FUNCTORS OVER ROOTED SMALL CATEGORIES

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1. INTRODUCTION

Triangular matrix rings

- R y S rings.
- $U \in R$-Mod-S.

$$T = \begin{pmatrix} R & U \\ 0 & S \end{pmatrix}$$

Right modules over T (Green, 1982)

Triples (X, Y, f) with:

- $X \in$ Mod-R, $Y \in$ Mod-S.
- $f : X \otimes_R U \to Y$ is a S-linear map.

Other interpretation

There is a complete family of orthogonal idempotents of T, $\{e_1, e_2\}$ such that

- $e_1 T e_1 = R$.
- $e_2 T e_2 = S$.
- $e_1 T e_2 = U$.

> We give a description of T-modules in terms of $e_1 T e_1$-modules and $e_2 T e_2$-modules.

What about rings with enough idempotents?

A ring A (without unit) has enough idempotents if there is a family $\{e_i \mid i \in I\}$ of orthogonal idempotents such that

$$T = \bigoplus_{i \in I} Te_i = \bigoplus_{i \in I} e_i T$$

Functor categories **(Mitchell, 1972)**

There is a bijective correspondence between Morita equivalence classes of rings with enough idempotents and of preadditive categories

Main question

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Can we apply these ideas to study the category of additive functors
from a small preadditive category to the category of abelian groups?
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2. GREEN'S THEOREM FOR FUNCTOR CATEGORIES

Small preadditive categories

Let $\mathbf{C}$ be a small preadditive category.

- $R_c = End_{\mathbf{C}}(c)$ is a ring.
- $R_{ab} = Hom_{\mathbf{C}}(a, b) \in R_a$-Mod-$R_b$.
- Composition induces $t_{abc} : R_{ab} \otimes_{R_b} R_{bc} \to R_{ac}$ in R_a-Mod-R_c.

Functor categories

Let $(\mathbf{C}, \mathbf{Ab})$ be the category of additive covariant functors from $\mathbf{C}$ to the category of abelian groups $\mathbf{Ab}$.

Properties of functors

Given $F : \mathbf{C} \to \mathbf{Ab}$

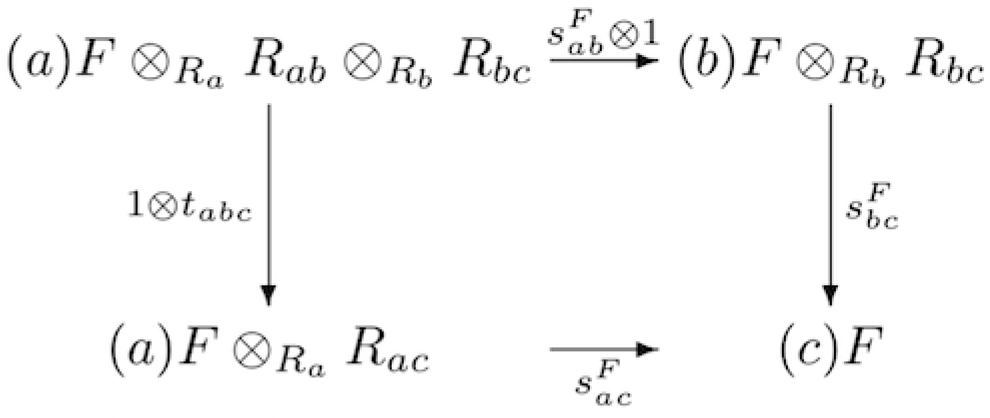
- For any $a \in \mathbf{C}$, $(a)F \in \mathrm{Mod}\text{-}R_a$ via

$$x \cdot f = (x)(f)F, \quad x \in (a)F, \quad f \in R_a$$

- For any $a, b \in \mathbf{C}$ it is induced

$$s^F_{ab} : (a)F \otimes_{R_b} R_{bc} \to (b)F, \quad s^F_{ab}(x \otimes f) = (x)(f)F$$

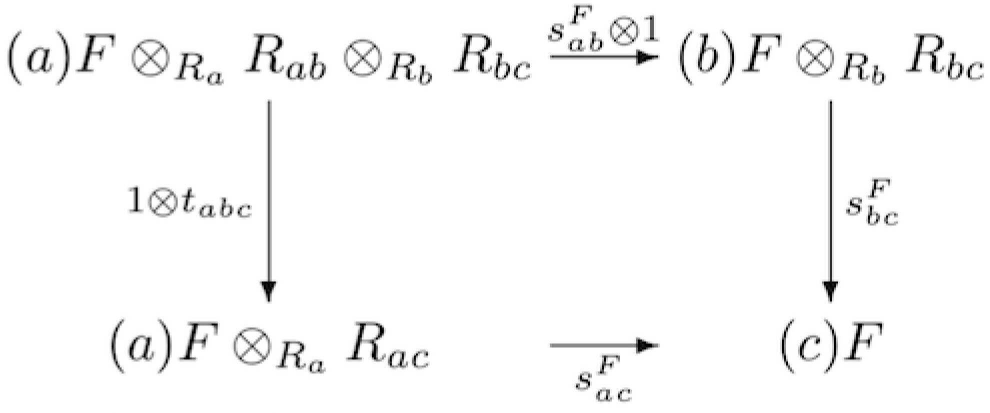
- The following diagram commutes:



Description of functors

A functor is determined by a tuple $(M_a, s_{ab}^M)_{a,b\in\mathbf{C}}$ such that

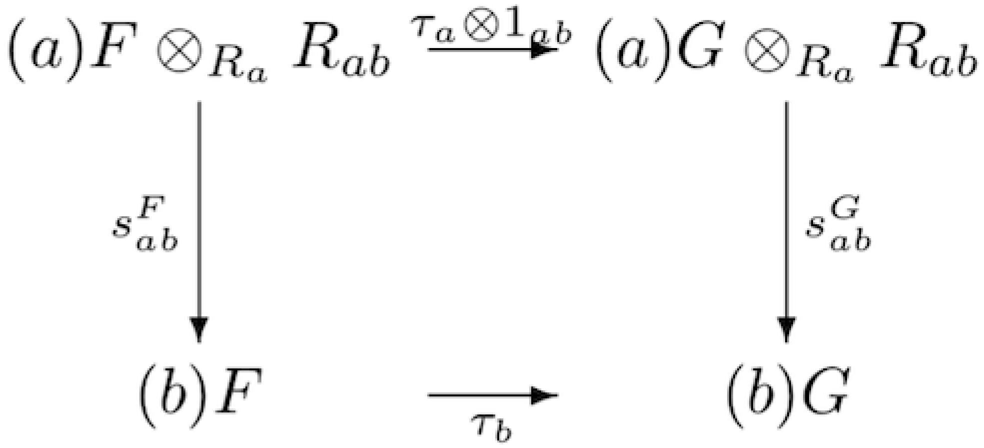
- For any $a \in \mathbf{C}$, $M_a \in \text{Mod-}R_a$.
- For any $a, b \in \mathbf{C}$, $s_{ab}^F : M_a \otimes_{R_b} R_{bc} \to M_b$
- The preceding diagram commutes.



Natural transformations

A natural transformation $\tau : (M_a, s_{ab}^M)_{a,b\in\mathbf{C}} \to (N_a, s_{ab}^M)_{a,b\in\mathbf{C}}$ is determined by a tuple $(\tau_a)_{a\in\mathbf{C}}$ such that:

- $\tau_a : M_a \to N_a$ is R_a-linear.
- The following diagram commutes:



Equivalence of categories

The category $(\mathbf{C}, \mathbf{Ab})$ is equivalent to the category $\mathrm{Rep}(\mathcal{S})$ of representations of the **spices** $\mathcal{S}$:

- $(R_a, R_{ab})_{a,b}$ **(Gabriel, 1973)**
- With a commutativity condition t_{abc}, **(Simson, 1970)**.

3. ROOTED SMALL CATEGORIES

The quiver of a small category

Given a small category $\mathbf{C}$, we define the quiver $Q(\mathbf{C})$ as follows:

- Vertices of $Q(\mathbf{C}) = \mathbf{C}$.
- There is an arrow $\varphi : a \to b$ if $\mathrm{Hom}_{\mathcal{C}}(a, b) \neq 0$.

Left rooted quivers

A quiver is left rooted if it does not have a path of the form

$$\cdots \bullet \to \bullet \to \bullet \to \bullet$$

Properties of left rooted quivers

We can make transfinite induction on the sets

- V_0 = vertices with no arrow ending in them.
- V_1 = vertices b with arrow $a \to b$ with $a \in V_0$.
- And so on...

Left rooted categories

$\mathbf{C}$ is left rooted if $Q(\mathbf{C})$ is left rooted.

4. PROJECTIVE FUNCTORS AND FLAT FUNCTORS

> $\mathbf{C}$ is a rooted small preadditive category.

The functor $\mathbf{p}$

The functor $\mathbf{p} : \prod_{c\in\mathbf{C}} \text{Mod-}R_c \to (\mathbf{C}, \mathbf{AB})$ is defined $\mathbf{p}(M_c)_{c\in\mathbf{C}} = (Q_a, s^Q_{ab})$ where

- $Q_a = \bigoplus_{c\in\mathbf{C}} M_c \otimes_{R_c} R_{ca}$.
- $s^Q_{ab} : \bigoplus_{c\in\mathbf{C}} M_c \otimes_{R_c} R_{ca} \otimes_{R_a} R_{ab} \to \bigoplus_{c\in\mathbf{C}} M_c \otimes_{R_c} R_{cb}$ is $(1 \otimes t_{cab})_{c\in\mathbf{C}}$.

Projective functors 1 (extrinsic)

Let $P = (P_a, s^P_{ab})_{a,b\in\mathbf{C}}$ be a functor. Then P is projective if and only if $P = \mathbf{p}(M_c)_{c\in\mathbf{C}}$ with $M_c \in \text{Proj-}R_c$.

Projective functors 2 (intrinsic)

Let $P = (P_a, s^P_{ab})_{a,b \in \mathbf{C}}$ be a functor. Then P is projective if and only if

- For each a, $P_a = S^P_a \oplus K_a$ in $\mathrm{Mod}\text{-}R_a$ with $P_a \in \mathrm{Proj}\text{-}R_a$, where

$$S^P_a = \sum_{c \neq a} \mathrm{Im}\ s^P_{ca}$$

- For each $b \in \mathbf{C}$, $\bigoplus_{c \neq b} \overline{s}^P_{cb}$ is monic, where $\overline{s}^P_{ab}$ is the composition

$$K_a \otimes_{R_a} R_{ab} \to P_a \otimes_{R_a} R_{ab} \to P_b$$

Flat functors

Let $F = (F_a, s^F_{ab})_{a,b \in \mathbf{C}}$ be a functor. The following are equivalent:

1. They are verified:
    - For each a, $F_a = S^F_a \oplus K_a$ in $\mathrm{Mod}\text{-}R_a$ with $K_a \in \mathrm{Flat}\text{-}R_a$.
    - For each $b \in \mathbf{C}$, $\bigoplus_{c \neq b} \overline{s}^F_{cb}$ is monic.
2. F is flat and belongs to $\mathrm{Im}\ \mathbf{p}$.

4. APPLICATIONS

Right perfect small preadditive categories

- The small category $\mathbf{C}$ is right perfect if every flat functor in $(\mathbf{C}, \mathbf{Ab})$ is projective.
- The following are equivalent for $\mathbf{C}$ (**Simson, 1974**)

  i. $\mathbf{C}$ is right perfect.

  ii. Every flat functor of the simple form is projective.

  iii. $\mathrm{End}_{\mathbf{C}}(c)$ is right perfect for each $c \in \mathbf{C}$.

Strong generating sets in additive categories

If $\mathbf{C}$ is additive, a strong generating subcategory is a full subcategory $\mathbf{N}$ such that $\mathbf{C} = \mathrm{add}(M)$.

- $(\mathbf{C}, \mathbf{Ab})$ and $(\mathbf{N}, \mathbf{Ab})$ are equivalent (**Mitchell, 1972**).

Weak left rooted categories

$\mathbf{C}$ is weak left rooted if it has a strong generating subcategory $\mathbf{N}$ which is left rooted.

When weak left rooted additive categories are right perfect

The same characerization as above can be proved.

Weak left fp-rooted finitely accesible categories

A finitely accessible category $\mathbf{A}$ is left fp-rooted (resp. weak fp-rooted) if the full subcategory of all finitely presented objects is left rooted (resp. weak rooted).

Examples

Take R a ring with complete set of idempotents $\{e_i \mid i \in I\}$.

- Then $\mathbf{N} = \{Re_i \mid i \in I\}$ is a strong generating subcategory of $\text{proj-}R$.
- If $\mathbf{N}$ is left rooted (for instance, if R is a triangular matrix ring), then $\text{proj-}R$ is left weak rooted.
- In this case, $\text{Flat-}R$ is a finitely accessible additive category which is left weak fp-rooted.

Pure semisimple categories

The following assertions are equivalent for $\mathbf{A}$ finitely accessible and left weak fp-rooted:

1. $\mathbf{A}$ is pure semisimple.
2. There exists a strong generating subcategory $\mathbf{N}$ of $fp(\mathbf{A})$ such that $\mathrm{End}_{\mathbf{A}}(N)$ is right perfect.
3. Each finitely presented object F satisfies that $\mathrm{End}_{\mathbf{A}}(N)$ is right perfect.

5. FUTURE RESEARCH

EI-categories

An *EI-category* is a category $\mathbf{C}$ such that every endomorphism is an isomorphism.

There is recent interest in $(E, \mathrm{Mod}\text{-}R)$ where E is a finite EI-categories, i. e., with a finite number of morphisms, and R is commutative.

1. All results in the previos section work for E an R-linear category.
2. E induces a (group) spices
   - $E_a = \mathrm{End}_{\mathbf{C}}(a)$ is a group.
   - $E_{ab} = \mathrm{Hom}_{\mathbf{C}}(a, b)$ is a (E_a, E_b)-biset.
   - $t_{abc} : E_{ab} \times_{E_c} E_{bc} \to E_{ac}$ is the commutativity condition.

where $E_{ab} \times_{E_c} E_{bc}$ is the composition of the bisets.

3. (E, Set) are the representations of these spices.

Cotorsion pairs in functor categories

In a recent paper, Holm and Jorgensen have induced cotorsion pairs in the category $(\mathbf{C}, \text{Mod-}R)$.

These cotorsion pairs allow them to construct model structures in the functor category.

They begin with cotorsion pairs in $Mod - R$ and induce them in $(\mathbf{C}, \text{Mod-}R)$.

> Can we induce cotorsion pairs in $(\mathbf{C}, \text{Mod-}R)$ using our description of functor categories?