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Short communication

Kitainik axioms do not characterize the class of inclusion measures based on contrapositive fuzzy implications

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Abstract

In this short communication, we refute the conjecture by Fodor and Yager from [5] that the class of inclusion measures proposed by Kitainik coincides with that of inclusion measures based on contrapositive fuzzy implications. In particular, we show that the conjecture only holds when the considered universe of discourse is finite.

Keywords: Kitainik axioms; Fuzzy inclusion; Fuzzy implications

1. Introduction

The generalization of the binary relation of inclusion between fuzzy sets has taken the attention of the scientific community since the seminal paper of Lotfi Zadeh [10]. The reader can find many different ways in literature to design such extensions under the name of *measures of inclusion*, but in this paper we focus on two of them: the ones constructed by fuzzy implications [1,9] and those that satisfy the Kitainik axioms [6]. Both families of measures of inclusion are related. Actually, in 2000 János Fodor and Ronald R. Yager formulated a conjecture in [5] that stated that *"the Kitainik axioms generalize the measures of inclusion based on contrapositive implication"*. Such a conjecture was formulated as a theorem and its proof was not included because *"[...] can be easily derived from Theorem 6.3 in (Kitainik, 1993) and the above Theorem 23"*.[1] This conjecture has had a significant impact in the literature and has been assumed to be true by many authors (e.g., it appears in [2,3,7]). By the results of [4], it also pertains to Sinha and Dougherty's axioms [8], which define a class of inclusion measures that is strictly included into the one characterised by Kitainik axioms. Sadly, as we will demonstrate in this short communication, Fodor and Yager's conjecture fails in general when the universe of discourse in which fuzzy sets are defined is infinite.

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[1] In this quote the references are kept as originally in [5].

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The remainder of this paper is structured as follows: in Section 2, some preliminary definitions are recalled and then, in Section 3, a counterexample of the mentioned conjecture is introduced. Moreover, the precise reason why Fodor and Yager's argument fails is identified, and it is shown that the conjecture holds in general only for finite universes.

2. Preliminaries

2.1. Fuzzy sets

A fuzzy set A is a pair $(\mathscr{U}, \mu_A)$ where $\mathscr{U}$ is a set (called universe) and μ_A is a mapping from $\mathscr{U}$ to [0, 1] (called membership function). $\mathscr{F}(\mathscr{U})$ denotes the set of fuzzy sets on the universe $\mathscr{U}$. Since the universe is always prefixed, for the sake of simplicity, we identify fuzzy sets with membership functions (i.e., $A = \mu_A$). On $\mathscr{F}(\mathscr{U})$, we can extend the usual crisp operations of union, intersection and complement as follows. Given two fuzzy sets A and B, we define

- (union) $(A \cup B)(u) = \max\{A(u), B(u)\}$
- (intersection) $(A \cap B)(u) = \min\{A(u), B(u)\}$
- (complement) $A^c(u) = 1 - A(u)$.

Certainly, there are many other options to extend those operators in fuzzy set theory (e.g., by using t-norms, t-conorms, etc.) but these are the most common and the ones used in the fuzzy frameworks related to this paper, that is in [5] and [6].

2.2. Measures of inclusion based on implications and on Kitainik's axioms

In the eighties, many different authors supported the idea of measuring the inclusion between two fuzzy sets by means of a fuzzy implication [1,9]. Specifically, a fuzzy implication is an operator $I\colon [0,1] \times [0,1] \to [0,1]$ such that is decreasing in the first component, increasing in the second and satisfies the boundary conditions $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$. A fuzzy implication is called contrapositive if $I(x, y) = I(1-y, 1-x)$ for all $x, y \in [0, 1]$.

Definition 1. Given two fuzzy sets $A, B \in \mathscr{F}(\mathscr{U})$, the measure of inclusion of A in B with respect to a fuzzy implication I is given by:

$$\mathscr{I}_I(A, B) = \inf_{u \in \mathscr{U}} I(A(u), B(u)).$$

Note that under the previous definition, two fuzzy implications define different measures of inclusion.

On the other hand, in 1987 Leonid Kitainik proposed a set of axioms aimed at characterizing those inclusion measures based on implications, that is, inclusion measures defined by Definition 1. Those axioms are the following:

Definition 2 *([6])*. A binary relation $S\colon \mathscr{F}(\mathscr{U}) \times \mathscr{F}(\mathscr{U}) \to [0, 1]$ is called a *K-measure of inclusion* if it satisfies the following axioms for all fuzzy sets A, B and C:

(K1) $S(A, B) = S(B^c, A^c)$.
(K2) $S(A, B \cap C) = \min\{S(A, B), S(A, C)\}$.
(K3) If $T\colon \mathscr{U} \to \mathscr{U}$ is a bijective transformation on the universe, then $S(A, B) = S(T(A), T(B))$, where $T(A)$ is the fuzzy set defined by $T(A)(u) = A(T(u))$ for all $u \in \mathscr{U}$.
(K4) If A and B are crisp then $S(A, B) = 1$ if and only if $A \subseteq B$.
(K5) If A and B are crisp then $S(A, B) = 0$ if and only if $A \not\subseteq B$.

Independently from Kitainik's work, in 1993 Sinha and Dougherty [8] also proposed a set of axioms for fuzzy inclusion. It was shown in [4] that these axioms coincide with Kitainik's, with the exception that in Sinha and Dougherty's framework, (K4) and (K5) are enforced for arbitrary fuzzy sets. Hence, the class of inclusion measures (referred to as SD-measures in the remainder of this paper) defined by it is a (strict) subclass of the class of K-measures.

3. A counter example of the Fodor-Yager conjecture

In 2000 Fodor and Yager stated the following conjecture [5, Theorem 24]:

Fodor-Yager conjecture:
A binary relation $R\colon \mathscr{F}(\mathscr{U}) \times \mathscr{F}(\mathscr{U}) \to [0,1]$ is a K-measure of inclusion S if and only if there exists a contrapositive fuzzy implication I such that

$$S(A,B) = \mathscr{I}_I(A,B) = \inf_{u\in\mathscr{U}} I(A(u),B(u)),$$

for all fuzzy sets A and B.

In other words, Fodor-Yager ensured that the axiomatic definition of Kitainik (Definition 2) is a subclass of the measures of inclusion based on infimum of fuzzy implications (Definition 1). However, the authors did not provide an explicit proof of that statement and, as we show in the following example, the conjecture is not true for fuzzy sets defined on infinite universes.

Example 1. Let us consider the universe $\mathscr{U} = [0,1]$ and the measure of inclusion defined for any pair of fuzzy sets $A, B \in \mathscr{F}([0,1])$ as:

$$S(A,B) = \begin{cases} 0 & \text{if there exists } u \in \mathscr{U} \text{ such that } A(u) = 1 \text{ and } B(u) = 0 \\ \frac{1}{2} & \begin{array}{l}\text{if there does not exist } u \in \mathscr{U} \text{ such that } A(u) = 1 \text{ and } B(u) = 0 \\ \text{AND} \\ \text{there exists a sequence } \{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{U} \text{ such that } \lim_{n\to\infty} A(u_n) = 1 \text{ and } \lim_{n\to\infty} B(u_n) = 0\end{array} \\ 1 & \text{otherwise.} \end{cases}$$

Firstly, let us prove that S satisfies the five axioms of the Kitainik measures of inclusion.

(K1) Let us prove that $S(A,B) = S(B^c, A^c)$ by distinguishing the three possible cases.

– Let us assume that $S(A,B) = 0$, then:

$$\begin{aligned} S(A,B) = 0 &\iff \text{there exists } u \in \mathscr{U} \text{ such that } A(u) = 1 \text{ and } B(u) = 0 \\ &\iff \text{there exists } u \in \mathscr{U} \text{ such that } B^c(u) = 1 \text{ and } A^c(u) = 0 \\ &\iff S(B^c, A^c) = 0 \end{aligned}$$

– Let us assume that $S(A,B) = \frac{1}{2}$, then:

$$\begin{aligned} S(A,B) = \frac{1}{2} &\iff \begin{cases} \text{there does not exist } u \in \mathscr{U} \text{ such that } A(u) = 1 \text{ and } B(u) = 0 \\ \text{AND} \\ \text{there exists a sequence } \{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{U} \text{ such that} \\ \lim_{n\to\infty} A(u_n) = 1 \text{ and } \lim_{n\to\infty} B(u_n) = 0 \end{cases} \\ &\iff \begin{cases} \text{there does not exist } u \in \mathscr{U} \text{ such that } B^c(u) = 1 \text{ and } A^c(u) = 0 \\ \text{AND} \\ \text{there exists a sequence } \{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{U} \text{ such that} \\ \lim_{n\to\infty} B^c(u_n) = 1 \text{ and } \lim_{n\to\infty} A^c(u_n) = 0 \end{cases} \\ &\iff S(B^c, A^c) = \frac{1}{2} \end{aligned}$$

– Let us assume that $S(A,B) = 1$, then

$$S(A,B) = 1 \iff S(A,B) \neq 0 \text{ and } S(A,B) \neq \frac{1}{2}$$

$$\iff S(B^c, A^c) \neq 0 \text{ and } S(B^c, A^c) \neq \frac{1}{2}$$
$$\iff S(B^c, A^c) = 1$$

(K2) Let us prove that $S(A, B \cap C) = \min\{S(A, B), S(A, C)\}$ by distinguishing the three possible cases.
– Let us assume that $S(A, B \cap C) = 0$, then:

$$\begin{aligned} S(A, B \cap C) = 0 &\iff \text{there exists } u \in \mathscr{U} \text{ such that } A(u) = 1 \text{ and } (B \cap C)(u) = 0 \\ &\iff \begin{cases} \text{there exists } u \in \mathscr{U} \text{ such that } A(u) = 1 \text{ and } B(u) = 0 \\ OR \\ \text{there exists } u \in \mathscr{U} \text{ such that } A(u) = 1 \text{ and } C(u) = 0 \end{cases} \\ &\iff \begin{cases} S(A, B) = 0 \\ OR \\ S(A, C) = 0 \end{cases} \\ &\iff \min\{S(A, B), S(A, C)\} = 0 \end{aligned}$$

– Let us assume that $S(A, B \cap C) = \frac{1}{2}$, then:

$$\begin{aligned} S(A, B \cap C) = \frac{1}{2} &\iff \begin{cases} \text{there does not exist } u \in \mathscr{U} \text{ such that } A(u) = 1 \text{ and } (B \cap C)(u) = 0 \\ \text{AND} \\ \text{there exists a sequence } \{u_n\}_{n \in \mathbb{N}} \subseteq \mathscr{U} \text{ such that} \\ \lim_{n\to\infty} A(u_n) = 1 \text{ and } \lim_{n\to\infty} B \cap C(u_n) = 0 \end{cases} \\ &\iff \begin{cases} \text{for all } u \in \mathscr{U} \text{ such that } A(u) = 1 \text{ then } B(u) \neq 0 \text{ and } C(u) \neq 0 \\ \text{AND} \\ \text{there exists a sequence } \{u_n\}_{n \in \mathbb{N}} \subseteq \mathscr{U} \text{ such that} \\ \begin{cases} \lim_{n\to\infty} A(u_n) = 1 \text{ and } \lim_{n\to\infty} B(u_n) = 0 \\ OR \\ \lim_{n\to\infty} A(u_n) = 1 \text{ and } \lim_{n\to\infty} C(u_n) = 0 \end{cases} \end{cases} \\ &\iff \begin{cases} S(A, B) = \frac{1}{2} \\ AND \\ S(A, C) \neq 0 \end{cases} \quad OR \quad \begin{cases} S(A, B) \neq 0 \\ AND \\ S(A, C) = \frac{1}{2} \end{cases} \\ &\iff \min\{S(A, B), S(A, C)\} = \frac{1}{2} \end{aligned}$$

Note that in the last equivalence we have used that $S(A, B) \neq 0$ implies $S(A, B) \geq \frac{1}{2}$.
– Let us assume that $S(A, B \cap C) = 1$, then

$$\begin{aligned} S(A, B \cap C) = 1 &\iff S(A, B \cap C) \neq 0 \text{ and } S(A, B \cap C) \neq \frac{1}{2} \\ &\iff \min\{S(A, B), S(A, C)\} \neq 0 \text{ and } \min\{S(A, B), S(A, C)\} \neq \frac{1}{2} \\ &\iff \min\{S(A, B), S(A, C)\} = 1 \end{aligned}$$

(K3) Let $T : \mathscr{U} \to \mathscr{U}$ be a bijective transformation on the universe and let us prove that $S(T(A), T(B)) = S(A, B)$. By definition and bijectivity of T we have that

$$S(A, B) = \begin{cases} 0 & \text{if there exists } u \in \mathscr{U} \text{ such that } A(u) = 1 \text{ and } B(u) = 0 \\ \frac{1}{2} & \begin{array}{l} \text{if there does not exist } u \in \mathscr{U} \text{ such that } A(u) = 1 \text{ and } B(u) = 0 \\ \text{AND} \\ \text{there exists a sequence } \{u_n\}_{n \in \mathbb{N}} \subseteq \mathscr{U} \text{ such that} \\ \lim_{n\to\infty} A(u_n) = 1 \text{ and } \lim_{n\to\infty} B(u_n) = 0 \end{array} \\ 1 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 0 & \text{if there exists } u \in \mathscr{U} \text{ such that } T(A(u)) = 1 \text{ and } T(B(u)) = 0 \\ \frac{1}{2} & \begin{array}{l}\text{if there does not exist } u \in \mathscr{U} \text{ such that } T(A(u)) = 1 \text{ and } T(B(u)) = 0 \\ \text{AND} \\ \text{there exists a sequence } \{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{U} \text{ such that} \\ \lim_{n\to\infty} T(A(u_n)) = 1 \text{ and } \lim_{n\to\infty} T(B(u_n)) = 0 \end{array} \\ 1 & \text{otherwise.} \end{cases}$$

$$= S(T(A), T(B))$$

(K4) Let us assume that A and B are crisp and let us prove that $S(A, B) = 1$ if and only if $A \subseteq B$. Let us firstly note that, if A and B are crisp sets then $S(A, B) \neq \frac{1}{2}$. The reason is because the existence of a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{U}$ such that $\lim_{n\to\infty} A(u_n) = 1$ and $\lim_{n\to\infty} B(u_n) = 0$, implies the existence of $n_0 \in \mathbb{N}$ such that $A(u_n) = 1$ and $B(u_n) = 0$ for all $n \geq n_0$. Then, if such a sequence exists, the first of the conditions to be $S(A, B) = \frac{1}{2}$ does not hold (i.e., the condition *"if there does not exist $u \in \mathscr{U}$ such that $A(u) = 1$ and $B(u) = 0$"*).
Once we know that for crisp sets $S(A, B) \neq \frac{1}{2}$, then we have: $S(A, B) = 1$ if and only if $S(A, B) \neq 0$ if and only if for all $u \in \mathscr{U}$ such that $A(u) = 1$ then $B(u) \neq 0$ if and only if (since B is crisp) for all $u \in \mathscr{U}$ such that $A(u) = 1$ then $B(u) = 1$ if and only if $A \subseteq B$.

(K5) Let us assume that A and B are crisp and let us prove that $S(A, B) = 0$ if and only if $A \nsubseteq B$. By definition, $S(A, B) = 0$ if and only if there exists $u \in \mathscr{U}$ such that $A(u) = 1$ and $B(u) = 0$, which is equivalent to say that $A \nsubseteq B$.

Therefore, we have that S satisfies the five axioms of Kitainik and S is a K-measure of inclusion.

Let us prove now that there is no fuzzy implication $I\colon [0, 1] \times [0, 1] \to [0, 1]$ such that:

$$S(A, B) = \inf_{u\in\mathscr{U}} I(A(u), B(u)).$$

Let us proceed by reductio ad absurdum and let us assume that such an implication I exists. Then, I has to satisfy the following property for all pairs $x, y \in [0, 1]$: given the fuzzy sets $A_x(u) = x$ and $A_y(u) = y$ for all $u \in \mathscr{U}$, then

$$S(A_x, A_y) = \inf_{u\in\mathscr{U}} I(A_x(u), A_y(u)) = \inf_{u\in\mathscr{U}} I(x, y) = I(x, y).$$

By definition of S, the equality $S(A_x, A_y) = 0$ holds only if $A_x = A_1$ and $A_y = A_0$. As a result, we have $I(x, y) = 0$ if and only if $x = 1$ and $y = 0$. On the other hand, note that the condition $S(A_x, A_y) = \frac{1}{2}$ never holds for those "constant" fuzzy sets A_x and A_y, since the existence of the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{U}$ such that $\lim_{n\to\infty} A_x(u_n) = 1$ and $\lim_{n\to\infty} A_y(u_n) = 0$, implies that $\lim_{n\to\infty} A_x(u_n) = x = 1$ and $\lim_{n\to\infty} A_y(u_n) = y = 0$, which contradicts the first condition of that case (i.e., there is not $u \in \mathscr{U}$ such that $A_x(u) = 1$ and $A_y(u) = 0$). Then, if $x \neq 1$ or $y \neq 0$, necessarily $S(A_x, A_y) = 1 = I(x, y)$ and vice-versa. In other words, the only possible implication is the drastic implication

$$I(x, y) = \begin{cases} 0 & \text{if } x = 1 \text{ and } y = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Consequently, if the Fodor-Yager conjecture holds, necessarily we have that $S(A, B) \neq \frac{1}{2}$ for every pair of fuzzy sets A and B. Finally, we reach the contradiction by showing an example of two fuzzy sets A and B such that $S(A, B) = \frac{1}{2}$.

Let us recall that $\mathscr{U} = [0, 1]$. Let A and B be the fuzzy sets defined by $A(u) = 1$ for all $u \in [0, 1]$ and

$$B(u) = \begin{cases} u & \text{if } u \neq 0 \\ \frac{1}{2} & \text{if } u = 0 \end{cases}$$

Then, it is obvious that for all $u \in \mathscr{U}$ such that $A(u) = 1$ then, $B(u) \neq 0$ and moreover, the sequence $u_n = \left\{\frac{1}{n+1}\right\}_{n\in\mathbb{N}} \subseteq [0, 1]$ satisfies that $\lim_{n\to\infty} A(u_n) = \lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} B(u_n) = \lim_{n\to\infty} \frac{1}{n+1} = 0$. Then, we can conclude that $S(A, B) = \frac{1}{2}$ and that the Fodor-Yager conjecture does not hold for S. □

The previous counterexample states that the Fodor-Yager conjecture is wrong in general. Taking a look at [5], the impression is that the conjecture may originate from a previous theorem that fails (in an item) when the universe is infinite. Firstly, let us recall a family of fuzzy sets that is necessary to understand the theorem: given $\alpha \in [0,1]$ and $u_0 \in \mathscr{U}$, the fuzzy set $\alpha_{\chi\{u_0\}}$ is defined as

$$\alpha_{\chi\{u_0\}} = \begin{cases} \alpha & \text{if } u = u_0 \\ 0 & \text{otherwise.} \end{cases}$$

Secondly, let us recall the statement of the mentioned result:

> [5, Theorem 23] For any K-measure of inclusion S on $\mathscr{F}(\mathscr{U})$ the following conditions are satisfied[2]: [...]
> (iv) $S(A,B) = \inf_{u\in\mathscr{U}} S\left(A(u)_{\chi\{u\}}, (1-B(u)_{\chi\{u\}})^c\right)$

Example 1 is also a counterexample of [5, Theorem 23 item (iv)].

Example 2. Let us consider again the K-measure of inclusion given in Example 1 and the fuzzy sets A and B given at the end of the mentioned example. On the one hand, as we saw previously, $S(A,B) = \frac{1}{2}$. On the other hand, since for all $u_0 \in (0,1]$ we have

$$A(u_0)_{\chi\{u_0\}}(u) = \begin{cases} 1 & \text{if } u = u_0 \\ 0 & \text{otherwise.} \end{cases} \qquad (1 - B(u_0)_{\chi\{u_0\}})^c(u) = \begin{cases} u & \text{if } u = u_0 \\ 1 & \text{otherwise,} \end{cases}$$

we have $S\big(A(u)_{\chi\{u\}}, (1-B(u))^c_{\chi\{u\}}\big) = 1$ for all $u \in (0,1]$. In addition, for $u = 0$, it is easy to check that $S\big(A(u)_{\chi\{u\}}, (1-B(u))^c_{\chi\{u\}}\big) = 1$ as well. As a result, we have that

$$\inf_{u\in[0,1]} S\big(A(u)_{\chi(u)}, (1-B(u))^c_{\chi(u)}\big) = \inf_{u\in[0,1]} S\big(A(u)_{\chi(u)}, (1-B(u))^c_{\chi(u)}\big) = 1 \neq \frac{1}{2} = S(A,B) \quad \square$$

To end the paper, two things are worth mentioning. First, we may see that [5, Theorem 23 item (iv)] is true for fuzzy sets defined on a finite universe $\mathscr{U}$, since for all pairs of fuzzy sets A and B we have that $A(u) = \cup_{u\in\mathscr{U}} A(u)_{\chi(u)}$, $B(u) = \cap_{u\in\mathscr{U}} (1-B(u))^c_{\chi(u)}$, and then:

$$S(A,B) = S\left(\bigcup_{u\in\mathscr{U}} A(u)_{\chi(u)}, \bigcap_{u\in\mathscr{U}} (1-B(u))^c_{\chi(u)}\right) = \inf_{u\in\mathscr{U}} S\left(A(u)_{\chi(u)}, (1-B(u))^c_{\chi(u)})\right).$$

Note that in the last equality we have used that $\mathscr{U}$ is finite to apply axiom (K2). From this last mentioned property, the Fodor-Yager conjecture can easily be derived for the case where fuzzy sets are defined on finite universes.

Finally, the analogous characterization of Sinha-Dougherty axioms obtained in [4] is not affected by the above counterexample, since the finiteness of U was included as hypothesis in the statement:

Theorem 1. *[4] Given a finite universe U, a binary relation $R\colon \mathscr{F}(\mathscr{U}) \times \mathscr{F}(\mathscr{U}) \to [0,1]$ is an SD-measure of inclusion S if and only if there exists a contrapositive fuzzy implication I that additionally satisfies, for x, y in $[0,1]$,*

(I1) $x \le y \Leftrightarrow I(x,y) = 1$
(I2) $x = 1$ *and* $y = 0 \Leftrightarrow I(x,y) = 0$

such that

$$S(A,B) = \mathscr{I}_I(A,B) = \inf_{u\in\mathscr{U}} I(A(u),B(u)),$$

for all fuzzy sets A and B.

[2] Only the relevant item is included.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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