} \bigwedge_{y\in A} \big(X(z) \wedge Y(y) \to (X(z) \wedge \rho(x_0, y))\big)\,.\end{aligned}$$

In addition, since $X(z) \le \rho(z, x_0)$, it holds due to the transitivity of ρ that

$$X(z) \wedge \rho(x_0, y) \le \rho(z, x_0) \wedge \rho(x_0, y) \le \rho(z, y)\,,$$

and, hence, using (12), that

$$\rho_\propto(a, Y) \wedge X(a) \le \bigwedge_{z\in A} \bigwedge_{y\in A} \big(X(z) \wedge Y(y) \to \rho(z, y)\big) = \rho_\propto(X, Y)\,.$$

(iv) The proof is similar to that of the previous item. □

Corollary 1. *Consider a fuzzy T-digraph $\langle A, \rho\rangle$ and fuzzy sets $X, Y\colon A \to \mathbb{L}$. If X and Y are normal cliques, then for all $x_0 \in \mathrm{Core}(X)$ and $y_0 \in \mathrm{Core}(Y)$, it holds that*

$$\rho_\propto(X, Y) = \rho_\propto(x_0, Y) = \rho_\propto(X, y_0) = \rho(x_0, y_0)\,.$$

In order to extend the definition of a relational Galois connection to the fuzzy framework considered in this paper, we need the notions of antitone and inflationary fuzzy relations between fuzzy T-digraphs. The following definition states these notions in terms of the Plotkin fuzzy powering $\propto$.

Definition 4. Consider two fuzzy T-digraphs $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$. A fuzzy relation $\mu\colon A \times B \to \mathbb{L}$ is said to be:

- *antitone* if $\rho_A(a_1, a_2) \wedge \mu(a_1, b_1) \wedge \mu(a_2, b_2) \le \rho_B(b_2, b_1)$, for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Equivalently, $\rho_A(a_1, a_2) \le \rho_{B\propto}(a_2^\mu, a_1^\mu)$.

- *isotone* if $\rho_A(a_1,a_2) \wedge \mu(a_1,b_1) \wedge \mu(a_2,b_2) \le \rho_B(b_1,b_2)$, for all $a_1,a_2 \in A$ and $b_1,b_2 \in B$. Equivalently, $\rho_A(a_1,a_2) \le \rho_{B\propto}(a_1^{\mu}, a_2^{\mu})$.

Definition 5. Let $\langle A, \rho\rangle$ be a fuzzy T-digraph. A fuzzy relation $\mu\colon A \times A \to \mathbb{L}$ is said to be:

- *inflationary* if $\mu(a_1,a_2) \le \rho(a_1,a_2)$, for all $a_1,a_2 \in A$. Equivalently, $\rho_{\propto}(a, a^{\mu}) = \top$.
- *idempotent* if $\rho_{\propto}(a^{\mu\circ\mu}, a^{\mu}) = \top$ and $\rho_{\propto}(a^{\mu}, a^{\mu\circ\mu}) = \top$, for all $a \in A$.

3. Fuzzy relational Galois connections between fuzzy T-digraphs

In this section, we introduce the central notion of a fuzzy relational Galois connection as a natural generalization of the notion of relational Galois connection to the present fuzzy framework.

Definition 6. Consider two fuzzy T-digraphs[2] $\langle A, \rho\rangle$ and $\langle B, \rho\rangle$ and two total fuzzy relations $\mu\colon A \times B \to \mathbb{L}$ and $\nu\colon B \times A \to \mathbb{L}$. We say that the couple (μ, ν) is a *fuzzy relational Galois connection* if both μ and ν are antitone and both $\mu \circ \nu$ and $\nu \circ \mu$ are inflationary.

In order to evaluate the appropriateness of this definition, we aim for a characterization in terms of a suitable generalization of the classical Galois condition.

Definition 7. Consider two fuzzy T-digraphs $\langle A, \rho\rangle$ and $\langle B, \rho\rangle$ and two total fuzzy relations $\mu\colon A \times B \to \mathbb{L}$ and $\nu\colon B \times A \to \mathbb{L}$. We say that the couple (μ, ν) satisfies the *fuzzy Galois condition* if the following holds for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$:

(i) $\rho(a_1,a_2) \wedge \mu(a_1,b_1) \wedge \nu(b_2,a_2) \le \rho(b_2,b_1)$;
(ii) $\rho(b_2,b_1) \wedge \mu(a_1,b_1) \wedge \nu(b_2,a_2) \le \rho(a_1,a_2)$;

or, equivalently, for all $a \in A$ and $b \in B$:

(i) $\rho_H(a, b^{\nu}) \le \rho_S(b, a^{\mu})$;
(ii) $\rho_H(b, a^{\mu}) \le \rho_S(a, b^{\nu})$.

Note that the conditions in the above definition are related to the compatibility of fuzzy relations studied in [21]. We already proved in [22, Theorem 1] that the above fuzzy Galois condition provides a characterization of fuzzy relational Galois connections (Definition 6) in the framework of fuzzy preposets. However, this is not the case for fuzzy T-digraphs, as the following example illustrates.

Example 1. Consider the fuzzy T-digraphs $\mathbb{A} = \langle\{a_1,a_2,a_3,a_4\}, \rho_A\rangle$ and $\mathbb{B} = \langle\{b_1,b_2,b_3\}, \rho_B\rangle$, and the fuzzy relations $\mu : A \times B \to [0,1]$ and $\nu : B \times A \to [0,1]$ defined below:

ρ_A	a_1	a_2	a_3	a_4
a_1	0.5	1	1	0
a_2	0.5	1	1	0
a_3	0.5	0.5	1	0
a_4	0	0	0	1

ρ_B	b_1	b_2	b_3
b_1	1	0	0
b_2	0	1	1
b_3	0	1	1

μ	b_1	b_2	b_3
a_1	1	0.5	0
a_2	1	0.4	0
a_3	1	0.3	0
a_4	0	0	1

ν	a_1	a_2	a_3	a_4
b_1	0	0.5	1	0
b_2	0	0	0	1
b_3	0	0	0	1

It is routine to check that (μ, ν) satisfies the fuzzy Galois condition, although it is not a fuzzy relational Galois connection, since $\mu \circ \nu$ is not inflationary. Indeed, for instance, $(\mu \circ \nu)(a_1, a_4) = 0.5 \not\le \rho_A(a_1, a_4) = 0$.

[2] Formally, we should write $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$, but we will often abuse the notation whenever there is no risk of confusion.

Example 2. Consider Belnap's diamond as the underlying algebra of truth values $\mathfrak{B} = \{\bot, t, f, \top\}$, the fuzzy T-digraphs $\mathbb{A} = \langle\{a_1, a_2, a_3, a_4\}, \rho_A\rangle$ and $\mathbb{B} = \langle\{b_1, b_2, b_3\}, \rho_B\rangle$, and the fuzzy relations $\mu\colon A \times B \to \mathfrak{B}$ and $\nu\colon B \times A \to \mathfrak{B}$ defined below:

ρ_A	a_1	a_2	a_3	a_4
a_1	t	$\top$	$\top$	$\bot$
a_2	t	$\top$	$\top$	$\bot$
a_3	t	t	$\top$	$\bot$
a_4	$\bot$	$\bot$	$\bot$	$\top$

ρ_B	b_1	b_2	b_3
b_1	$\top$	$\bot$	$\bot$
b_2	$\bot$	$\top$	$\top$
b_3	$\bot$	$\top$	$\top$

μ	b_1	b_2	b_3
a_1	$\top$	t	$\bot$
a_2	$\top$	f	$\bot$
a_3	$\top$	t	$\bot$
a_4	$\bot$	$\bot$	$\top$

ν	a_1	a_2	a_3	a_4
b_1	$\bot$	f	$\top$	$\bot$
b_2	$\bot$	$\bot$	$\bot$	$\top$
b_3	$\bot$	$\bot$	$\bot$	$\top$

It is routine to check that (μ, ν) satisfies the fuzzy Galois condition, although it is not a fuzzy relational Galois connection, since $\mu \circ \nu$ is not inflationary. Indeed, for instance, $(\mu \circ \nu)(a_1, a_4) = t \nleq \rho_A(a_1, a_4) = \bot$.

The following theorem states that the fuzzy Galois condition needs to be complemented by a clique condition in order to extend the characterization of fuzzy relational Galois connections to the framework of fuzzy T-digraphs.

Theorem 1. *Consider two fuzzy T-digraphs $\langle A, \rho\rangle$ and $\langle B, \rho\rangle$ and two total fuzzy relations $\mu\colon A \times B \to \mathbb{L}$ and $\nu\colon B \times A \to \mathbb{L}$. The couple (μ, ν) is a fuzzy relational Galois connection between $\langle A, \rho\rangle$ and $\langle B, \rho\rangle$ if and only if the following conditions hold:*

(i) *(μ, ν) satisfies the fuzzy Galois condition;*
(ii) *a^μ and b^ν are cliques, for all $a \in A$, $b \in B$.*

Proof. Assume that (μ, ν) is a fuzzy relational Galois connection. Consider $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Since μ is total, we can choose $b_3 \in B$ such that $\mu(a_2, b_3) = \top$. As μ is antitone and $\nu \circ \mu$ is inflationary, we obtain

$$\begin{aligned}
&\rho(a_1, a_2) \wedge \mu(a_1, b_1) \wedge \nu(b_2, a_2) \\
&\qquad = \rho(a_1, a_2) \wedge \mu(a_1, b_1) \wedge \mu(a_2, b_3) \wedge \nu(b_2, a_2) \wedge \mu(a_2, b_3) \\
&\qquad \le \rho(b_2, b_3) \wedge \rho(b_3, b_1) \le \rho(b_2, b_1)\,.
\end{aligned}$$

The proof of the second part of the fuzzy Galois condition is similar.

Let us now prove that a^μ is a clique, for all $a \in A$. As $\mu \circ \nu$ is inflationary, for all $a, x \in A$ and $b \in B$, it holds that $\mu(a, b) \wedge \nu(b, x) \le \rho(a, x)$, which is equivalent to $\mu(a, b) \le \nu(b, x) \to \rho(a, x)$, and, hence, $\mu(a, b) \le \rho_S(a, b^\nu)$. Now, by Lemma 1(i) and the fuzzy Galois condition, we obtain

$$\mu(a, b) \le \rho_S(a, b^\nu) \le \rho_H(a, b^\nu) \le \rho_S(b, a^\mu)\,,$$

which implies that a^μ is a clique, because $\mu(a, b) \le \mu(a, x) \to \rho(b, x)$ is equivalent to $\mu(a, b) \wedge \mu(a, x) \le \rho(b, x)$. The proof that b^ν is a clique, for all $b \in B$, is similar.

Conversely, assume that conditions (i) and (ii) hold. Let us prove first that $\mu \circ \nu$ is inflationary. Consider $a_1, a_2 \in A$. Since μ is total, we can choose $b' \in B$ such that $\mu(a_1, b') = \top$. Using the fact that a^μ is a clique and the fuzzy Galois condition, we get

$$\begin{aligned}
(\mu \circ \nu)(a_1, a_2) &= \bigvee_{b \in B} \Big(\mu(a_1, b) \wedge \mu(a_1, b') \wedge \mu(a_1, b') \wedge \nu(b, a_2)\Big) \\
&\le \bigvee_{b \in B} \Big(\rho(b, b') \wedge \mu(a_1, b') \wedge \nu(b, a_2)\Big) \le \rho(a_1, a_2)\,,
\end{aligned}$$

which shows that $\mu \circ \nu$ is inflationary. The proof that $\nu \circ \mu$ is inflationary is similar.

Now, before proving that μ is antitone, we show that, for all $a_1, a_2 \in A$ and $b \in B$, there exists $a' \in A$ such that

$$\rho(a_1, a_2) \wedge \mu(a_2, b) \le \rho(a_1, a') \wedge \nu(b, a')\,. \tag{14}$$

To do this, since ν is total, we can choose $a' \in A$ such that $\nu(b, a') = \top$. Now, as $\mu \circ \nu$ is inflationary and ρ is transitive, we obtain for all $b_1, b_2 \in B$:

$$\begin{aligned}\rho(a_1,a_2)\wedge\mu(a_2,b) &\le \rho(a_1,a_2)\wedge\mu(a_2,b)\wedge\nu(b,a')\wedge\nu(b,a')\\ &\le \rho(a_1,a_2)\wedge\rho(a_2,a')\wedge\nu(b,a')\\ &\le \rho(a_1,a')\wedge\nu(b,a')\,.\end{aligned}$$

Using this inequality and the fuzzy Galois condition, we obtain

$$\begin{aligned}\rho(a_1,a_2)\wedge\mu(a_1,b_1)\wedge\mu(a_2,b_2) &\le \rho(a_1,a')\wedge\mu(a_1,b_1)\wedge\nu(b_2,a')\\ &\le \rho(b_2,b_1)\,,\end{aligned}$$

which shows that μ is antitone. The proof that ν is antitone is similar. □

As a consequence of this theorem, given a fuzzy relational Galois connection (μ,ν), we have that the aftersets a^μ and b^ν are normal cliques for all $a \in A$ and $b \in B$. Hence, taking into account Lemma 1, we have that $\rho_S(a,b^\nu) = \rho_\propto(a,b^\nu) = \rho_H(a,b^\nu)$. As a result, the inequalities in the definition of the fuzzy Galois condition reduce to equalities, as is expressed in the following corollary.

Corollary 2. *Consider two fuzzy T-digraphs $\langle A,\rho\rangle$ and $\langle B,\rho\rangle$ and two total fuzzy relations $\mu\colon A\times B\to\mathbb{L}$ and $\nu\colon B\times A\to\mathbb{L}$. The couple (μ,ν) is a fuzzy relational Galois connection between $\langle A,\rho\rangle$ and $\langle B,\rho\rangle$ if and only if the following conditions hold, for all $a\in A$, $b\in B$:*

(i) $\rho_\propto(a,b^\nu)=\rho_\propto(b,a^\mu)$;
(ii) *a^μ and b^ν are cliques.*

The clique condition arising in the characterization of fuzzy relational Galois connections in terms of the fuzzy Galois connection is noteworthy: the aftersets of both components (i.e., the fuzzy relations) need to be normal cliques. These conditions can be related to the work of Demirci [23] on fuzzy functions: the clique condition corresponds to the fuzzy relations being *partial fuzzy functions*, while the normality (the fuzzy relations being total) further restricts the partial fuzzy functions to being *perfect*. Hence, this characterization catapults us back to a functional framework.

4. Fuzzy closure relations

In this section, within the proposed framework of fuzzy relational Galois connections, we explore the counterpart of the classical relationship between Galois connections and closure relations as well as the interplay between closure relations and closure systems.

4.1. Fuzzy relational Galois connections and fuzzy closure relations

To start, let us introduce the notion of fuzzy closure relation.

Definition 8. Consider a fuzzy T-digraph $\langle A,\rho\rangle$. A fuzzy relation $\kappa\colon A\times A\to\mathbb{L}$ is called a *fuzzy closure relation* on A if it is total, isotone, inflationary and idempotent.

Remark 1. Note that the idempotence (as in Definition 5) of a fuzzy closure relation κ is equivalent to demand that $\rho_\propto(a^{\kappa\circ\kappa},a^\kappa)=\top$, for all $a\in A$.

In the characterizing Theorem 1, we already saw that the aftersets of the components of a fuzzy relational Galois connection are cliques. The following result shows that also the aftersets of fuzzy closure relations are cliques.

Lemma 4. *Consider a fuzzy T-digraph $\langle A,\rho\rangle$ and a fuzzy closure relation $\kappa\colon A\times A\to\mathbb{L}$ on A. For all $a\in A$, we have:*

(i) *a^κ and $a^{\kappa\circ\kappa}$ are cliques;*

(ii) $a^\kappa(x) \leq \rho_\propto(x^\kappa, x)$ *for all* $x \in A$.

Proof. Since κ is total, we can choose $y_0 \in \mathrm{Core}(a^\kappa)$ and $z \in \mathrm{Core}(y_0^\kappa)$ such that $z \in \mathrm{Core}(a^{\kappa\circ\kappa})$. First, since κ is idempotent, for all $x, y \in A$, we have

$$a^\kappa(x) \wedge a^{\kappa\circ\kappa}(y) \leq \rho(x, y) \wedge \rho(y, x)\,.$$

We then obtain that a^κ is a clique because

$$\begin{aligned} a^\kappa(x) \wedge a^\kappa(y) &= a^\kappa(x) \wedge a^{\kappa\circ\kappa}(z) \wedge a^{\kappa\circ\kappa}(z) \wedge a^\kappa(y) \\ &\leq \rho(x, z) \wedge \rho(z, y) \leq \rho(x, y)\,. \end{aligned}$$

On the other hand, we obtain that $a^{\kappa\circ\kappa}$ is a clique because

$$\begin{aligned} a^{\kappa\circ\kappa}(x) \wedge a^{\kappa\circ\kappa}(y) &= a^{\kappa\circ\kappa}(x) \wedge a^\kappa(y_0) \wedge a^\kappa(y_0) \wedge a^{\kappa\circ\kappa}(y) \\ &\leq \rho(x, y_0) \wedge \rho(y_0, y) \leq \rho(x, y)\,. \end{aligned}$$

Second, the idempotence of κ implies that

$$a^\kappa(x) \wedge x^\kappa(z) \wedge a^\kappa(y) \leq a^{\kappa\circ\kappa}(z) \wedge a^\kappa(y) \leq \rho(z, y) \wedge \rho(y, z)\,.$$

We then obtain

$$a^\kappa(x) \wedge x^\kappa(z) \wedge a^\kappa(y_0) \leq a^\kappa(x) \wedge x^\kappa(z) \leq \rho(z, y_0)\,.$$

On the other hand, since a^κ is a clique, we can also write

$$a^\kappa(x) \wedge x^\kappa(z) \wedge a^\kappa(y_0) \leq a^\kappa(x) \wedge a^\kappa(y_0) \leq \rho(y_0, x)\,.$$

Combining the above, and recalling $y_0 \in \mathrm{Core}(a^\kappa)$, we have

$$a^\kappa(x) \wedge x^\kappa(z) \leq \rho(z, y_0) \wedge \rho(y_0, x) \leq \rho(z, x)\,,$$

consequently, $a^\kappa(x) \leq \rho_\propto(x^\kappa, x)$. □

The main result in this subsection is that the compositions of the components of a fuzzy relational fuzzy Galois connection effectively yield fuzzy closure relations. To that end, we need the following lemma.

Lemma 5. *If* (μ, ν) *is a fuzzy relational Galois connection between two fuzzy T-digraphs* $\langle A, \rho\rangle$ *and* $\langle B, \rho\rangle$*, then for all* $a \in A$ *the following chain of equalities holds:*

$$\rho_\propto(a^\mu, a^{\mu\circ\nu\circ\mu}) = \top = \rho_\propto(a^{\mu\circ\nu\circ\mu}, a^\mu)\,.$$

Proof. First of all, we note that the first inequality $\rho_\propto(a^\mu, a^{\mu\circ\nu\circ\mu}) = \top$ is equivalent to the inequality $a^\mu(z) \wedge a^{\mu\circ\nu\circ\mu}(w) \leq \rho(z, w)$, for all $z, w \in B$. In order to prove this inequality, we consider the following facts:

(a) Since $\nu \circ \mu$ is inflationary, we have $\rho_\propto(y, y^{\nu\circ\mu}) = \top$, for all $y \in B$, or, equivalently, $y^{\nu\circ\mu}(w) \leq \rho(y, w)$, for all $y, w \in B$.
(b) Since a^μ is a clique, we have $\rho_\propto(a^\mu, a^\mu) = \top$, or, equivalently, $a^\mu(z) \wedge a^\mu(y) \leq \rho(z, y)$ for all $y, z \in B$.

From these two inequalities, we obtain for all $y \in B$:

$$a^\mu(z) \wedge a^\mu(y) \wedge y^{\nu\circ\mu}(w) \leq \rho(z, y) \wedge \rho(y, w) \leq \rho(z, w)\,,$$

and thus

$$\bigvee_{y\in B} \Big(a^\mu(z) \wedge a^\mu(y) \wedge y^{\nu\circ\mu}(w)\Big) \leq \rho(z, w)\,.$$

By distributivity, we obtain for all $z, w \in B$:

$$a^\mu(z) \wedge \bigvee_{y\in B} \Big(a^\mu(y) \wedge y^{\nu\circ\mu}(w)\Big) = a^\mu(z) \wedge a^{\mu\circ\nu\circ\mu}(w) \leq \rho(z,w)\,.$$

Similarly, the second equality $\rho_\propto(a^{\mu\circ\nu\circ\mu}, a^\mu) = \top$ is equivalent to the inequality $a^{\mu\circ\nu\circ\mu}(z) \wedge a^\mu(w) \leq \rho(z,w)$, for all $z, w \in B$. In order to prove this inequality, we consider the following facts:

(a) Since $\mu \circ \nu$ is inflationary, we have $\rho_\propto(a, a^{\mu\circ\nu}) = \top$, or, equivalently, $a^{\mu\circ\nu}(x) \leq \rho(a,x)$, for all $x \in A$.
(b) Since μ is antitone, for all $x \in A$ and all $z, w \in B$, we have

$$\rho(a,x) \leq \rho_\propto(x^\mu, a^\mu) \leq x^\mu(z) \wedge a^\mu(w) \to \rho(z,w)\,.$$

From the above inequalities, we obtain for all $z, w \in B$:

$$a^{\mu\circ\nu}(x) \leq x^\mu(z) \wedge a^\mu(w) \to \rho(z,w)\,,$$

which is equivalent to

$$a^{\mu\circ\nu}(x) \wedge x^\mu(z) \leq a^\mu(w) \to \rho(z,w)\,.$$

Since the latter holds for all $x \in A$, we have

$$a^{\mu\circ\nu\circ\mu}(z) = \bigvee_{x\in A} \Big(a^{\mu\circ\nu}(x) \wedge x^\mu(z)\Big) \leq a^\mu(w) \to \rho(z,w)\,,$$

or, equivalently, $a^{\mu\circ\nu\circ\mu}(z) \wedge a^\mu(w) \leq \rho(z,w)$. □

Proposition 1. *Consider two fuzzy T-digraphs $\langle A, \rho\rangle$ and $\langle B, \rho\rangle$, and two total fuzzy relations $\mu\colon A \times B \to \mathbb{L}$ and $\nu\colon B \times A \to \mathbb{L}$. If (μ, ν) is a fuzzy relational Galois connection between $\langle A, \rho\rangle$ and $\langle B, \rho\rangle$, then $\mu \circ \nu$ is a fuzzy closure relation on A and $\nu \circ \mu$ is a fuzzy closure relation on B.*

Proof. We only give the proof that $\mu \circ \nu$ is a fuzzy closure relation on A, the other proof being similar. It is straightforward that $\mu \circ \nu$ is total. By definition of a fuzzy relational Galois connection, it holds that $\mu \circ \nu$ is inflationary. Let us now prove that $\rho(a_1, a_2) \leq \rho_\propto(a_1^{\mu\circ\nu}, a_2^{\mu\circ\nu})$, i.e., $\mu \circ \nu$ is isotone. Successively using that μ and ν are antitone, we obtain for all $y_1, y_2 \in B$:

$$\begin{aligned}\rho(a_1,a_2) \wedge a_2^\mu(y_1) \wedge a_1^\mu(y_2) &\leq \rho(y_1,y_2)\\ &\leq \rho_\propto(y_2^\nu, y_1^\nu)\\ &= \bigwedge_{x_1,x_2\in A} \big(y_2^\nu(x_1) \wedge y_1^\nu(x_2) \to \rho(x_1,x_2)\big)\,.\end{aligned}$$

In particular, for all $x_1, x_2 \in A$, we obtain

$$\rho(a_1,a_2) \wedge a_2^\mu(y_1) \wedge a_1^\mu(y_2) \leq y_2^\nu(x_1) \wedge y_1^\nu(x_2) \to \rho(x_1,x_2)\,.$$

The adjointness property leads to:

$$\rho(a_1,a_2) \wedge a_2^\mu(y_1) \wedge a_1^\mu(y_2) \wedge y_2^\nu(x_1) \wedge y_1^\nu(x_2) \leq \rho(x_1,x_2)\,,$$

and hence

$$\rho(a_1,a_2) \wedge \bigvee_{y_1\in B} \big(a_2^\mu(y_1) \wedge y_1^\nu(x_2)\big) \wedge \bigvee_{y_2\in B} \big(a_1^\mu(y_2) \wedge y_2^\nu(x_1)\big) \leq \rho(x_1,x_2)\,.$$

Using the definition of fuzzy relational composition, we obtain

$$\rho(a_1,a_2) \wedge a_1^{\mu\circ\nu}(x_1) \wedge a_2^{\mu\circ\nu}(x_2) \leq \rho(x_1,x_2)\,,$$

which gives isotonicity.

Finally, in order to prove idempotence, it suffices to show that $a^{\mu\circ\nu\circ\mu\circ\nu}(x_1) \wedge a^{\mu\circ\nu}(x_2) \leq \rho(x_1,x_2)$. Lemma 5 and the fact that ν is antitone lead to

$$
\begin{aligned}
a^{\mu}(y_1) \wedge a^{\mu\circ\nu\circ\mu}(y_2) &\le \rho(y_1, y_2) \\
&\le \rho_{\propto}(y_2^{\nu}, y_1^{\nu}) \\
&= \bigwedge_{x_1, x_2 \in A} \left(y_2^{\nu}(x_1) \wedge y_1^{\nu}(x_2) \to \rho(x_1, x_2) \right) .
\end{aligned}
$$

Hence, for all $x_1, x_2 \in A$, we have

$$a^{\mu}(y_1) \wedge a^{\mu\circ\nu\circ\mu}(y_2) \le y_2^{\nu}(x_1) \wedge y_1^{\nu}(x_2) \to \rho(x_1, x_2) ,$$

which is equivalent to

$$a^{\mu}(y_1) \wedge a^{\mu\circ\nu\circ\mu}(y_2) \wedge y_2^{\nu}(x_1) \wedge y_1^{\nu}(x_2) \le \rho(x_1, x_2) .$$

Taking the supremum over y_1, y_2, we obtain

$$a^{\mu\circ\nu\circ\mu\circ\nu}(x_1) \wedge a^{\mu\circ\nu}(x_2) \le \rho(x_1, x_2) . \quad \square$$

4.2. Fuzzy closure relations and fuzzy closure systems

In order to generalize the construction of fuzzy closure systems to the framework of fuzzy T-digraphs, we need an appropriate notion of fuzzy minimum. The following definition, inspired by [24], turns out to do the job.

Definition 9. Given a fuzzy T-digraph $\langle A, \rho\rangle$ and a fuzzy set $X\colon A \to \mathbb{L}$. The minimum of X is the fuzzy set $m(X)\colon A \to \mathbb{L}$ defined by $m(X)(a) = X(a) \wedge \rho_{\propto}(a, X)$, for all $a \in A$.

First, we prove that $m(X)$ is also a clique. This will turn out to be helpful later on for proving that a fuzzy closure system generates a fuzzy closure relation.

Lemma 6. *Consider a fuzzy T-digraph $\langle A, \rho\rangle$ and a fuzzy set $X\colon A \to \mathbb{L}$. Then $m(X)$ is a clique.*

Proof. By definition of $m(X)$, for all $w \in A$, we have

$$
\begin{aligned}
m(X)(a) &= X(a) \wedge \rho_{\propto}(a, X) \\
&= X(a) \wedge \bigwedge_{w \in A} (X(w) \to \rho(a, w)) \\
&\le X(w) \to \rho(a, w) .
\end{aligned}
$$

The adjointness property then leads to

$$m(X)(a) \wedge X(w) \le \rho(a, w) .$$

The proof that $m(X)$ is a clique then goes as follows:

$$m(X)(a_1) \wedge m(X)(a_2) \le m(X)(a_1) \wedge X(a_2) \le \rho(a_1, a_2) ,$$

for all $a_1, a_2 \in A$. $\square$

Next, we introduce the notion of a fuzzy closure system.

Definition 10. Consider a fuzzy T-digraph $\langle A, \rho\rangle$. A fuzzy set $C\colon A \to \mathbb{L}$ is called a *fuzzy closure system* if $m(a^{\rho} \cap C)$ is normal, for all $a \in A$.

The following theorem states that a fuzzy closure relation generates a fuzzy closure system.

Theorem 2. *Consider a fuzzy T-digraph $\langle A, \rho\rangle$ and a fuzzy closure relation $\kappa\colon A \times A \to \mathbb{L}$ on A. Then the fuzzy set $C_{\kappa}\colon A \to \mathbb{L}$ defined by $C_{\kappa}(x) = \rho_{\propto}(x^{\kappa}, x)$, for all $x \in A$, satisfies $a^{\kappa}(x) \le m(a^{\rho} \cap C_{\kappa})(x)$ for all $a, x \in A$. Therefore, C_{κ} is a fuzzy closure system.*

Proof. For all $a, x \in A$, since κ is inflationary, we have $a^\kappa(x) \le \rho(a, x) = a^\rho(x)$. By Lemma 4(ii), we also have $a^\kappa(x) \le \rho_\propto(x^\kappa, x) = C_\kappa(x)$. Combined, we obtain $a^\kappa(x) \le (a^\rho \cap C_\kappa)(x)$.

For all $y \in A$, since κ is isotone, we have

$$\rho(a, y) \le \rho_\propto(a^\kappa, y^\kappa) = \bigwedge_{x,z\in A} \left(a^\kappa(x) \wedge y^\kappa(z) \to \rho(x, z)\right) ,$$

which, using the adjointness property, leads to

$$\rho(a, y) \wedge a^\kappa(x) \wedge y^\kappa(z) \le \rho(x, z) .$$

Moreover, by definition of C_κ and Lemma 3(iii), we have

$$C_\kappa(y) \wedge y^\kappa(z) = \rho_\propto(y^\kappa, y) \wedge y^\kappa(z) \le \rho(z, y) ,$$

and, thus,

$$\rho(a, y) \wedge a^\kappa(x) \wedge y^\kappa(z) \wedge C_\kappa(y) \le \rho(x, z) \wedge \rho(z, y) \le \rho(x, y) .$$

Again, since y^κ is total, we can choose z_0 such that $y^\kappa(z_0) = \top$ and

$$\rho(a, y) \wedge a^\kappa(x) \wedge C_\kappa(y) = a^\kappa(x) \wedge (a^\rho \cap C_\kappa)(y) \le \rho(x, y) .$$

As a result, we obtain $a^\kappa(x) \le \rho_\propto(x, a^\rho \cap C_\kappa)$.

Finally, taking into account the results in the two previous paragraphs, we can write

$$a^\kappa(x) \le (a^\rho \cap C_\kappa)(x) \wedge \rho_\propto(x, a^\rho \cap C_\kappa) = m(a^\rho \cap C_\kappa)(x) .$$

Since κ is total, it is then immediate that m is a fuzzy closure system. □

The following theorem is somehow a converse of the previous one, in the sense that a fuzzy closure system is shown to generate a fuzzy closure relation.

Theorem 3. *Consider a fuzzy T-digraph $\langle A, \rho\rangle$ and a fuzzy closure system $C\colon A \to \mathbb{L}$. The fuzzy relation $\kappa_C\colon A \times A \to \mathbb{L}$ defined by $\kappa_C(a, x) = a^{\kappa_C}(x) = m(a^\rho \cap C)(x)$, for all $a, x \in A$, is a fuzzy closure relation.*

Proof. We have to prove that κ_C is total, inflationary, isotone, and idempotent.

(i) Obviously, κ_C is total since $m(a^\rho \cap C)$ is normal.

(ii) κ_C is inflationary by definition of m, since $a^{\kappa_C}(x) = m(a^\rho \cap C)(x) \le \rho(a, x)$.

(iii) Using the definition of m and the transitivity of ρ, we get

$$\begin{aligned}\rho(a_1, a_2) \wedge {a_1}^{\kappa_C}(x) \wedge a_2^{\kappa_C}(y) &= \rho(a_1, a_2) \wedge m(a_1^\rho \cap C)(x) \wedge m(a_2^\rho \cap C)(y)\\ &= \rho(a_1, a_2) \wedge (a_1^\rho \cap C)(x) \wedge \rho_\propto(x, a_1^\rho \cap C) \wedge m(a_2^\rho \cap C)(y)\\ &\le \rho(a_1, a_2) \wedge \rho_\propto(x, a_1^\rho \cap C) \wedge \rho(a_2, y) \wedge C(y)\\ &\le \rho_\propto(x, a_1^\rho \cap C) \wedge \rho(a_1, y) \wedge C(y)\\ &= \rho_\propto(x, a_1^\rho \cap C) \wedge (a_1^\rho \cap C)(y) .\end{aligned}$$

Now, using Lemma 3(ii), we obtain

$$\rho(a_1, a_2) \wedge a_1^{\kappa_C}(x) \wedge a_2^{\kappa_C}(y) \le \rho(x, y) .$$

Hence, for all $x, y \in A$, it holds that

$$\rho(a_1, a_2) \le a_1^{\kappa_C}(x) \wedge a_2^{\kappa_C}(y) \to \rho(x, y) ,$$

which implies

$$\rho(a_1, a_2) \le \rho_\propto(a_1^{\kappa_C}, a_2^{\kappa_C}) .$$

(iv) For the idempotence of κ_C, we need to show that $\rho_\propto(a^{\kappa_C \circ \kappa_C}, a^{\kappa_C}) = \top$. Since $a^{\kappa_C}(w) = m(a^\rho \cap C)(w) \leq C(w)$ and a^{κ_C} is a clique due to Lemma 6, and using Lemma 3(ii), for all $x, y, w \in A$, we have

$$\begin{aligned} a^{\kappa_C}(x) \wedge x^{\kappa_C}(y) \wedge a^{\kappa_C}(w) &\leq a^{\kappa_C}(x) \wedge x^{\kappa_C}(y) \wedge a^{\kappa_C}(w) \wedge C(w) \\ &\leq \rho(x, w) \wedge x^{\kappa_C}(y) \wedge C(w) \\ &\leq \rho_\propto(y, x^\rho \cap C) \wedge \rho(x, w) \wedge C(w) \\ &\leq \rho_\propto(y, w) = \rho(y, w) . \end{aligned}$$

Therefore

$$\left(\bigvee_{x \in A} (a^{\kappa_C}(x) \wedge x^{\kappa_C}(y)) \right) \wedge a^{\kappa_C}(w) = a^{\kappa_C \circ \kappa_C}(y) \wedge a^{\kappa_C}(w) \leq \rho(y, w) ,$$

which is, precisely, $\rho_\propto(a^{\kappa_C \circ \kappa_C}, a^{\kappa_C}) = \top$. □

5. Construction of the right adjoint

This section constitutes the core contribution of this paper. We provide necessary and sufficient conditions under which it is possible to build a right adjoint for a given fuzzy relation. To that end, we first introduce the notion of compatibility of a fuzzy closure system with a fuzzy relation μ, more specifically, with its fuzzy kernel relation, which is defined below.

Definition 11. Consider a fuzzy T-digraph $\langle A, \rho \rangle$, a set B and a fuzzy relation $\mu \colon A \times B \to \mathbb{L}$. The *fuzzy kernel relation* $\equiv_\mu \colon A \times A \to \mathbb{L}$ is defined by, for all $a_1, a_2 \in A$:

$$(a_1 \equiv_\mu a_2) = \bigvee_{b \in B} (a_1^\mu(b) \wedge a_2^\mu(b)) .$$

A fuzzy closure system $C \colon A \to \mathbb{L}$ is called *compatible* with μ if $\rho_\propto(a^{\equiv_\mu}, a^\rho \cap C) = \top$, for all $a \in A$, or, equivalently, for all $a_1, a_2 \in A$, it holds:

$$\bigvee_{b \in B} (a_1^\mu(b) \wedge a_2^\mu(b)) \leq \rho_\propto(a_1, a_2^\rho \cap C) .$$

The following technical lemma will be used later on in order to prove Proposition 2.

Lemma 7. *Consider a fuzzy relational Galois connection (μ, ν) between two fuzzy T-digraphs $\langle A, \rho \rangle$ and $\langle B, \rho \rangle$. We have:*

(i) $\rho_\propto(a^\mu, b) \leq \rho_\propto(b^\nu, a^{\mu \circ \nu})$, *for all $a \in A$, $b \in B$;*
(ii) $\rho_\propto(a_1^\mu, a_2^\mu) \leq \rho_\propto(a_2, a_1^{\mu \circ \nu})$, *for all $a_1, a_2 \in A$.*

Proof. Let us prove the first inequality. For all $x \in A$ and $y \in B$, we have

$$\begin{aligned} \rho_\propto(a^\mu, b) \wedge b^\nu(x) \wedge a^{\mu \circ \nu}(y) &= \rho_\propto(a^\mu, b) \wedge b^\nu(x) \wedge \left(\bigvee_{z \in B} (a^\mu(z) \wedge z^\nu(y)) \right) \\ &= \bigvee_{z \in B} \left(\rho_\propto(a^\mu, b) \wedge b^\nu(x) \wedge a^\mu(z) \wedge z^\nu(y) \right) . \end{aligned}$$

Using Lemma 3(i) with $X = a^\mu$, we obtain

$$\bigvee_{z \in B} \left(\rho_\propto(a^\mu, b) \wedge b^\nu(x) \wedge a^\mu(z) \wedge z^\nu(y) \right) \leq \bigvee_{z \in B} \left(\rho(z, b) \wedge b^\nu(x) \wedge z^\nu(y) \right) .$$

Since ν is antitone, for all $x, y \in A$ and $z \in B$ it holds that:

$$\rho(z,b) \le \rho_\propto(b^\nu, z^\nu) = \bigwedge_{x\in A}\bigwedge_{y\in A}\big(b^\nu(x)\wedge z^\nu(y) \to \rho(x,y)\big)$$
$$\le b^\nu(x)\wedge z^\nu(y) \to \rho(x,y)\,.$$

As a consequence, we have

$$\bigvee_{z\in B}\big(\rho(z,b)\wedge b^\nu(x)\wedge z^\nu(y)\big) \le \rho(x,y)\,.$$

Hence, $\rho_\propto(a^\mu, b)\wedge b^\nu(x)\wedge a^{\mu\circ\nu}(y) \le \rho(x,y)$ for all $x, y \in A$, and, using the adjointness property, we obtain

$$\rho_\propto(a^\mu, b) \le \bigwedge_{x\in A}\bigwedge_{y\in A}\big(b^\nu(x)\wedge a^{\nu\circ\mu}(y) \to \rho(x,y)\big) = \rho_\propto(b^\nu, a^{\mu\circ\nu})\,.$$

Next, we prove the second inequality. Since a_1^μ is normal, we can choose $y_0 \in B$ such that $a_1^\mu(y_0) = \top$. Since a_1^μ and a_2^μ are normal cliques, Corollaries 1 and 2 yield

$$\rho_\propto(a_1^\mu, a_2^\mu) = \rho_\propto(y_0, a_2^\mu) = \rho_\propto(a_2, y_0^\nu)\,.$$

Due to Corollary 1, we also have $\rho_\propto(a_1^\mu, y_0) = \top$. Together with the first inequality proven and Lemma 2, we then obtain

$$\begin{aligned}\rho_\propto(a_1^\mu, a_2^\mu) &= \rho_\propto(a_2, y_0^\nu)\wedge\rho_\propto(a_1^\mu, y_0)\\ &\le \rho_\propto(a_2, y_0^\nu)\wedge\rho_\propto(y_0^\nu, a_1^{\mu\circ\nu})\\ &\le \rho_\propto(a_2, a_1^{\mu\circ\nu})\,. \quad \square\end{aligned}$$

The following proposition provides important information for the construction of the right adjoint.

Proposition 2. *Consider a fuzzy relational Galois connection (μ,ν) between two fuzzy T-digraphs $\langle A,\rho\rangle$ and $\langle B,\rho\rangle$, then the fuzzy closure system $C_{\mu\circ\nu}\colon A \to \mathbb{L}$ defined by $C_{\mu\circ\nu}(x) = \rho_\propto(x^{\mu\circ\nu}, x)$, for all $x \in A$, is compatible with μ.*

Proof. Consider $a_1, a_2 \in A$. Since ν is antitone, for all $x \in A$ and $b \in B$, it holds that

$$\begin{aligned}&a_1^\mu(b)\wedge a_2^\mu(b)\wedge a_2^\rho(x)\wedge C_{\mu\circ\nu}(x)\\ &= a_1^\mu(b)\wedge a_2^\mu(b)\wedge \rho(a_2,x)\wedge\rho_\propto(x^{\mu\circ\nu},x)\\ &\le a_1^\mu(b)\wedge a_2^\mu(b)\wedge \rho_\propto(x^\mu,a_2^\mu)\wedge\rho_\propto(x^{\mu\circ\nu},x)\,.\end{aligned}$$

According to Lemma 3(iii), we have $a_2^\mu(b)\wedge\rho_\propto(x^\mu,a_2^\mu) \le \rho_\propto(x^\mu,b)$. Hence,

$$a_1^\mu(b)\wedge a_2^\mu(b)\wedge\rho_\propto(x^\mu,a_2^\mu)\wedge\rho_\propto(x^{\mu\circ\nu},x) \le a_1^\mu(b)\wedge\rho_\propto(x^\mu,b)\wedge\rho_\propto(x^{\mu\circ\nu},x)\,.$$

Applying Lemma 3(iii) and Lemma 2 (since $x^{\mu\circ\nu}$ is normal), we have $a_1^\mu(b)\wedge\rho_\propto(x^\mu,b)\le\rho_\propto(x^\mu,a_2^\mu)$ and thus

$$a_1^\mu(b)\wedge\rho_\propto(x^\mu,b)\wedge\rho_\propto(x^{\mu\circ\nu},x) \le \rho_\propto(x^\mu,a_1^\mu)\wedge\rho_\propto(x^{\mu\circ\nu},x)\,.$$

Applying Lemma 7, we obtain

$$\rho_\propto(x^\mu,a_1^\mu)\wedge\rho_\propto(x^{\mu\circ\nu},x) \le \rho_\propto(a_1,x^{\mu\circ\nu})\wedge\rho_\propto(x^{\mu\circ\nu},x) \le \rho_\propto(a_1,x) = a_1^\rho(x)\,.$$

Finally, from $a_1^\mu(b)\wedge a_2^\mu(b)\wedge a_2^\rho(x)\wedge C_{\mu\circ\nu}(x) \le a_1^\rho(x)$, we obtain

$$\bigvee_{b\in B}(a_1^\mu(b)\wedge a_2^\mu(b)) \le \bigwedge_{x\in A}(a_1^\rho(x)\wedge C_{\mu\circ\nu}(x) \to a_2^\rho(x)) = \rho_\propto(a_2, a_1^\rho\cap C)\,,$$

which concludes the proof. $\square$

Before proceeding to prove sufficiency, we need the following technical results in Lemmas 8 and 9.

Lemma 8. *Consider a fuzzy T-digraph $\langle A, \rho\rangle$ and a fuzzy set $C : A \to \mathbb{L}$. For all $a_1, a_2, a_3 \in A$, it holds that*

$$\rho_\propto(a_1, a_2 \cap C) \wedge \rho_\propto(a_2, a_3 \cap C) \leq \rho_\propto(a_1, a_3 \cap C)\,.$$

Proof. The claim follows from the following chain:

$$\begin{aligned}
&\rho_\propto(a_1, a_2^\rho \cap C) \wedge \rho_\propto(a_2, a_3^\rho \cap C)\\
&= \bigwedge_{x\in A} \big(a_2^\rho(x) \wedge C(x) \to a_1^\rho(x)\big) \wedge \bigwedge_{x\in A} \big(a_3^\rho(x) \wedge C(x) \to a_2^\rho(x)\big)\\
&= \bigwedge_{x\in A} \Big(\big(a_3^\rho(x) \wedge C(x) \to a_2^\rho(x)\big) \wedge \big(a_2^\rho(x) \wedge C(x) \to a_1^\rho(x)\big)\Big)\\
&\overset{(9)}{\leq} \bigwedge_{x\in A} \Big(\big(a_3^\rho(x) \wedge C(x) \to (a_2^\rho(x) \wedge C(x))\big) \wedge \big(a_2^\rho(x) \wedge C(x) \to a_1^\rho(x)\big)\Big)\\
&\overset{(8)}{\leq} \bigwedge_{x\in A} \big(a_3^\rho(x) \wedge C(x) \to a_1^\rho(x)\big) = \rho_\propto(a_1, a_3^\rho \cap C)\,. \quad \square
\end{aligned}$$

Lemma 9. *Consider a fuzzy T-digraph $\langle A, \rho\rangle$, a set B, a fuzzy relation $\mu\colon A \times B \to \mathbb{L}$ and a fuzzy closure system $C\colon A \to \mathbb{L}$ that is compatible with μ. For all $a_1, a_2, x \in A$ and $b \in B$, we have:*

(i) $(a_1 \equiv_\mu a_2) \wedge m(a_1^\rho \cap C)(x) \leq m(a_2^\rho \cap C)(x)$.
(ii) *If* $a_1^\mu(b) = \top$, *then* $a_2^\mu(b) \wedge m(a_1^\rho \cap C)(x) \leq m(a_2^\rho \cap C)(x)$.
(iii) *If* $a_1^\mu(b) = a_2^\mu(b) = \top$, *then* $m(a_1^\rho \cap C) = m(a_2^\rho \cap C)$.

Proof. We prove the first item, the other two items being immediate consequences of it. For all $a_1, a_2, x \in A$, since C is compatible with μ, we have

$$(a_1 \equiv_\mu a_2)(x) \leq \rho_\propto(a_1, a_2^\rho \cap C) \wedge \rho_\propto(a_2, a_1^\rho \cap C)\,.$$

Using the definition of m, we obtain

$$\begin{aligned}
&(a_1 \equiv_\mu a_2) \wedge m(a_1^\rho \cap C)(x) \leq\\
&\quad = \rho_\propto(a_1, a_2^\rho \cap C) \wedge \rho_\propto(a_2, a_1^\rho \cap C) \wedge (a_1^\rho \cap C)(x) \wedge \rho_\propto(x, a_1^\rho \cap C)\,.
\end{aligned}$$

Lemma 8 then implies

$$\rho_\propto(x, a_1^\rho \cap C) \wedge \rho_\propto(a_1, a_2^\rho \cap C) \leq \rho_\propto(x, a_2^\rho \cap C)\,,$$

and thus

$$\begin{aligned}
&\rho_\propto(a_1, a_2^\rho \cap C) \wedge \rho_\propto(a_2, a_1^\rho \cap C) \wedge (a_1^\rho \cap C)(x) \wedge \rho_\propto(x, a_1^\rho \cap C)\\
&\leq \rho_\propto(x, a_2^\rho \cap C) \wedge (a_1^\rho \cap C)(x) \wedge \rho_\propto(a_2, a_1^\rho \cap C)\\
&\leq \rho_\propto(x, a_2^\rho \cap C) \wedge (a_1^\rho \cap C)(x) \wedge \big((a_1^\rho \cap C)(x) \to a_2^\rho(x)\big)\\
&= \rho_\propto(x, a_2^\rho \cap C) \wedge (a_1^\rho \cap C)(x) \wedge (a_1^\rho \cap C)(x) \wedge \big((a_1^\rho \cap C)(x) \to a_2^\rho(x)\big)\\
&\leq \rho_\propto(x, a_2^\rho \cap C) \wedge (a_1^\rho \cap C)(x) \wedge a_2^\rho(x) \leq \rho_\propto(x, a_2^\rho \cap C) \wedge C(x) \wedge a_2^\rho(x)\\
&= m(a_2^\rho \cap C)(x)\,. \quad \square
\end{aligned}$$

The following proposition states an additional necessary condition.

Proposition 3. *If (μ, ν) is a fuzzy relational Galois connection between two fuzzy T-digraphs $\langle A, \rho\rangle$ and $\langle B, \rho\rangle$, with* $\mathrm{rng}(\mu) \neq B$, *then there exist a crisp function $\gamma\colon B \smallsetminus \mathrm{rng}(\mu) \to A$ and a fuzzy closure system $C\colon A \to \mathbb{L}$, such that for all $a, x \in A$ and $b \in B \smallsetminus \mathrm{rng}(\mu)$, it holds that*

$$a^\mu(b) \wedge \gamma(b)^{\kappa_C}(x) \leq a^{\kappa_C \circ \kappa_C}(x)\,. \tag{15}$$

Proof. Since ν is total and $\operatorname{rng}(\mu) \neq B$, we can define $\gamma\colon B \smallsetminus \operatorname{rng}(\mu) \to A$ such that $b^{\nu}(\gamma(b)) = \top$, for all $b \in B \smallsetminus \operatorname{rng}(\mu)$.

From Propositions 1 and 2, and Theorems 2 and 3, we know that $C := C_{\mu\circ\nu}$ is a fuzzy closure system that is compatible with μ, such that $a^{\mu\circ\nu}(x) \leq a^{\kappa_C}(x)$, for all $x \in A$. Using the latter, the definition of γ and standard properties, for all $a, x \in A$ and $b \in B \smallsetminus \operatorname{rng}(\mu)$, we get

$$\begin{aligned}
a^{\kappa_C\circ\kappa_C}(x) &= \bigvee_{z\in A} (a^{\kappa_C}(z) \wedge z^{\kappa_C}(x)) \\
&\geq a^{\kappa_C}(\gamma(b)) \wedge \gamma(b)^{\kappa_C}(x) \\
&\geq a^{\mu\circ\nu}(\gamma(b)) \wedge \gamma(b)^{\kappa_C}(x) \\
&= \left(\bigvee_{y\in B} (a^{\mu}(y) \wedge y^{\nu}(\gamma(b))\right) \wedge \gamma(b)^{\kappa_C}(x) \\
&\geq a^{\mu}(b) \wedge b^{\nu}(\gamma(b)) \wedge \gamma(b)^{\kappa_C}(x) \\
&= a^{\mu}(b) \wedge \gamma(b)^{\kappa_C}(x)\,. \quad \square
\end{aligned}$$

The following proposition shows that the conditions in Propositions 2 and 3 are also sufficient.

Proposition 4. *Consider a fuzzy T-digraph $\langle A, \rho\rangle$, a set B, a fuzzy relation $\mu\colon A \times B \to \mathbb{L}$ and a fuzzy closure system C that is compatible with μ. Assume that either* $\operatorname{rng}(\mu) = B$ *or there exists a crisp function $\gamma\colon B \smallsetminus \operatorname{rng}(\mu) \to A$ that satisfies condition* (15). *Then there exist a transitive fuzzy relation ρ' on B and a fuzzy relation $\nu\colon B \times A \to \mathbb{L}$ such that (μ, ν) is a fuzzy relational Galois connection between the T-digraphs $\langle A, \rho\rangle$ and $\langle B, \rho'\rangle$.*

Proof. First of all, due to the axiom of choice, we can define the mapping $\xi\colon B \to A$ as follows:

$$\xi(b) = \begin{cases} a & \text{, if there exists } a \in A \text{ such that } a^{\mu}(b) = \top \\ \gamma(b) & \text{, otherwise.} \end{cases}$$

Now define $\nu : B \times A \to \mathbb{L}$ as

$$b^{\nu} = m(\xi(b)^{\rho} \cap C) = \xi(b)^{\kappa_C}$$

and $\rho' : B \times B \to \mathbb{L}$ as:

$$\rho'(b_1, b_2) = \rho_{\propto}(b_2^{\nu}, b_1^{\nu})\,.$$

Note that the definition of ν does not depend on the choice of the mapping ξ, thanks to Lemma 9(iii). Moreover, ν is total since C is a fuzzy closure system.

Using Lemma 2, we obtain

$$\begin{aligned}
\rho'(b_1, b_2) \wedge \rho'(b_2, b_3) &= \rho_{\propto}(b_2^{\nu}, b_1^{\nu}) \wedge \rho_{\propto}(b_3^{\nu}, b_2^{\nu}) \\
&\leq \rho_{\propto}(b_3^{\nu}, b_1^{\nu}) = \rho'(b_1, b_3)\,.
\end{aligned}$$

In order to prove that (μ, ν) is a fuzzy relational Galois connection, we will use the characterization given in Theorem 1. Let us first prove the clique conditions. By definition of ν and Lemma 6, it is clear that b^{ν} is a clique, for all $b \in B$. In order to prove that a^{μ} is a clique, for all $a \in A$, we first note that by Lemma 9(ii), for all $a, x \in A$ and $b \in \operatorname{rng}(B)$, we have

$$a^{\mu}(b) \wedge \xi(b)^{\kappa_C}(x) \leq a^{\kappa_C}(x)\,. \tag{16}$$

We thus have to consider three cases depending on whether b_1 or b_2 belong to $\operatorname{rng}(\mu)$.

(1) Let $b_1, b_2 \in \operatorname{rng}(\mu)$. Using (16) and the fact that a^{κ_C} is a clique, we obtain

$$a^{\mu}(b_1) \wedge a^{\mu}(b_2) \wedge \xi(b_2)^{\kappa_C}(x) \wedge \xi(b_1)^{\kappa_C}(y) \leq a^{\kappa_C}(x) \wedge a^{\kappa_C}(y) \leq \rho(x, y)\,,$$

and, hence,

$$a^{\mu}(b_1) \wedge a^{\mu}(b_2) \le \bigwedge_{x,y\in A} \big(\xi(b_2)^{\kappa_C}(x) \wedge \xi(b_1)^{\kappa_C}(y) \to \rho(x,y)\big)$$
$$= \rho_{\propto}(\xi(b_2)^{\kappa_C}, \xi(b_1)^{\kappa_C}) = \rho'(b_1, b_2)\,.$$

(2) Let $b_1 \notin \mathrm{rng}(\mu)$ and $b_2 \in \mathrm{rng}(\mu)$. Using (15) and (16), and the fact that κ_C is idempotent, we obtain

$$a^{\mu}(b_1) \wedge a^{\mu}(b_2) \wedge \xi(b_1)^{\kappa_C}(x) \wedge \xi(b_2)^{\kappa_C}(y) \le a^{\kappa_C \circ \kappa_C}(x) \wedge a^{\kappa_C}(y) \le \rho(x,y)\,.$$

(3) Finally, let $b_1, b_2 \notin \mathrm{rng}(\mu)$. Using (15) and Lemma 4, we obtain

$$a^{\mu}(b_1) \wedge a^{\mu}(b_2) \wedge \xi(b_2)^{\kappa_C}(x) \wedge \xi(b_1)^{\kappa_C}(y) \le a^{\kappa_C \circ \kappa_C}(x) \wedge a^{\kappa_C \circ \kappa_C}(y) \le \rho(x,y)\,.$$

This indeed proves that a^{μ} is a clique.

Finally, let us prove the fuzzy Galois condition, i.e., $\rho_{\propto}(a, b^{\nu}) = \rho'_{\propto}(b, a^{\mu})$, for all $a \in A$ and $b \in B$. Since μ is total, we can choose y_0 such that $a^{\mu}(y_0) = \top$. We then have

$$\rho'_{\propto}(b, a^{\mu}) = \bigwedge_{y\in B} \big(a^{\mu}(y) \to \rho'(b,y)\big) \le \rho'(b, y_0) = \rho_{\propto}(y_0^{\nu}, b^{\nu})$$
$$\overset{(*)}{\le} \rho_S(y_0^{\nu}, b^{\nu}) = \bigwedge_{w\in A} \left(b^{\nu}(w) \to \bigvee_{z\in A} (y_0^{\nu}(z) \wedge \rho(z,w))\right),$$

where $(*)$ follows from Lemma 1(i). Note that $a^{\mu}(y_0) = \top$ implies that $y_0^{\nu} = \xi(y_0)^{\kappa_C} = a^{\kappa_C}$ and thus $y_0^{\nu}(z) = m(a^{\rho} \cap C)(z) \le \rho(a,z)$. We then continue

$$\rho'_{\propto}(b, a^{\mu}) \le \bigwedge_{w\in A} \left(b^{\nu}(w) \to \bigvee_{z\in A} (y_0^{\nu}(z) \wedge \rho(z,w))\right)$$
$$\le \bigwedge_{w\in A} \left(b^{\nu}(w) \to \bigvee_{z\in A} (\rho(a,z) \wedge \rho(z,w))\right)$$
$$\le \bigwedge_{w\in A} \big(b^{\nu}(w) \to \rho(a,w)\big) = \rho_{\propto}(a, b^{\nu})\,.$$

In order to prove the converse inequality, we first prove that if $b^{\nu}(x_0) = \top$, then $x_0^{\mu_C}(x_0) = \top$. Since b^{ν} is a clique, we have $b^{\nu}(x_0) \le \rho(x_0, x_0)$. Moreover, since $b^{\nu} = m(\xi(b)^{\rho} \cap C)$, we have $b^{\nu}(x_0) \le C(x_0)$. Hence, we obtain

$$b^{\nu}(x_0) \le (x_0^{\rho} \cap C)(x_0)\,. \tag{17}$$

On the other hand, it trivially holds that $b^{\nu}(x_0) \wedge \rho(x_0,z) \wedge C(z) \le \rho(x_0,z)$, or, equivalently, $b^{\nu}(x_0) \le (x_0^{\rho} \cap C)(z) \to \rho(x_0,z)$, for all $z \in A$. Therefore

$$b^{\nu}(x_0) \le \bigwedge_{z\in A} \big((x_0^{\rho} \cap C)(z) \to \rho(x_0,z)\big) = \rho_{\propto}(x_0, x_0^{\rho} \cap C)\,. \tag{18}$$

Now, using $b^{\nu}(x_0) = \top$, (17), (18) and the definition of $m(x_0^{\rho} \cap C)$, we obtain

$$\top = b^{\nu}(x_0) \le m(x_0^{\rho} \cap C)(x_0) = x_0^{\kappa_C}(x_0)\,, \tag{19}$$

and hence $x_0^{\kappa_C}(x_0) = \top$.

We are now ready to complete the proof. Since κ_C is isotone, we obtain using (19)

$$\rho_{\propto}(a, b^{\nu}) = \bigwedge_{x\in A} \big(b^{\nu}(x) \to \rho(a,x)\big)$$
$$\le \rho(a, x_0) \le \rho_{\propto}(a^{\kappa_C}, x_0^{\kappa_C})$$

$$= \bigwedge_{z,w\in A} \big(a^{\kappa C}(z) \wedge x_0^{\kappa C}(w) \to \rho(z,w)\big)$$
$$\overset{(\star)}{\leq} \bigwedge_{z\in A} \big(a^{\kappa C}(z) \to \rho(z,x_0)\big) = \rho_\propto(a^{\kappa C}, x_0)\,.$$

Using Corollary 1 and Lemma 1(iii), we then get

$$\rho_\propto(a, b^\nu) \leq \rho_\propto(a^{\kappa C}, x_0) = \rho_\propto(a^{\kappa C}, b^\nu) = \rho_\propto(y_0^\nu, b^\nu)$$
$$= \rho'(b, y_0) = \rho'_\propto(b, a^\mu)\,.$$

This concludes the proof. □

As a consequence of Propositions 2, 3 and 4, we get the following result.

Theorem 4. *Consider a fuzzy T-digraph $\langle A, \rho\rangle$ and a fuzzy relation $\mu\colon A \times B \to \mathbb{L}$. Then there exists a transitive fuzzy relation ρ' on B and a fuzzy relation $\nu\colon B \times A \to \mathbb{L}$ such that (μ, ν) is a fuzzy relational Galois connection between $\langle A, \rho\rangle$ and $\langle B, \rho'\rangle$ if and only if there exists a fuzzy closure system C that is compatible with μ and, in case* $\mathrm{rng}(\mu) \neq B$*, there exists a crisp function $\gamma\colon B \smallsetminus \mathrm{rng}(\mu) \to A$ that satisfies condition (15).*

Note that the theorem above states an existence condition but, indeed, the proof has been constructive and leads to a procedure to actually build the right adjoint which is illustrated in the following examples.

Example 3. Consider the Heyting chain $\mathbb{H} = \langle H, \leq\rangle$ with $H = \{\frac{i}{10} \mid 0 \leq i \leq 10, i \in \mathbb{N}\}$, the sets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$, and the fuzzy relations $\rho\colon A \times A \to H$ and $\mu\colon A \times B \to H$ defined below:

ρ	a_1	a_2	a_3
a_1	1	1	0.8
a_2	1	1	0.8
a_3	0.8	0.8	1

μ	b_1	b_2	b_3	b_4
a_1	0	0.8	0.8	1
a_2	1	0	0.2	0.5
a_3	0.5	0.4	1	0.3

Consider also the fuzzy set $C = \{(a_1, 0.5), (a_2, 1), (a_3, 1)\}$. It is easy to check that C is a fuzzy closure system that is compatible with μ. Moreover, $\gamma\colon B \smallsetminus \mathrm{rng}(\mu) \to A$ given by $\gamma(b_2) = a_1$ satisfies condition (15). Theorem 4 then ensures that there exist $\rho'\colon B \times B \to H$ and $\nu\colon B \times A \to H$ such that (μ, ν) is a fuzzy relational Galois connection between $\langle A, \rho\rangle$ and $\langle B, \rho'\rangle$. One possible construction, according to Proposition 4, is given by

ρ'	b_1	b_2	b_3	b_4
b_1	1	1	0.8	1
b_2	1	1	0.8	1
b_3	0.8	0.8	1	0.8
b_4	1	1	0.8	1

ν	a_1	a_2	a_3
b_1	0.5	1	0.8
b_2	0.5	1	0.8
b_3	0.5	0.8	1
b_4	0.5	1	0.8

Example 4. Consider Belnap's diamond as the underlying algebra of truth values $\mathfrak{B} = \{\bot, t, f, \top\}$, the sets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$, and the fuzzy relations $\rho\colon A \times A \to \mathfrak{B}$ and $\mu\colon A \times B \to \mathfrak{B}$ defined below:

ρ	a_1	a_2	a_3
a_1	$\top$	f	f
a_2	f	f	$\top$
a_3	f	f	$\top$

μ	b_1	b_2	b_3	b_4
a_1	f	$\bot$	$\top$	t
a_2	$\top$	$\top$	$\bot$	$\bot$
a_3	t	$\top$	f	$\bot$

Consider also the fuzzy set $C = \{(a_1, \top), (a_2, f), (a_3, \top)\}$. It is easy to check that C is a fuzzy closure system that is compatible with μ. Moreover, $\gamma\colon B \smallsetminus \mathrm{rng}(\mu) \to A$ given by $\gamma(b_4) = a_1$ satisfies condition (15). Theorem 4 then ensures that there exist $\rho'\colon B \times B \to \mathfrak{B}$ and $\nu\colon B \times A \to \mathfrak{B}$ such that (μ, ν) is a fuzzy relational Galois connection between $\langle A, \rho\rangle$ and $\langle B, \rho'\rangle$. One possible construction is given by

ρ'	b_1	b_2	b_3	b_4
b_1	$\top$	$\top$	f	f
b_2	$\top$	$\top$	f	f
b_3	f	f	$\top$	$\top$
b_4	f	f	$\top$	$\top$

ν	a_1	a_2	a_3
b_1	f	$\bot$	$\top$
b_2	f	$\bot$	$\top$
b_3	$\top$	$\bot$	f
b_4	$\top$	$\bot$	f

Example 5. Consider the Heyting chain $\mathbb{H} = \langle H, \leq\rangle$ with $H = \{0, 0.5, 1\}$, the sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$, and the fuzzy relations $\rho\colon A \times A \to H$ and $\mu\colon A \times B \to H$ defined below:

ρ	a_1	a_2
a_1	1	0
a_2	0.5	1

μ	b_1	b_2	b_3
a_1	0.5	0.5	1
a_2	0.5	1	0.5

It is a matter of calculation to check that there is no fuzzy closure system that is compatible with μ and no mapping $\gamma: B \smallsetminus \mathrm{rng}(\mu) \to A$ that satisfies condition (15). Theorem 4 then ensures that there exists no transitive fuzzy relation $\rho'\colon B \times B \to H$ and no fuzzy relation $\nu\colon B \times A \to H$ such that (μ, ν) is a fuzzy relational Galois connection between $\langle A, \rho\rangle$ and $\langle B, \rho'\rangle$.

Example 6. Consider Belnap's diamond as the underlying algebra of truth values $\mathfrak{B} = \{\bot, t, f, \top\}$, the sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$, and the fuzzy relations $\rho\colon A \times A \to \mathfrak{B}$ and $\mu\colon A \times B \to \mathfrak{B}$ defined below:

ρ	a_1	a_2
a_1	$\top$	$\bot$
a_2	t	$\top$

μ	b_1	b_2	b_3
a_1	t	f	$\top$
a_2	t	$\top$	f

It is a matter of calculation to check that there is no fuzzy closure system that is compatible with μ and no mapping $\gamma: B \smallsetminus \mathrm{rng}(\mu) \to A$ that satisfies condition (15). Theorem 4 then ensures that there exists no transitive fuzzy relation $\rho'\colon B \times B \to \mathfrak{B}$ and no fuzzy relation $\nu\colon B \times A \to \mathfrak{B}$ such that (μ, ν) is a fuzzy relational Galois connection between $\langle A, \rho\rangle$ and $\langle B, \rho'\rangle$.

6. Conclusions and future work

In this paper, we have introduced the notion of fuzzy relational Galois connection between fuzzy T-digraphs in case the underlying algebra of truth values is a complete Heyting algebra. Similarly to [17–19], our definition departs from one of several standard equivalent definitions in the crisp case, namely, the antitonicity of the components of the Galois connection and the inflationarity of their compositions. For this notion of fuzzy relational Galois connection, we have characterised the existence of a right adjoint for a given fuzzy relation between a fuzzy T-digraph and an unstructured set.

Concerning the particular notion of fuzzy relational Galois connection proposed and investigated here, it is worth mentioning that we have studied the minimal properties needed to characterize this notion in terms of a natural Galois condition [25]. These properties turn out to be related to the framework of perfect fuzzy functions [23].

In future work, on the one hand, we will further explore the relationship with perfect fuzzy functions somehow envisaged in our previous works [16]. On the other hand, it is worth to investigate the possibility of dropping the Heyting restriction as general hypothesis for the characterisation of the existence and the construction of the right adjoint. It would be also interesting to explore the problem of existence of a right adjoint from the point of view of the associated fuzzy topology. Some recent work about the relationship between fuzzy Galois connections and topology can be found in [4].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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