




# Two-dimensional perfect evolution algebras over domains

Yolanda Cabrera Casado<sup>1</sup> · Dolores Martín Barquero<sup>2</sup> ·  
Cándido Martín González<sup>3</sup> 

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## Abstract

We will study evolution algebras  $A$  that are free modules of dimension *two* over domains. We start by making some general considerations about algebras over domains: They are sandwiched between a certain essential  $D$ -submodule and its scalar extension over the field of fractions of the domain. We introduce the notion of quasiperfect algebras and we characterize the perfect and quasiperfect evolution algebras in terms of the determinant of its structure matrix. We classify the two-dimensional perfect evolution algebras over domains parametrizing the isomorphism classes by a convenient moduli set.

**Keyword** Perfect evolution algebras, Isomorphism class, Moduli set, Domain, Graph

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✉ Cándido Martín González  
candido\_m@uma.es

Yolanda Cabrera Casado  
yolandacc@uma.es

Dolores Martín Barquero  
dmartin@uma.es

- <sup>1</sup> Departamento de Matemática Aplicada, Escuela de Ingenierías Industriales, Universidad de Málaga, Campus de Teatinos, 29071 Malaga, Spain
- <sup>2</sup> Departamento de Matemática Aplicada, Escuela de Ingenierías Industriales, Universidad de Málaga, Campus de Teatinos, 29071 Malaga, Spain
- <sup>3</sup> Departamento de Álgebra, Geometría y Topología, Facultad de ciencias, Universidad de Málaga, Campus de Teatinos, 29010 Malaga, Spain

## 1 Introduction

There is a large number of publications studying two-dimensional evolution algebras (see [1, 2, 4–6, 8, 9], etc.). Among them, we could highlight those works that deal with their classification. So, in the works [6, 8], the evolution algebras of dimension two over the reals and complex numbers are, respectively, classified. The paper [5] contains the classification of the evolution algebras of dimension two and three over the real and complex field. The work [1] is addressed to the case in which the ground field  $\mathbb{K}$  is algebraically closed. In [2], the classification of the evolution algebras of dimension two over arbitrary fields is provided. In [3], we can find more information about the evolution of the research in the field of evolution algebras defined in [10]. Our contribution in this work is the study of two-dimensional perfect evolution algebras over domains.

In this paper, the word domain will stand for an integral domain, i.e., a commutative ring such that  $xy = 0$  implies  $x = 0$  or  $y = 0$ . If  $D$  is a domain, an *evolution algebra* over  $D$  is a  $D$ -algebra that is free as  $D$ -module and has a basis  $B = \{e_i\}_{i \in I}$  such that  $e_i e_j = 0$  for any  $i \neq j$ . Such basis  $B$  is called a *natural basis*. Of course, any one-dimensional  $D$ -algebra is an evolution algebra. Evolution algebras over domains are much more complex than evolution algebras over fields: As we will see, there may be an infinite family of non-isomorphic one-dimensional evolution algebras over certain domains (while in the case of fields, we only have two isomorphism classes). One of the elements of this study is the use of moduli sets from certain classes of algebras: The idea is to parametrize the algebras of a class by tuples of parameters ranging in a given space. It turns out that in some cases, the tuples range in curves or surfaces or other varieties. The different algebras in the same isomorphism class may happen to be in a curve of an affine plane and the different curves fill the space modulo the restrictions on the parameters imposed by the class of algebras. This may be seen as a bundle in the category of sets.

This paper is organized as follows. In Sect. 2, we prove some results on algebras over domains in Lemma 1 and Proposition 1. We introduce the class of quasiperfect algebras. We give necessary and sufficient conditions for an evolution algebra over a domain to be perfect or quasiperfect in terms of its structure matrix and we prove in both cases that the natural basis is unique up to permutations or multiplication by invertible scalars (in Proposition 2). Furthermore, we associate a colored directed graph to a quasiperfect evolution  $D$ -algebra and we prove that this graphical colored representation is unique in Lemma 2. Next, in Sect. 3, we define the required terminology to be able to classify our  $D$ -algebras, so we introduce some definitions and tools for the classification task like moduli sets, direct limits, etc. Finally, in Sect. 4, we give the classification of the two-dimensional perfect evolution algebras over domains (Theorem 1) parametrizing the isomorphism classes by a corresponding moduli set.

## 2 Preliminaries and previous results

As previously mentioned, the classification of evolution algebras over domains is much richer than the classification over fields. For instance, there are two one-dimensional

evolution algebras over fields up to isomorphism: the ground field with zero multiplication and the ground field with its usual multiplication. But, if you consider a domain  $D$  and a one-dimensional evolution  $D$ -algebra, there are more isomorphism classes of evolution algebras. Ruling out the trivial one, we have defined in  $D$  a multiplication with  $1^2 = d$ , where  $d \in D^* := D \setminus \{0\}$ . Denote this algebra by  $D_d$ . The product in  $D_d$  is  $x \cdot y = xyd$  for any  $x, y \in D$ . Now, let us define  $D_e$  similarly. If  $f : D_d \rightarrow D_e$  is an isomorphism, it is a  $D$ -module isomorphism hence  $f(a) = af(1)$  for any  $a$ . So,  $f$  is the multiplication times an element  $x := f(1)$ . Consequently,  $x \in D^\times$  (the group of invertibles of  $D$ ). But also  $f(uvd) = f(u)f(v)e$  whence  $f(d) = f(1)^2e$  or  $d = xe$ . Thus, the isomorphic condition is  $D_d \cong D_e$  if and only if there exists  $x \in D^\times$  such that  $d = xe$ . So, the isomorphism classes of nontrivial one-dimensional evolution algebras over domains with nonzero product are in one-to-one correspondence with the set  $D^*/D^\times$ , that is, the set of equivalence classes of  $D^*$  modulo the action  $D^\times \times D^* \rightarrow D^*$  such that  $x \cdot d = xd$  for any  $x \in D^\times, d \in D^*$ . In particular, if  $D = \mathbb{Z}$ , we have  $\mathbb{Z}^*/\{\pm 1\} \cong \mathbb{N}^* = \{1, 2, \dots\}$ . Thus, there are countably many isomorphism classes of one-dimensional evolution algebras over certain domains. If we consider the domain  $D := \mathbb{K}[x]$  of polynomials in one indeterminate over the field  $\mathbb{K}$ , then the isomorphism classes of one-dimensional evolution  $D$ -algebras are in one-to-one correspondence with the set of monic polynomials of  $\mathbb{K}[x]$ .

Next, we introduce some definitions and properties that we will need later. If  $D$  is a (commutative) domain, we will denote by  $Q := Q(D)$  the field of fractions of  $D$ . For any  $D$ -module  $M$ , we will construct  $M_Q$  the  $Q$ -module of fractions  $M_Q := \{\frac{m}{d} : m \in M, d \in D \setminus \{0\}\}$  with the usual operations of sum, product and product by elements of  $Q$ . To be more specific, we consider the set of couples  $(m, d)$  with  $m \in M, 0 \neq d \in D$  modulo the equivalence relation

$$(m, d) \equiv (m', d') \text{ iff } \exists t \in D \setminus \{0\}, t(d'm - dm') = 0.$$

Then, we denote the equivalence class of  $(m, d)$  in the usual way:  $\frac{m}{d}$  and  $M_Q$  are a  $Q$ -vector space relative to the usual sum of fractions and  $\frac{d_1}{d_2} \frac{m}{d_3} := \frac{d_1 m}{d_2 d_3}$ . If  $M$  is a torsion-free  $D$ -module, there is a canonical monomorphism of  $D$ -modules  $M \rightarrow M_Q$  such that  $m \mapsto \frac{m}{1}$ . Usually, we will denote  $\frac{m}{1} := m$ , so that the elements of  $M$  and their images in  $M_Q$  will be identified. Unless otherwise stated, we will work throughout this paper with torsion-free  $D$ -modules.

A well-known property of  $M$  is that a set of vectors  $\{e_i\} \subset M$  is linearly independent if and only if its image in  $M_Q$  is  $D$ -linearly independent. As well, if  $\{u_i\}_{i \in I}$  is linearly independent in the  $Q$ -vector space  $M_Q$  and  $u_i = \frac{m_i}{d_i}$  for any  $i \in I$ , then the set of numerators  $\{m_i\}_{i \in I}$  is linearly independent in  $M$ . We also have

**Lemma 1** *Let  $D$  be a domain,  $Q$  its field of fractions,  $M$  a torsion-free  $D$ -module and  $M_Q$  the  $Q$ -module of fractions of  $M$ .*

- (a) *If  $\{u_i\}_{i \in I}$  is a basis of  $M_Q$  (as a  $Q$ -vector space) and  $u_i = \frac{m_i}{d_i}$  for  $i \in I$ , then  $\{m_i\}_{i \in I}$  is also a basis of  $M_Q$  and a maximal linearly independent subset of  $M$ .*
- (b) *A set  $\{m_i\}_{i \in I} \subset M$  is a maximal linearly independent subset of  $M$  if and only if  $\{m_i\}_{i \in I}$  is a basis of the  $Q$ -vector space  $M_Q$ .*

- (c) If  $M$  is a  $D$ -module and  $\{m_i\}_{i \in I}$  a maximal linearly independent subset of  $M$ , then  $\oplus_i Dm_i$  is an essential submodule of  $M$ .
- (d) Assume that  $M$  is a free  $D$ -module with a finite basis  $\{e_i\}_{i \in I}$ . A maximal linearly independent subset  $\{u_i\}_{i \in I}$  of  $M$  is a basis of  $M$  if and only if the determinant of the change of basis matrix is in  $D^\times$ .

**Proof** For the first assertion, observe that the set of the  $m_i$ 's is linearly independent also in  $M_Q$ . Indeed, if we have  $\sum \lambda_i m_i = 0$ , then we can write  $\sum \lambda_i d_i \frac{m_i}{d_i} = \sum \lambda_i d_i u_i = 0$ , hence, for any  $i$ , we have  $\lambda_i d_i = 0$ , which implies  $\lambda_i = 0$ . The set  $\{m_i\}_{i \in I}$  is also a system of generators of the  $Q$ -vector space  $M_Q$  because any  $x \in M_Q$  can be written as

$$x = \sum \lambda_i u_i = \sum \lambda_i \frac{m_i}{d_i} = \sum \lambda_i \frac{1}{d_i} m_i.$$

Let us prove now that  $\{m_i\}_{i \in I}$  is maximal among the linearly independent subsets of  $M$ : If  $\{m_i\}_{i \in I} \subsetneq T$ , for a linearly independent subset  $T \subset M$ , then  $T$  contains properly a basis of  $M_Q$  and is linearly independent, a contradiction.

For the second assertion, take a maximal linearly independent set  $\{m_i\}_{i \in I} \subset M$ . We know that  $\{m_i\}_{i \in I}$  is linearly independent also in  $M_Q$ . If this set is not a basis, there is some  $x \in M_Q$  such that  $\{m_i\}_{i \in I} \cup \{x\}$  is again linearly independent. If  $x = \frac{z}{d}$ , then  $\{m_i\}_{i \in I} \cup \{z\}$  is a linearly independent subset of  $M$ , contradicting the maximality of  $\{m_i\}_{i \in I}$ . Reciprocally, if  $\{m_i\}_{i \in I} \subset M$  is a basis of  $M_Q$ , we know that  $\{m_i\}_{i \in I}$  is linearly independent in  $M$ . To prove the third assertion, we take a nonzero submodule  $N$  of  $M$ . We must prove that  $N \cap (\oplus_i Dm_i) \neq 0$ . Take  $0 \neq n \in N$ . Since  $\{m_i\}_{i \in I}$  is a basis of  $M_Q$ , we have  $dn = \sum_i d_i m_i$  for some  $d, d_i \in D$  (and  $d \neq 0$ ). Thus,  $0 \neq dn \in N \cap (\oplus_i Dm_i)$ . Finally, we prove the fourth assertion. We have  $u_i = a_i^j e_j$  (using Einstein summation convention) for any  $i \in I$ . If  $\det[(a_i^j)] \in D^\times$ , then there are scalars  $b_i^j \in D$  such that  $e_i = b_i^j e_j$  for any  $i$ . Hence,  $\{u_i\}_{i \in I}$  is a basis of  $M$ . Reciprocally, if  $\{u_i\}_{i \in I}$  turns out to be a basis, we may write  $e_i = b_i^j e_j$  for suitable scalars  $b_i^j \in D$ . But, then the matrices  $(a_i^j)$  and  $(b_i^j)$  have product 1, that is,  $(a_i^j)(b_i^j) = 1$ . This implies that the determinant of each such matrix is an invertible element of  $D$ . □

**Proposition 1** *Let  $A$  be a  $D$ -algebra, then there is a maximal linearly independent subset  $\{a_i\}$  of  $A$  (in fact a basis of the  $Q$ -vector space  $A_Q$ ) such that  $A$  is contained as  $D$ -module in a sandwich*

$$\oplus D a_i \subset A \subset A_Q,$$

and  $\oplus D a_i$  is an essential  $D$ -module of  $A$ .

**Proof** Take a basis  $\{u_i\}$  of  $A_Q$  as a  $Q$ -vector space. If  $u_i = a_i/d_i$ , then  $\{a_i\}$  is a maximal linearly independent subset of  $A$  by Lemma 1(a). Also,  $\oplus D a_i$  is essential as a  $D$ -submodule of  $A$  by Lemma 1(c). Note that,  $A$  is a torsion-free  $D$ -module: If  $da = 0$  for some nonzero  $d \in D$ , write  $a = \sum_i q_i u_i$  as a linear combination of the

$u_i$ 's. Then, we have  $0 = d \sum_i q_i u_i$  whence  $dq_i = 0$  for any  $i$ . Since  $d \neq 0$ , we have  $q_i = 0$ . □

In the situation above,  $A \subset A_Q$ , if  $A$  is perfect ( $A^2 = A$ ), then  $A_Q$  is also perfect. However, we may have  $A_Q$  perfect and  $A$  not. For instance, consider  $A = 2\mathbb{Z} \times \mathbb{Z}$  with componentwise multiplication. Then,  $A_{\mathbb{Q}} \cong \mathbb{Q} \times \mathbb{Q}$ , which is perfect but  $A$  is not.

An example of the situation described in Proposition 1 is given by taking  $D = \mathbb{Z}$  and  $A = \{(\frac{x}{2}, y) : x, y \in \mathbb{Z}\} = \mathbb{Z}(\frac{1}{2}, 0) \oplus \mathbb{Z}(0, 1)$ . Then,  $A$  is a two-dimensional free  $\mathbb{Z}$ -module and  $A_{\mathbb{Q}} = \mathbb{Q}(1, 0) \oplus \mathbb{Q}(0, 1)$  being

$$\mathbb{Z}(1, 0) \oplus \mathbb{Z}(0, 1) \subsetneq A \subsetneq \mathbb{Q}(1, 0) \oplus \mathbb{Q}(0, 1).$$

**Definition 1** An algebra  $A$  over a domain  $D$  will be termed *quasiperfect* if  $A_Q^2 = A_Q$ .

We will need the following proposition for the classification task of the two-dimensional perfect evolution algebras over domains, whose first item is exactly the same as in the case of evolution algebras over fields:

**Proposition 2** Assume that  $\mathcal{E}$  is an evolution algebra over a domain  $D$  with a finite natural basis  $\{e_i\}_{i \in I}$ . Let  $\omega_i^j \in D$  be the structure constants, that is,  $e_i^2 = \omega_i^j e_j$  (using Einstein summation convention). Then, we have

- (1)  $\mathcal{E}^2 = \mathcal{E}$  if and only if the matrix  $(\omega_i^j)$  is invertible. Moreover, for any other natural basis  $\{f_i\}_{i \in I}$ , there is a permutation  $\sigma$  of  $I$  such that  $f_i = k_i e_{\sigma(i)}$  and each  $k_i \in D^\times$ .
- (2)  $\mathcal{E}$  is quasiperfect if and only if the determinant of  $(\omega_i^j)$  is nonzero. As in the previous case, for any other natural basis  $\{f_i\}_{i \in I}$ , there is a permutation  $\sigma$  of  $I$  such that  $f_i = k_i e_{\sigma(i)}$  and each  $k_i \in D^\times$ .

**Proof** From  $\mathcal{E} = \mathcal{E}^2$ , we deduce that  $e_i = x_i^j e_j^2$  for any  $i \in I$ . Then,  $e_i = x_i^j \omega_j^k e_k$  so that  $x_i^j \omega_j^k = \delta_i^k$  (Kronecker Delta). Thus, the matrix  $(\omega_i^j)$  is invertible. Reciprocally, if  $(\omega_i^j)$  is invertible, the linear map such that  $e_i \mapsto e_i^2$  is an isomorphism, whence  $\mathcal{E}^2 = \mathcal{E}$ . Assuming the perfectness of  $\mathcal{E}$ , if  $\{f_i\}$  is another natural basis and we write  $f_i = a_i^j e_j$ , then for  $i \neq j$ , we have  $0 = f_i f_j = a_i^k e_k a_j^q e_q = a_i^k a_j^q e_k e_q = a_i^k a_j^q \delta_{kq} \omega_k^s e_s = a_i^k a_j^q \omega_k^s e_s$ , whence  $a_i^k a_j^q \omega_k^s = 0$  for any  $s$  and any couple  $(i, j)$  with  $i \neq j$ . Since the matrix  $(\omega_i^j)$  is invertible, we consider its inverse matrix  $(\tilde{\omega}_i^j)$ , so we have  $\omega_i^j \tilde{\omega}_j^k = \delta_i^k$ . Then, from  $a_i^k a_j^q \omega_k^s = 0$ , we get  $a_i^k a_j^q \omega_k^s \tilde{\omega}_s^q = 0$  or  $a_i^k a_j^q \delta_k^q = 0$ . Thus,  $a_i^q a_j^q = 0$  for any  $q$  provided  $i \neq j$ . So, in each column and each row of the matrix  $(a_i^j)$ , there is a unique nonzero element. Consequently,  $f_i = k_i e_{\sigma(i)}$  for a certain permutation  $\sigma$  of  $I$ . Now, the coefficients  $k_i$  are invertible in  $D$  since the determinant of the matrix of basis change is invertible. Let us prove now the second assertion. If we have  $\mathcal{E}_Q^2 = \mathcal{E}_Q$ , the matrix  $(\omega_i^j)$  is invertible in  $Q$ . Its determinant is a nonzero element of  $D$ . Reciprocally, if  $\det(\omega_i^j) \neq 0$ , then it is invertible in  $Q$  so that  $\mathcal{E}$  is quasiperfect. In this case, if  $\{e_i\}$  and  $\{f_i\}$  are natural bases of  $\mathcal{E}$ , then there is a

permutation  $\sigma$  and nonzero elements  $k_i \in Q$  such that  $f_i = k_i e_{\sigma(i)}$  for any  $i$ . Also,  $f_i = a_i^j e_j$  for certain  $a_i^j \in D$ , hence  $a_i^{\sigma(i)} = k_i$  and  $a_i^j = 0$  if  $j \neq \sigma(i)$ . In any case  $k_i \in D^\times$ .  $\square$

**Corollary 1** *Any perfect evolution algebra over a domain is quasiperfect.*

We prove, in the next lemma, a result that generalizes the situation of perfect evolution algebras over fields. This lemma is the heart of the classification of the perfect algebras over domains in the sense that we can associate a graphical representation of a given evolution algebra. This graphical representation is similar to the one introduced in [7] with a fundamental variation: Edges representing invertible coefficients are drawn in black, while those representing nonzero and noninvertible coefficients are dotted drawn.

**Lemma 2** *Assume that  $\mathcal{E}$  is a quasiperfect evolution algebra over a domain  $D$  with a finite natural basis  $\{e_i\}_{i \in I}$ . Then, the following numbers do not depend on the natural basis chosen:*

- (1) *The number of nonzero entries in the structure matrix  $(\omega_i^j)$ .*
- (2) *The number of nonzero entries in the diagonal of  $(\omega_i^j)$ .*
- (3) *The number of invertible elements in  $(\omega_i^j)$ .*
- (4) *The number of invertible elements in the diagonal of  $(\omega_i^j)$ .*

**Proof** For our original natural basis, we have  $e_i^2 = \omega_i^j e_j$ . Take now any other natural basis  $\{f_i\}$ . There is a permutation  $\sigma$  of  $I$  such that  $f_i = k_i e_{\sigma(i)}$  for some invertible elements  $k_i \in D^\times$  by item (2), Proposition 2. It follows that  $f_i^2 = \tau_i^q f_q$ , where

$$\tau_i^q = \frac{k_i^2}{k_q} \omega_{\sigma(i)}^{\sigma(q)}.$$

So, the number of nonzero (respectively invertible) elements in the matrix  $(\omega_i^j)$  coincides with the number of nonzero (respectively, invertible) elements in the matrix  $(\tau_i^j)$ , similarly for the diagonal elements.  $\square$

What the lemma tells is that this graphical colored representation is unique in the quasiperfect case.

**Proposition 3** *Let  $D$  be a domain and  $Q$  its field of fractions. If  $\mathcal{E}$  is a quasiperfect evolution algebra over  $D$  of finite dimension and  $\{m_i\}$  is maximal linearly independent subset of  $\mathcal{E}$  such that  $m_i m_j = 0$  for  $i \neq j$ , then  $\{m_i\}$  is a natural basis of  $\mathcal{E}$ .*

**Proof** Fix a natural basis  $\{e_i\}$  of  $\mathcal{E}$ . We know that  $\{m_i\}$  is a basis of  $\mathcal{E}_Q$  by Lemma 1 and it is a natural basis. Then,  $m_i = k_i e_{\sigma(i)}$  for some  $k_i \in Q^\times$  and a permutation  $\sigma$ . On the other hand, we can write  $m_i = a_i^j e_j$ , where  $a_i^j \in D$ . Consequently,  $a_i^{\sigma(i)} = k_i$  so that, each  $k_i \in D^\times$ . Furthermore,  $a_i^j = 0$  if  $j \neq \sigma(i)$ . Thus,  $\{m_i\}$  is a natural basis of  $\mathcal{E}$ .  $\square$

### 3 The moduli set

In this section, we introduce some terminology in order to present what we will call the moduli set for the different classes of algebras. Given a class  $C$  of algebras, we will say that a set  $S$  is a *moduli* for  $C$  (or a *moduli set*) if there is a one-to-one correspondence between the isomorphism classes of algebras of  $C$  and the elements of  $S$ . In some occasions, the moduli of a class will have an additional algebraic or geometric structure. In the category of sets, we recall that a *bundle* is an epimorphism  $\pi : E \rightarrow B$ , where  $E$  is called *total set*,  $B$  is the *base set* and for every  $b \in B$ ,  $\pi^{-1}(b)$  is the *fiber* over  $b$ . By a *cross section* of  $\pi$ , we understand a right inverse  $s : B \rightarrow E$ . We have an example of bundle if we consider as the total set a family of algebras  $C$  and as the base set the set  $C / \cong$  of isomorphism classes of the algebras in  $C$ . Then,  $\pi$  maps any algebra to its isomorphism class. The fiber of an element represents an isomorphism class of algebras. If we specify a cross section of the bundle, then this is equivalent to give a representative of the isomorphism class of any element of  $C$ . Thus, the classification problem of  $C$  under isomorphism consists just in giving a cross section of the corresponding bundle. The moduli set of the classification is the base set of the corresponding bundle.

#### 3.1 Direct limits

Some of the cases of our classification are based on direct limits. If  $M$  is an abelian monoid and  $n \in \mathbb{Z}$ , we consider the direct system

$$M \xrightarrow{(\cdot)^n} M \xrightarrow{(\cdot)^n} \dots \rightarrow M \xrightarrow{(\cdot)^n} \dots$$

where  $(\cdot)^n$  is the homomorphism such that  $g \mapsto g^n$ , then we will denote the direct limit of such system by  $\lim_{\rightarrow n} M$ . Recall that this monoid can be described as follows: Consider the sequence of monoids  $\{M_i\}_{i \in \mathbb{N}}$  such that  $M_i := M$  for any  $i$ . Then, in the disjoint union  $\sqcup_i M_i$ , we define an equivalence relation: If  $x \in M_i$  and  $y \in M_j$ , we say that  $x \sim y$  if and only if  $y^{n^k} = x^{n^h}$  for some naturals  $k, h$ . We denote the equivalence class of  $g \in M$  by  $[g]$ . So,  $\lim_{\rightarrow n} M$  is the quotient of  $\sqcup_i M_i$  modulo  $\sim$ . A particular case of this arises if we take  $M := D^\times / (D^\times)^{[q]}$ , where  $(D^\times)^{[q]} := \{x^q : x \in D^\times\}$  with  $q \in \mathbb{N}^*$ . This specific  $M$  is a group and its elements are equivalence classes  $\bar{\lambda}$  with  $\lambda \in D^\times$ . We have  $\bar{\lambda} = \bar{\mu}$  if and only if  $\lambda = \mu r^q$  for some  $r \in D^\times$ . We can consider  $\lim_{\rightarrow n} M$  whose elements are the equivalence classes  $[\bar{\lambda}]$  with  $\bar{\lambda} \in M$ .

We have a canonical group homomorphism  $D^\times \rightarrow \lim_{\rightarrow n} M$  such that  $\lambda \rightarrow [\bar{\lambda}]$ . Two elements  $\lambda, \mu \in D^\times$  are said to have the same image in  $\lim_{\rightarrow n} M$  if  $[\bar{\lambda}] = [\bar{\mu}]$ . For instance, consider the group  $\lim_{\rightarrow 2} D^\times / (D^\times)^{[3]}$ . Since 2 and 3 are coprime, the transition homomorphisms  $(\cdot)^2$  are isomorphisms. Then,  $\lambda$  and  $\mu$  have the same image in  $\lim_{\rightarrow 2} D^\times / (D^\times)^{[3]}$  if  $\bar{\lambda}^{2^\alpha} = \bar{\mu}^{2^\beta}$  for some naturals  $\alpha, \beta \geq 0$ . If  $\alpha = \beta$ , then  $\bar{\lambda} = \bar{\mu}$ . If  $\alpha > \beta$ , then  $\bar{\mu} = (\bar{\lambda})^{2^{\alpha-\beta}}$  and if  $\alpha < \beta$ , we have  $\bar{\lambda} = (\bar{\mu})^{2^{\beta-\alpha}}$ . In each case, replacing  $2^{\alpha-\beta}$  or  $2^{\beta-\alpha}$  with its remainder  $\xi$  modulo 3, we have  $\bar{\mu} = \bar{\lambda}^\xi$  or  $\bar{\lambda} = \bar{\mu}^\xi$ , where  $\xi = 1, 2$ . So, there is some  $r \in D^\times$  such that  $\mu = \lambda r^3$  or  $\mu = \lambda^2 r^3$  (observe that the

other possibility  $\lambda = \mu^2 s^3$  is a consequence of  $\mu = \lambda^2 r^3$ ). Summarizing, we have the following lemma.

**Lemma 3** *The elements  $\lambda, \mu \in D^\times$  have the same image in the group  $\lim_{\rightarrow 2} D^\times / (D^\times)^{[3]}$  if and only if  $\mu = \lambda r^3$  or  $\mu = \lambda^2 r^3$  for some  $r \in D^\times$ .*

In this case, we have a bundle  $\pi: D^\times \rightarrow \lim_{\rightarrow 2} D^\times / (D^\times)^{[3]}$  such that  $\lambda \mapsto [\bar{\lambda}]$ . The fiber of some  $[\bar{\lambda}]$  is the set of all  $\mu \in D^\times$  such that  $\mu = \lambda r^3$  or  $\mu = \lambda^2 r^3$  for some  $r \in D^\times$ .

### 3.2 Algebraic sets

Assume that we have an action  $D^\times \times X \rightarrow X$ , where  $X$  is some subset of  $D^n$ . So, for  $t \in D^\times$  and  $a = (a_1, \dots, a_n) \in X$  we might have  $t \cdot (a_1, \dots, a_n) = (x_1, \dots, x_n)$ , where each  $x_i$  is a polynomial in  $t$  with coefficients in  $D$ . Consequently, we may write  $x_i = p_i(t)$ , where  $p_i$  is the mentioned polynomial. The orbit of  $a$  under the above action is in a curve  $x_i = p_i(t)$ . More precisely, the orbit is contained in the image of the map  $c: Q \rightarrow Q^n$  such that  $t \mapsto (p_1(t), \dots, p_n(t))$ , where  $Q$  is the field of fractions of  $D$ . Then, the Zariski closure of the image of  $c$  is an algebraic set  $V \subset Q^n$  and the orbit of  $a \in X$  is just  $V \cap X$ . This setting will appear in our classification of evolution algebras.

## 4 The perfect case

In this section, we give the classification of two-dimensional perfect evolution algebras over domains. For this task, we analyze the corresponding moduli sets. Consider a two-dimensional evolution algebra  $\mathcal{E}$  over the domain  $D$  with a natural basis  $\{e_1, e_2\}$  and assume that  $\mathcal{E}$  is perfect. We study the different cases.

### 4.1 Case $e_1^2 = \alpha e_1, e_2^2 = \beta e_2$

Then,  $\alpha, \beta \in D^\times$ . We can define  $f_1 = \alpha^{-1}e_1$  and  $f_2 = \beta^{-1}e_2$ . So, we have  $f_1^2 = \alpha^{-2}\alpha e_1 = f_1$  and  $f_2^2 = \beta^{-2}\beta e_2 = f_2$ . All the algebras in this case are isomorphic to  $D \times D$  with componentwise operations. Then, up to isomorphism, there is only one algebra of this type.

### 4.2 Case $e_1^2 = \alpha e_2, e_2^2 = \beta e_1$

Again  $\alpha, \beta \in D^\times$ . We can define  $f_1 = e_1$  and  $f_2 = \alpha e_2$ . So, we get  $f_1^2 = \alpha e_2 = f_2$  and  $f_2^2 = \alpha^2 \beta f_1$ . Thus, we have a one-parametric family of algebras given by the multiplication table  $e_1^2 = e_2$  and  $e_2^2 = \alpha e_1$ , where  $\alpha \in D^\times$ . Denote the above algebra by  $A_{2,\alpha}$ . We analyze the isomorphism question  $A_{2,\alpha} \cong A_{2,\beta}$ . If the isomorphism exists we have bases  $\{e_1, e_2\}$  and  $\{u_1, u_2\}$  of  $A_{2,\alpha}$  such that  $e_1^2 = e_2, e_2^2 = \alpha e_1$  and  $u_1^2 = u_2, u_2^2 = \beta u_1$ . Then, we have two possibilities:



- (1)  $u_i = k_i e_i, k_i \in D^\times$  for  $i = 1, 2$ . This gives  $\beta\alpha^{-1} = k_1^3$  and  $k_2 = k_1^2$ . So  $\beta\alpha^{-1} \in (D^\times)^3$ .
- (2)  $u_1 = ke_2$  and  $u_2 = he_1$  for some  $k, h \in D^\times$ . In this case  $k^3 = \beta\alpha^{-2}, h = k^2\alpha$  and so  $\beta\alpha^{-2} \in (D^\times)^3$ .

Therefore, we have  $A_{2,\alpha} \cong A_{2,\beta}$  if and only if  $\beta\alpha^{-1} \in (D^\times)^3$  or  $\beta\alpha^{-2} \in (D^\times)^3$ . But, I ask the reader to check the implication  $\Leftarrow$ . A moduli set for the class of algebras  $A_{2,\alpha}$  is the group  $\lim_{\rightarrow 2} D^\times / (D^\times)^{[3]}$ .

### 4.3 Case $e_1^2 = \alpha e_1, e_2^2 = \beta e_1 + \delta e_2, \beta \neq 0$

In this case,  $\alpha, \delta \in D^\times$ . We define  $f_1 = \alpha^{-1}e_1$  and  $f_2 = \delta^{-1}e_2$ . Then,  $f_1^2 = f_1$  and  $f_2^2 = \lambda f_1 + f_2$  for a certain  $\lambda \in D^*$ . We get a one-parameter family of algebras  $A_{3,\lambda}$  with  $\lambda \neq 0$  and product  $e_1^2 = e_1, e_2^2 = \lambda e_1 + e_2$ . We investigate the isomorphism  $A_{3,\lambda} \cong A_{3,\mu}$ . As before, we have two natural basis  $\{e_1, e_2\}$  and  $\{u_1, u_2\}$  such that  $e_1^2 = e_1, e_2^2 = \lambda e_1 + e_2$  and  $u_1^2 = u_1, u_2^2 = \mu u_1 + u_2$ . Then,

- (1) If  $u_i = k_i e_i$  for  $i = 1, 2$ , after some computations, we get  $k_2 = k_1 = 1$ , thus  $\lambda = \mu$ .
- (2) If  $u_1 = ke_2$  and  $u_2 = he_1$ , we get an inconsistent system of equations.

Thus,  $A_{3,\lambda} \cong A_{3,\mu}$  if and only if  $\lambda = \mu$ . The moduli set for the class  $A_{3,\lambda}$  is  $D^*$ , which is a monoid.

### 4.4 Case $e_1^2 = \alpha e_2, e_2^2 = \beta e_1 + \delta e_2, \delta \neq 0$

In this case,  $\alpha, \beta \in D^\times$ . We have a family  $A(\alpha, \beta, \delta)$  of algebras with  $\alpha, \beta \in D^\times$  and  $\delta \in D^*$  (and the multiplication above). It is straightforward to prove that there is no possible isomorphism  $A(\alpha, \beta, \delta) \cong A(\alpha', \beta', \delta')$  when  $\delta \in D^\times$  but  $\delta' \notin D^\times$ . So, let us investigate the algebras  $A(\alpha, \beta, \delta)$  with  $\delta \in D^\times$ . We will prove that in this case,  $A(\alpha, \beta, \delta) \cong A(\lambda, 1, 1)$  for a suitable  $\lambda \in D^\times$ . Indeed, defining  $f_1 = \beta\delta^{-2}e_1$  and  $f_2 = \delta^{-1}e_2$ , one gets  $f_1^2 = \lambda f_2$  and  $f_2^2 = f_1 + f_2$ , where  $\lambda = \alpha\beta^2\delta^{-3} \in D^\times$ . Furthermore, it is easy to check that  $A(\lambda, 1, 1) \cong A(\mu, 1, 1)$  if and only if  $\lambda = \mu$ .

Let us investigate now the other class of algebras:  $A(\alpha, \beta, \delta)$  with  $\delta \notin D^\times$ . By making the change of basis  $f_1 = k_1 e_1, f_2 = \alpha k_1^2 e_2$ , we get  $f_1^2 = k_1^2 \alpha e_2 = f_2$  and  $f_2^2 = \alpha^2 \beta k_1^3 f_1 + \delta \alpha k_1^2 f_2$  and  $\alpha^2 \beta k_1^3 \in D^\times$  while  $\delta \alpha k_1^2 \notin D^\times$ . Thus, any  $A(\alpha, \beta, \delta)$  with  $\delta$  not invertible is isomorphic to  $A(1, \lambda, \mu)$ , where  $\lambda \in D^\times$  and  $0 \neq \mu \notin D^\times$ . In addition,  $A(1, \lambda, \mu) \cong A(1, k^3 \lambda, k^2 \mu)$  for any  $k \in D^\times$  ( $\mu \notin D^\times$ ). Moreover,  $A(1, \lambda, \mu) \cong A(1, \lambda', \mu')$  (where  $\mu, \mu' \notin D^\times$ ) if and only if there is a  $k \in D^\times$  such that  $\lambda' = k^3 \lambda$  and  $\mu' = k^2 \mu$ .

Summarizing: The algebras in this case fall into two mutually non-isomorphism classes: those of the form  $A(\lambda, 1, 1)$  with  $\lambda \in D^\times$  and those of the form  $A(1, \lambda, \mu)$  with  $\lambda \in D^\times, 0 \neq \mu \notin D^\times$ . Also,

$$A(\lambda, 1, 1) \cong A(\mu, 1, 1) \text{ iff } \lambda = \mu \tag{1}$$

$$A(1, \lambda, \mu) \cong A(1, \lambda', \mu') \text{ iff } \exists k \in D^\times : \lambda' = k^3 \lambda, \mu' = k^2 \mu. \tag{2}$$

The algebras of the form  $A(1, \lambda, \mu)$  only exist over domains that are not fields. Note that, the condition (2) induces an action  $D^\times \times X_0 \rightarrow X_0$ , where  $X_0 = D^\times \times (D^* \setminus D^\times)$  given by  $k \cdot (\lambda, \mu) = (k^3\lambda, k^2\mu)$ . Note that, the isotropy subgroup of any  $(\lambda, \mu) \in X_0$  is trivial. This implies that the cardinal of each orbit agrees with that of  $D^\times$ .

For a fixed  $(\lambda, \mu) \in X_0$ , we can consider  $c: Q \rightarrow Q^2$  such that  $c(k) = (k^3\lambda, k^2\mu)$ . The Zariski closure of the image of  $c$  is  $V(I)$ , the algebraic set of zeros of the ideal  $I \triangleleft Q[x, y]$  generated by the polynomial  $\mu^3x^2 - \lambda^2y^3$ . So, it is a curve  $c_{\lambda, \mu}$  of  $Q^2$ . Thus, the cardinal of the orbit of  $(\lambda, \mu)$  agrees with that of the set of points of  $c_{\lambda, \mu}$  lying on  $X_0$ . Any point of  $X_0$  is in some curve  $c_{\lambda, \mu}$ , in fact,  $(\lambda, \mu) \in c_{\lambda, \mu}$ . Denote by  $c_{\lambda, \mu}^*$  the section  $c_{\lambda, \mu}^* := c_{\lambda, \mu} \cap X_0$ . Then, if  $(\lambda', \mu') \notin c_{\lambda, \mu}$  we have  $c_{\lambda, \mu}^* \cap c_{\lambda', \mu'}^* = \emptyset$ . So,  $X_0$  is the disjoint union of all sections  $c_{\lambda, \mu}^*$  and we have a bundle (in the category of sets)  $p: X_0 \rightarrow X_0/D^\times$ , in which  $p(\lambda, \mu) = \text{orb}(\lambda, \mu)$  can be identified with  $c_{\lambda, \mu}^*$ . The fibers of this bundle are the points in one specific curve, so the fibers represent classes of isomorphic algebras.

**Lemma 4** *The cardinal of each orbit of  $X_0/D^\times$  is  $|D^\times|$  and agrees with that of the set of points of the curve  $c_{\lambda, \mu} \equiv \mu^3x^2 - \lambda^2y^3 = 0$  in  $D^\times \times (D^* \setminus D^\times)$ . The orbit set  $X_0/D^\times$  is the base space of a bundle  $p: X_0 \rightarrow X_0/D^\times$ , where the fibers represent classes of isomorphic algebras. So,  $X$  is a disjoint union of sections  $c_{\lambda, \mu} \cap X_0$ .*

The previous bundle can be “lifted” to specific fields, for instance, in the real case, we may consider the plane with the axes removed:  $\Gamma := \mathbb{R}^\times \times \mathbb{R}^\times$ . Denote by  $\mathbf{O}$  the origin  $\mathbf{O} := (0, 0)$ . Consider also the curves  $c_{\lambda, \mu}$  each one of which is the zero set of  $\mu^3x^2 - \lambda^2y^3$ . Then,  $\Gamma = \sqcup_{\lambda, \mu} c_{\lambda, \mu}^*$  is the disjoint union of the perforated curves  $c_{\lambda, \mu}^* := c_{\lambda, \mu} \setminus \{\mathbf{O}\}$  with  $\lambda, \mu \neq 0$ . Each such curve  $c_{\lambda, \mu}$  cuts the line  $x = 1$  in a unique point:  $(1, \mu/\sqrt[3]{\lambda^2})$ . Then, we consider  $\pi: \mathbb{R}^\times \times \mathbb{R}^\times \rightarrow \mathbb{R}^\times$ , where  $\pi(\lambda, \mu)$  can be defined as the intersection of  $c_{\lambda, \mu}$  with the vertical line  $x = 1$ . In other words,  $\pi(\lambda, \mu) = \frac{\mu}{\sqrt[3]{\lambda^2}}$ . Since for any nonzero  $t$ , we have  $\pi(1, t) = t$ , the map  $\pi$  is surjective, and defining the curve  $\tilde{c}_t := c_{\lambda, \mu}$  for any  $(\lambda, \mu) \in \pi^{-1}(t)$ , we have

$$\mathbb{R}^\times \times \mathbb{R}^\times = \bigsqcup_{t \in \mathbb{R}^\times} \tilde{c}_t^*,$$

where  $\tilde{c}_t^* := \tilde{c}_t \setminus \{\mathbf{O}\}$ . We could paraphrase this by saying that the plane with the axes removed is a disjoint union (indexed in  $\mathbb{R}^\times$ ) of perforated curves (Figs. 1, 2).

If we consider, for instance, the domain  $D = \mathbb{Z}[\sqrt{3}] \subset \mathbb{R}$ , we know that  $D^\times$  consists of all  $a + b\sqrt{3} \in D$  such that  $a^2 - 3b^2 = 1$ , which has infinite cardinal. Thus, in each orbit of  $X_0 = \mathbb{Z}[\sqrt{3}]^\times \times (\mathbb{Z}[\sqrt{3}]^* \setminus \mathbb{Z}[\sqrt{3}]^\times)$  under the action of  $\mathbb{Z}[\sqrt{3}]^\times$ , there are infinitely many elements. The orbits of this action are in one-to-one correspondence with the points of the real line of the form  $\frac{\mu}{\sqrt[3]{\lambda^2}}$ , where  $\mu \in \mathbb{Z}[\sqrt{3}]^*$  and  $\lambda \in \mathbb{Z}[\sqrt{3}]^\times$ . So, we can consider the monoid  $\mathcal{M} := \{\frac{\mu}{\sqrt[3]{\lambda^2}}: \mu \in \mathbb{Z}[\sqrt{3}]^*, \lambda \in \mathbb{Z}[\sqrt{3}]^\times\} \subset \bar{\mathbb{Q}}$  and the orbits of the action  $\mathbb{Z}[\sqrt{3}]^\times \times X_0 \rightarrow X_0$  are in one-to-one correspondence with the monoid  $\mathcal{M}$ . Consequently, the isomorphism classes of algebras of type  $A(1, \lambda, \mu)$  are in one-to-one correspondence with  $\mathcal{M}$ .

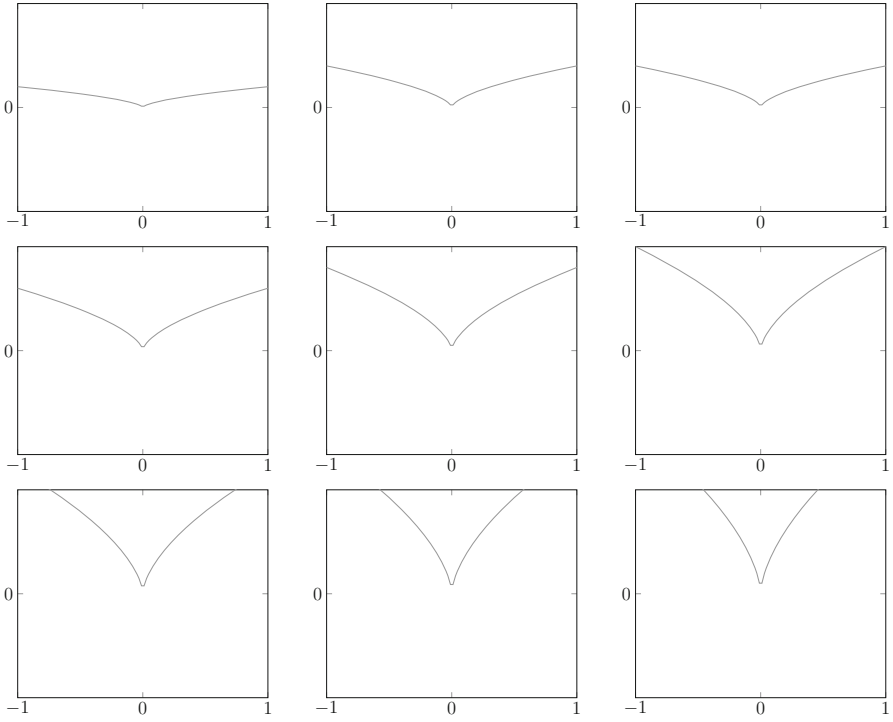
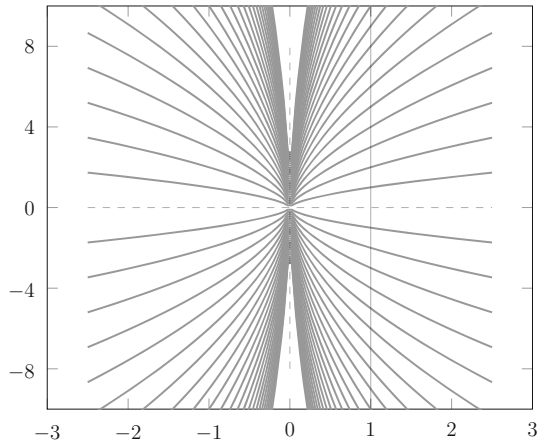


Fig. 1 Some of the curves  $c_{\lambda, \mu}$

Fig. 2 The family  $c_{\lambda, \mu}$



To manage the general case of Lemma 4, we have to make the following considerations. Pick a domain  $D$ , let  $Q$  be its field of fractions and  $\bar{Q}$  the algebraic closure of  $Q$ . Take the multiplicative monoid  $\mathcal{M} := \{\frac{\mu}{k} : \mu \in D^*, \exists \lambda \in D^\times, k^3 = \lambda^2\} \subset \bar{Q}$ . Then, we consider  $\mu_3(\mathcal{M}) := \{x \in \mathcal{M} : x^3 = 1\}$  and define the quotient monoid

$$\bar{\mathcal{M}} := \mathcal{M}/\mu_3(\mathcal{M}) \tag{3}$$

whose elements are equivalence classes  $[\lambda]$ . It is interesting to note that  $\mu_3(\mathcal{M}) = \mu_3(\bar{Q})$ : Indeed, if  $x \in \mu_3(\bar{Q})$ , we have  $x = \frac{1}{x^2} \in \mathcal{M}$ , since  $(x^2)^3 = 1^2$ . Furthermore, there is a bijection  $X_0/D^\times \cong \bar{\mathcal{M}}$  such that the equivalence class of  $(\lambda, \mu)$  in  $X_0$  maps to  $[\frac{\mu}{\sqrt[3]{\lambda^2}}]$ . Definitively, the isomorphism classes of algebras of type  $A(1, \lambda, \mu)$  over  $D$  are in bijective correspondence with the elements of  $\bar{\mathcal{M}}$ .

### 4.5 Case $e_1^2 = \alpha e_1 + \beta e_2, e_2^2 = \gamma e_1 + \delta e_2, \alpha, \beta, \gamma, \delta \neq 0$

In this case,  $\alpha\delta - \beta\gamma \in D^\times$ . Denote by  $B(\alpha, \beta, \gamma, \delta)$  the two-dimensional evolution algebra with natural basis  $\{e_1, e_2\}$  and multiplication  $e_1^2 = \alpha e_1 + \beta e_2, e_2^2 = \gamma e_1 + \delta e_2$  being  $\alpha\delta - \beta\gamma \in D^\times$  and  $\alpha, \beta, \gamma, \delta \neq 0$ . The change  $f_i = k_i e_i$  ( $i = 1, 2$ ) and  $k_i \in D^\times$  gives

$$\begin{cases} f_1^2 = k_1\alpha f_1 + \frac{k_1^2}{k_2}\beta f_2 \\ f_2^2 = \frac{k_2^2}{k_1}\gamma f_1 + k_2\delta f_2. \end{cases}$$

So,

$$B(\alpha, \beta, \gamma, \delta) \cong B\left(k_1\alpha, \frac{k_1^2}{k_2}\beta, \frac{k_2^2}{k_1}\gamma, k_2\delta\right) \tag{4}$$

for any  $k_i \in D^\times$ . On the other hand, the change  $f_1 = ke_2, f_2 = he_1$  with  $k, h \in D^\times$  produces

$$\begin{cases} f_1^2 = k\delta f_1 + \frac{k^2\gamma}{h} f_2 \\ f_2^2 = \frac{h^2\beta}{k} f_1 + h\alpha f_2. \end{cases}$$

This allows to conclude that  $B(\alpha, \beta, \gamma, \delta) \cong B(k\delta, \frac{k^2\gamma}{h}, \frac{h^2\beta}{k}, h\alpha)$  for any  $k, h \in D^\times$ . Any isomorphism between algebras of the type  $B(\alpha, \beta, \gamma, \delta)$  is of one of the previous forms. We now distinguish several mutually non-isomorphism classes:

#### 4.5.1 Both $\alpha$ and $\delta$ are invertible

We have the following isomorphism  $B(\alpha, \beta, \gamma, \delta) \cong B(1, \frac{\delta}{\alpha^2}\beta, \frac{\alpha}{\delta^2}\gamma, 1)$ . Thus, our algebra is of type  $B(1, \lambda, \mu, 1)$  with  $1 - \lambda\mu \in D^\times$  and  $\lambda, \mu \neq 0$ . Moreover, it

is easy to see that  $B(1, \lambda, \mu, 1) \cong B(1, \lambda', \mu', 1)$  if and only if  $(\lambda, \mu) = (\lambda', \mu')$  or  $(\lambda, \mu) = (\mu', \lambda')$ . The moduli set is the orbit set  $X/\mathbb{Z}_2$ , where  $X$  is the set of all  $(\lambda, \mu) \in D^* \times D^*$  with  $1 - \lambda\mu \in D^\times$  and the action  $\mathbb{Z}_2 \times X \rightarrow X$  is  $0 \cdot (\lambda, \mu) = (\lambda, \mu)$ ,  $1 \cdot (\lambda, \mu) = (\mu, \lambda)$ .

#### 4.5.2 Only one of $\alpha$ and $\delta$ is invertible

We may assume without loss of generality that  $\alpha \in D^\times$  but  $\delta \notin D^\times$ . Thus, our algebra is isomorphic to some  $B(1, \xi, v, \rho)$  with  $\rho - \xi v \in D^\times$ ,  $\xi, v, \rho \neq 0$  and  $\rho \notin D^\times$ . We focus in the class of algebras

$$\mathcal{C} := \{B(1, \xi, v, \rho) : \rho - \xi v \in D^\times, \xi, v, \rho \neq 0, \rho \notin D^\times\}.$$

But then,  $\mathcal{C} = \mathcal{C}_1 \sqcup \mathcal{C}_2 \sqcup \mathcal{C}_3$  (a disjoint union), where

$$\begin{aligned} \mathcal{C}_1 &:= \{B(1, \xi, v, \rho) \in \mathcal{C} : \xi, v \in D^\times, \rho \notin D^\times\}, \\ \mathcal{C}_2 &:= \{B(1, \xi, v, \rho) \in \mathcal{C} : \xi \in D^\times, v, \rho \notin D^\times\}, \\ \mathcal{C}_3 &:= \{B(1, \xi, v, \rho) \in \mathcal{C} : v \in D^\times, \xi, \rho \notin D^\times\}, \\ \mathcal{C}_4 &:= \{B(1, \xi, v, \rho) \in \mathcal{C} : \xi, v, \rho \notin D^\times\}. \end{aligned}$$

The algebras of the different classes are not isomorphic and any algebra in  $\mathcal{C}$  is isomorphic to some of  $\mathcal{C}_i$ , for  $i \in \{1, 2, 3, 4\}$ . Thus, we have to investigate the isomorphism question within each class  $\mathcal{C}_i$ .

- $\mathcal{C}_1$ : If  $k \in D^\times$ , we have  $B(1, \xi, v, \rho) \cong B(1, \frac{\xi}{k}, k^2 v, k\rho)$  using (4) and for  $k = \xi$  we have  $B(1, \xi, v, \rho) \cong B(1, 1, \xi^2 v, \xi\rho)$ . Then, any algebra in  $\mathcal{C}_1$  is isomorphic to  $B(1, 1, \lambda, \mu)$  with  $\lambda \in D^\times, \mu \notin D^\times, \mu \neq 0$  and  $\lambda - \mu \in D^\times$ . Moreover, two such algebras  $B(1, 1, \lambda, \mu)$  and  $B(1, 1, \lambda', \mu')$  are isomorphic if and only if  $(\lambda, \mu) = (\lambda', \mu')$ . For instance, over the integers, there are only two non-isomorphism classes in  $\mathcal{C}_1$ : the one of  $B(1, 1, 1, 2)$  and that of  $B(1, 1, -1, -2)$ .
- $\mathcal{C}_2$ : Any algebra in this class is isomorphic to some  $B(1, 1, \lambda, \mu)$  with  $\lambda, \mu \notin D^\times$  but  $\lambda - \mu \in D^\times$  and  $\lambda, \mu \neq 0$ . For any such algebras, we have  $B(1, 1, \lambda, \mu) \cong B(1, 1, \lambda', \mu')$  if and only if  $(\lambda, \mu) = (\lambda', \mu')$ .
- $\mathcal{C}_3$ : In this case, any algebra in this class is isomorphic to some  $B(1, \lambda, 1, \mu)$  with  $\lambda, \mu \notin D^\times$ , but  $\mu - \lambda \in D^\times$  and  $\lambda, \mu \neq 0$ . For any such algebras, we have  $B(1, \lambda, 1, \mu) \cong B(1, \lambda', 1, \mu')$  if and only if  $(\lambda, \mu) = (\lambda', \mu')$  or  $(\lambda, \mu) = (-\lambda', -\mu')$ .
- $\mathcal{C}_4$ : The isomorphism condition is  $B(1, \xi, v, \rho) \cong B(1, \xi', v', \rho')$  if and only if there is some  $k \in K^\times$  such that  $\xi' = \frac{1}{k}\xi, v' = k^2 v$  and  $\rho' = k\rho$  using (4). Denote by  $D^* \setminus D^\times$  the set of nonzero and noninvertible elements of  $D$  and recall that

$$\Omega_3 = \{(\xi, v, \rho) \in (D^* \setminus D^\times)^3 : \rho - \xi v \in D^\times\}.$$

Now, consider the class of *two-dimensional evolution algebras* (over the domain  $D$ ) given by  $C = \{B(1, \xi, v, \rho) : (\xi, v, \rho) \in \Omega_3\}$ . Observe that there is an action

$D^\times \times \Omega_3 \rightarrow \Omega_3$  given by  $k \cdot (\xi, v, \rho) = (\frac{\xi}{k}, k^2v, k\rho)$  for  $k \in D^\times$ . The set of orbits of  $\Omega_3$  under this action will be denoted by  $\Omega_3/D^\times$ . Then, the isomorphism classes in  $C$  are in one-to-one correspondence with the elements of the orbit set  $\Omega_3/D^\times$ . Alternatively, we can consider the curve  $xz = \xi\rho, \rho^2y = v z^2$  in  $Q^3$ , so that the isomorphism classes of algebras  $B(1, x, y, z)$  in  $C$  are in one-to-one correspondence with the points of intersection of  $\Omega_3$  with the curve

$$\begin{cases} xz = \xi\rho \\ \rho^2y = v z^2 \end{cases} \quad (\xi, v, \rho) \in \Omega_3.$$

Denote by  $\partial_{\xi,v,\rho}$  such curve in the affine space  $Q^3$  and  $\partial_{\xi,v,\rho}^* := \partial_{\xi,v,\rho} \cap \Omega_3$  its section with  $\Omega_3$ . Then, we have that  $(\xi', v', \rho') \notin \partial_{\xi,v,\rho}$  implies  $\partial_{\xi,v,\rho}^* \cap \partial_{\xi',v',\rho'}^* = \emptyset$ . Also  $(\xi, v, \rho) \in \partial_{\xi,v,\rho}$  for any  $(\xi, v, \rho) \in \Omega_3$ . Thus,  $\Omega_3$  is a disjoint union of sections  $\partial_{\xi,v,\rho}^*$  and we can define  $\mathfrak{d}$  as the set whose elements are the different curves  $\partial_{\xi,v,\rho}$ . Therefore, we get a bundle  $\Omega_3 \rightarrow \mathfrak{d}$  such that  $(\xi, v, \rho) \mapsto \partial_{\xi,v,\rho}$  and the isomorphism classes of algebras in  $C$  are in one-to-one correspondence with  $\mathfrak{d}$ . So, the isomorphism classes of algebras in  $C$  are indexed by the set of curves  $\mathfrak{d}$  or if we prefer, a moduli set for  $C$  is  $\mathfrak{d}$ : Each algebra in  $C$  is completely determined by a curve  $\partial_{\xi,v,\rho}$ . For instance, if  $D = \mathbb{Z}$ , an element in this class of algebras is  $B(1, 3, 2, 5)$  and its isomorphism class consists on itself and  $B(1, -3, 2, -5)$ .

### 4.5.3 Neither $\alpha$ nor $\delta$ are invertible

Again we distinguish two mutually non-isomorphism cases.

(1)  $\beta$  or  $\gamma$  is invertible. We may assume without loss of generality that  $\beta \in D^\times$ . Then,  $B(\alpha, \beta, \gamma, \delta) \cong B(k\alpha, 1, \beta^2k^3\gamma, \beta\delta k^2)$  for any  $k \in D^\times$  using (4). The algebras in this class are therefore of the form  $B(\mu, 1, \lambda, \omega)$  with  $\mu, \omega \notin D^\times$  and  $\mu, \lambda, \omega \neq 0, \mu\omega - \lambda \in D^\times$ .

(a) Assume  $\lambda \notin D^\times$ . Since  $B(\mu, 1, \lambda, \omega) \cong B(k\mu, 1, k^3\lambda, k^2\omega)$  for any  $k \in D^\times$ , we may define  $\Sigma_3 := \{(\mu, \lambda, \omega) \in (D^* \setminus D^\times)^3 : \mu\omega - \lambda \in D^\times\}$  and we have an action  $D^\times \times \Sigma_3 \rightarrow \Sigma_3$  given by  $k(\mu, \lambda, \omega) = (k\mu, k^3\lambda, k^2\omega)$ . The isomorphism classes of algebras of this kind are in one-to-one correspondence with the orbit set  $\Sigma_3/D^\times$ . Now, let  $C$  denote the class of algebras  $B(\mu, 1, \lambda, \omega)$  with  $\mu, \omega \notin D^\times$  and  $\mu, \lambda, \omega \neq 0, \mu\omega - \lambda \in D^\times$ . We have an action  $D^\times \times \Sigma_3 \rightarrow \Sigma_3$  given by  $k(\mu, \lambda, \omega) = (k\mu, k^3\lambda, k^2\omega)$ . The isomorphism classes of algebras of this kind are in one-to-one correspondence with the orbit set  $\Sigma_3/D^\times$ . Let  $\sigma_{\mu,\lambda,\omega}$  be the curve of  $Q^3$  given by

$$\begin{cases} \mu^3y = x^3\lambda \\ \mu^2z = x^2\omega \end{cases} \quad (\mu, \lambda, \omega) \in \Sigma_3. \tag{5}$$

As in previous cases denote by  $\sigma_{\mu,\lambda,\omega}^* = \sigma_{\mu,\lambda,\omega} \cap \Omega_3$ . Then,  $(\mu, \lambda, \omega) \in \sigma_{\mu,\lambda,\omega}^*$  for any  $(\mu, \lambda, \omega) \in \Sigma_3$ . As before, the different  $\sigma_{\mu,\lambda,\omega}^*$ s are pairwise disjoint and their disjoint union is  $\Sigma_3$ , so we get a bundle  $\Sigma_3 \rightarrow \sigma$ , where  $\sigma$  is the set

whose elements are the  $\sigma_{\mu,\lambda,\omega}$ s. Thus  $\Sigma_3/D^\times$  is one-to-one correspondence with the set  $\sigma$  and this is a moduli for the class  $C$ .

- (b) If  $\lambda \in (D^\times)^{[3]}$ , then  $\lambda = \epsilon^3$  for some invertible  $\epsilon$ . Since  $B(\mu, 1, \lambda, \omega) \cong B(k\mu, 1, k^3\lambda, k^2\omega)$  (for  $k \in D^\times$ ), we may take  $k = \epsilon^{-1}$  and then  $k^3\lambda = 1$ , so that  $B(\mu, 1, \lambda, \omega) \cong B(k\mu, 1, 1, k^2\omega)$ . Thus, the algebras in this case are all of the form  $B(\xi, 1, 1, \rho)$  with  $\xi, \rho \neq 0, \xi\rho - 1 \in D^\times$  and  $\xi, \rho \notin D^\times$ . We have  $B(\xi, 1, 1, \rho) \cong B(k\xi, 1, 1, k^{-1}\rho)$  whenever  $k \in D^\times$  satisfies  $k^3 = 1$ . So, denoting by  $\mu_3(D) := \{k \in D^\times : k^3 = 1\}$  and  $\Omega_2 := \{(\xi, \rho) \in (D^* \setminus D^\times)^2 : \xi\rho - 1 \in D^\times\}$ , we have the action  $\mu_3(D) \times \Omega_2 \rightarrow \Omega_2$  given by  $k \cdot (\xi, \rho) = (k\xi, k^{-1}\rho)$ . To analyze the orbit set  $\Omega_2/\mu_3(D)$  define for any  $(\xi, \rho) \in \Omega_2$  the curve  $h_{\xi,\rho}$  of  $Q^2$  given by  $xy = \xi\rho$ . Define also  $h_{\xi,\rho}^* = h_{\xi,\rho} \cap \Omega_2$ . We have  $(\xi, \rho) \in h_{\xi,\rho}$  and if  $(\xi', \rho') \notin h_{\xi,\rho}$ , then  $h_{\xi,\rho}^* \cap h_{\xi',\rho'}^* = \emptyset$ , so  $\Omega_2$  is the disjoint union of all  $h_{\xi,\rho}^*$ . Therefore,  $\Omega_2/\mu_3(D)$  is in one-to-one correspondence with the set  $h$  whose elements are the ‘‘hyperbolae’’  $h_{\xi,\rho}$ .
- (c) In case  $\lambda \in D^\times$  but not necessarily  $\lambda \in (D^\times)^{[3]}$ , since  $B(\mu, 1, \lambda, \omega) \cong B(k\mu, 1, k^3\lambda, k^2\omega)$  for any  $k \in D^\times$ , the only thing we can do is to consider the set  $S$  of all triples  $(\mu, \lambda, \omega) \in (D^* \setminus D^\times) \times (D^\times \setminus (D^\times)^{[3]}) \times (D^* \setminus D^\times)$ , such that  $\mu\omega - \lambda \in D^\times$  and the action  $D^\times \times S \rightarrow S$  given by  $k(\mu, \lambda, \omega) = (k\mu, k^3\lambda, k^2\omega)$ . Thus, the orbit set  $S/D^\times$  is in one-to-one correspondence with the set  $\sigma'$  whose elements are the sections  $\tilde{\sigma}_{\mu,\lambda,\omega} := \sigma_{\mu,\lambda,\omega} \cap S$  defined for  $(\mu, \lambda, \omega) \in S$  in the affine space  $Q^3$  by equations (5) by replacing  $(\mu, \lambda, \omega) \in \Sigma_3$  with  $(\mu, \lambda, \omega) \in S$ .

Note that, the case presented in ((b)), that is, when  $\lambda \in (D^\times)^{[3]}$ , is in fact a subcase of ((c)). We have specified it because if  $\lambda$  is a cube, one more 1 can be got in the structure matrix of the algebra.

- (2)  $\beta, \gamma \notin D^\times$ . So, we have the algebras  $B(\alpha, \beta, \gamma, \delta)$ , where the four scalars are nonzero and noninvertible but  $\alpha\delta - \beta\gamma \in D^\times$ . We will denote by  $\Omega_4$  as the set of all  $(\alpha, \beta, \gamma, \delta) \in (D^* \setminus D^\times)^4$  such that  $\alpha\delta - \gamma\beta \in D^\times$ . We have an isomorphism  $B(\alpha, \beta, \gamma, \delta) \cong B(k_1\alpha, \frac{k_1^2}{k_2}\beta, \frac{k_2^2}{k_1}\gamma, k_2\delta)$  for any  $k_1, k_2 \in D^\times$ . Thus, we have an action

$$(D^\times \times D^\times) \times \Omega_4 \rightarrow \Omega_4$$

given by  $(k_1, k_2) \cdot (\alpha, \beta, \gamma, \delta) := (k_1\alpha, \frac{k_1^2}{k_2}\beta, \frac{k_2^2}{k_1}\gamma, k_2\delta)$ . The isomorphism classes of algebras of this type are in one-to-one correspondence with the elements of the orbit set  $\Omega_4/(D^\times \times D^\times)$ . The orbit set  $\frac{\Omega_4}{(D^\times \times D^\times)}$  can be described defining (for every  $(\alpha, \beta, \gamma, \delta) \in \Omega_4$ ) the surface  $\omega_{\alpha,\beta,\gamma,\delta}$  of  $Q^4$  given by

$$\begin{cases} yt\alpha^2 = \beta\delta x^2 \\ zx\delta^2 = \alpha\gamma t^2. \end{cases}$$

Therefore,  $\bar{\omega}_{\alpha,\beta,\gamma,\delta} := \omega_{\alpha,\beta,\gamma,\delta} \cap \Omega_4$ , so that  $\Omega_4$  is a disjoint union of  $\bar{\omega}_{\alpha,\beta,\gamma,\delta}$ s and we have a bijection  $\frac{\Omega_4}{D^\times \times D^\times} \cong \omega$ , where the elements of  $\omega$  are the sections

$\bar{\omega}_{\alpha,\beta,\gamma,\delta}$ . Furthermore, this orbit set is in one-to-one correspondence with the  $Q$ -points of the surface

$$\begin{cases} yt\alpha^2 = \beta\delta x^2 \\ zx\delta^2 = \alpha\gamma t^2 \end{cases}$$

that are in  $\Omega_4$ . For instance, if  $D = \mathbb{Z}$ , the isomorphism class of  $B(2, 3, 3, 5)$  has the following four elements:  $B(2, 3, 3, 5)$ ,  $B(-2, 3, -3, 5)$ ,  $B(2, -3, 3, -5)$  and  $B(-2, -3, -3, -5)$ . Now, if we consider  $D = \mathbb{Z}[x, x^{-1}]$ , then  $D^\times = \{\pm x^n : n \in \mathbb{Z}\}$  and we have

$$B(7x^3 + 4x^2, 4x, 5x^5 + 3x^4, 3x^3) \cong B(7x + 4, 4, 5x + 3, 3)$$

being the elements  $(7x^3 + 4x^2, 4x, 5x^5 + 3x^4, 3x^3)$  and  $(7x + 4, 4, 5x + 3, 3)$  in the same orbit of  $\Omega_4$  under the action of  $D^\times \times D^\times$ . Indeed,  $(x^2, x^3) \cdot (7x + 4, 4, 5x + 3, 3) = (7x^3 + 4x^2, 4x, 5x^5 + 3x^4, 3x^3)$ .

Now, we can associate a colored directed graph to any quasiperfect evolution algebra over a domain. We fix a natural basis  $\{e_i\}$  of an evolution  $D$ -algebra  $\mathcal{E}$ . Let  $\omega_i^j$  be the structure constants. Then, we consider the graph whose vertices are in bijection with  $\{e_i\}$  and we draw a black edge from vertex  $i$  to vertex  $j$  if  $\omega_i^j \in D^\times$ . We draw a dotted edge from vertex  $i$  to vertex  $j$  if  $\omega_i^j \in D^* \setminus D^\times$ . The associated directed graph to a quasiperfect evolution algebra over a domain does not depend on the chosen natural basis because of Lemma 2.

Finally, we will collect the previous results in the following theorem in which the classification given is mutually exclusive thanks to Proposition 2.

**Theorem 1** *Let  $\mathcal{E}$  be a two-dimensional perfect evolution algebra over a domain  $D$ . Then, we have one and only one of the following possibilities:*

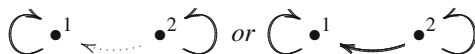
- (1)  $\mathcal{E} \cong A_1 = D \times D$  with product  $(x, y)(u, v) = (xu, yv)$ . The graph associated to the evolution algebra is of the form



- (2)  $\mathcal{E} \cong A_{2,\alpha}$ , where  $A_{2,\alpha} = D \times D$  with product  $(x, y)(u, v) = (\alpha yv, xu)$  and  $\alpha \in D^\times$ . These algebras are classified by the moduli  $\lim_{\rightarrow 2} D^\times / (D^\times)^{[3]}$ , that is,  $A_{2,\alpha} \cong A_{2,\beta}$  if and only if  $\beta\alpha^{-1} \in (D^\times)^3$  or  $\beta\alpha^{-2} \in (D^\times)^3$ . The graph is



- (3)  $\mathcal{E} \cong A_{3,\alpha} = D \times D$  with product  $(x, y)(u, v) = (xu + \alpha yv, yv)$  and  $\alpha \neq 0$ . These algebras are classified by the moduli  $D^*$ , that is,  $A_{3,\alpha} \cong A_{3,\beta}$  if and only if  $\alpha = \beta$ . The graph is one of





where the dotted arrow stands for the fact that  $\alpha$  is not invertible but it is nonzero.

(4) Denote by  $A(\alpha, \beta, \delta) = D \times D$  with multiplication  $(x, y)(u, v) = (\beta yv, \delta yv + \alpha xu)$ . Then,

(a)  $\mathcal{E} \cong A(\lambda, 1, 1) = D \times D$  with  $\lambda \in D^\times$ . The corresponding graph is



(b)  $\mathcal{E} \cong A(1, \lambda, \mu)$  with  $\lambda \in D^\times$  and  $0 \neq \mu \notin D^\times$ . The graph is

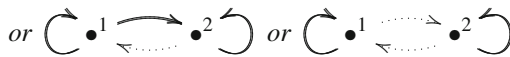
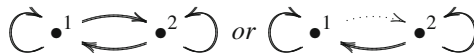


An algebra in one of the cases is not isomorphic to any algebra in other case. The isomorphism condition in each case is  $A(\lambda, 1, 1) \cong A(\mu, 1, 1)$  if and only if  $\lambda = \mu$ , that is, the moduli is  $D^\times$  and  $A(1, \lambda, \mu) \cong A(1, \lambda', \mu')$  if and only if  $\exists k \in D^\times, \lambda' = k^3\lambda, \mu' = k^2\mu$ . The moduli set is  $\bar{\mathcal{M}}$ .

(5) Denote by  $B(\alpha, \beta, \gamma, \delta) = D \times D$  with multiplication  $(x, y)(u, v) = (\alpha xu + \gamma yv, \beta xu + \delta yv)$ . Then

(5.1) Either  $\alpha$  or  $\delta$  is invertible. We have the following mutually excluding cases:

(i)  $\mathcal{E} \cong B(1, \lambda, \mu, 1), \lambda, \mu \neq 0, 1 - \lambda\mu \in D^\times$ . The graph is one of



(ii)  $\mathcal{E} \cong B(1, 1, \lambda, \mu)$  with  $\lambda \in D^\times, 0 \neq \mu \notin D^\times, \lambda - \mu \in D^\times$ . The corresponding graph is



(iii)  $\mathcal{E} \cong B(1, 1, \lambda, \mu)$  with  $\lambda, \mu \notin D^\times$  but  $\lambda - \mu \in D^\times, \lambda, \mu \neq 0$ . The graph is



(iv)  $\mathcal{E} \cong B(1, \lambda, 1, \mu)$  with  $\lambda, \mu \notin D^\times$  but  $\mu - \lambda \in D^\times, \lambda, \mu \neq 0$ . The associated graph is



(v)  $\mathcal{E} \cong B(1, \lambda, \mu, \gamma), \lambda, \mu, \gamma \neq 0, \gamma - \lambda\mu \in D^\times, \gamma, \lambda, \mu \notin D^\times$ . The graph is



An algebra in one of the cases is not isomorphic to any algebra in other case. In case (i),  $B(1, \lambda, \mu, 1) \cong B(1, \lambda', \mu', 1)$  if and only if  $(\lambda, \mu) = (\lambda', \mu')$  or  $(\lambda, \mu) = (\mu', \lambda')$ . In cases (ii) and (iii),  $B(1, 1, \lambda, \mu) \cong B(1, 1, \lambda', \mu')$  if and only if  $(\lambda, \mu) = (\lambda', \mu')$ . In case (iv),  $B(1, \lambda, 1, \mu) \cong B(1, \lambda', 1, \mu')$  if and only if  $(\lambda, \mu) = (\lambda', \mu')$  or  $(\lambda, \mu) = (-\lambda', -\mu')$ . In case (v),  $B(1, \lambda, \mu, \gamma) \cong B(1, \lambda', \mu', \gamma')$  if and only if  $\exists k \in D^\times, \lambda' = k^{-1}\lambda, \mu' = k^2\mu$  and  $\gamma' = k\gamma$ . The moduli set is  $\mathfrak{d}$ .

- (5.II) Neither  $\alpha$  nor  $\delta$  is invertible, but  $\beta$  or  $\delta$  is invertible. We have the possibilities:  
 (i)  $\mathcal{E} \cong B(\lambda, 1, \mu, \gamma), \lambda, \gamma, \mu \notin D^\times, \lambda, \mu, \gamma \neq 0, \gamma\lambda - \mu \in D^\times$ . The graph is



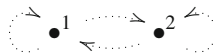
- (ii)  $\mathcal{E} \cong B(\lambda, 1, \mu, \gamma), \lambda, \gamma \notin D^\times, \mu \in D^\times, \lambda, \mu, \gamma \neq 0, \gamma\lambda - \mu \in D^\times$ . The corresponding graph is



In this case if  $\mu \in (D^\times)^{[3]}$ , we have  $\mathcal{E} \cong B(\lambda, 1, 1, \gamma)$ .

Again the cases are mutually excluding. In case (i), we have  $B(\lambda, 1, \mu, \gamma) \cong B(\lambda', 1, \mu', \gamma')$  if and only if  $\exists k \in D^\times, \lambda' = k\lambda, \mu' = k^3\mu, \gamma' = k^2\gamma$ . The moduli set is  $\sigma$ . In case (ii), when  $\mu \notin (D^\times)^{[3]}$ , we have  $B(\lambda, 1, \mu, \gamma) \cong B(\lambda', 1, \mu', \gamma')$  if and only if  $\exists k \in D^\times, \lambda' = k\lambda, \mu' = k^3\mu, \gamma' = k^2\gamma$ . The moduli set is  $\sigma'$ . In this case, when  $\mu \in (D^\times)^{[3]}$ , we get  $B(\lambda, 1, 1, \gamma) \cong B(\lambda', 1, 1, \gamma')$  if and only if  $\exists k \in D^\times, \lambda' = k\lambda$  and  $\mu' = k^{-1}\mu$ . The moduli set is  $\mathfrak{h}$ .

- (5.III) The elements  $\alpha, \delta, \beta$  and  $\delta$  are not invertible. Then,  $\mathcal{E} \cong B(\alpha, \beta, \gamma, \delta)$  and the graph is



We have that  $B(\alpha, \beta, \gamma, \delta) \cong B(\alpha', \beta', \gamma', \delta')$  if and only if  $\exists k_1, k_2 \in D^\times, \alpha' = k_1\alpha, \beta' = \frac{k_2}{k_1}\beta, \gamma' = \frac{k_2}{k_1}\gamma, \delta' = k_2\delta$ . The moduli set is  $\omega$ .

From this classification, we can recover the classification of two-dimensional perfect evolution algebras over arbitrary fields in [2].

**Corollary 2** If  $D$  is a field, then the two-dimensional perfect  $D$ -algebras are  $A_1, A_{2,\alpha}$  (for  $\alpha \neq 0$ ),  $A_{3,\alpha}$  ( $\alpha \neq 0$ ),  $A(\lambda, 1, 1)$  ( $\lambda \neq 0$ ) and  $B(1, \lambda, \mu, 1)$  ( $\lambda, \mu \neq 0, 1, \lambda\mu \neq 1$ ).

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**Data Availability Statement** The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

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