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Quasi-closed elements in fuzzy posets

Manuel Ojeda-Hernández *, Inma P. Cabrera, Pablo Cordero

Universidad de Málaga, Andalucía Tech, Spain

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ABSTRACT

We generalize the notion of quasi-closed element to fuzzy posets in two stages: First, in the crisp style in which each element in a given universe either is quasi-closed or not. Second, in the graded style by defining degrees to which an element is quasi-closed. We discuss the different possible definitions and comparing them with each other. Finally, we show that the most general one has good properties to be used when we have a complete fuzzy lattice as a frame.

1. Introduction

In this paper, we investigate which is the appropriate generalization of the notion of quasi-closedness when working on fuzzy posets. In the classical case, this notion is key to knowledge representation ensuring non-redundancy. However, obtaining an adequate generalization for fuzzy environments that guarantees similar properties, remains an open problem.

One of the more (if not the most) usual way to represent knowledge is through if-then rules that can be seen as *implications* between conjunctions of attributes or Horn formulas. The mathematical theory of implications plays a central role in different research areas as Relational Databases, Formal Concept Analysis, Frequent Set Mining, and Learning Spaces. [1]. Implications also play central role in logic programming languages and logic query systems [2], knowledge-based and expert systems [3], and classification [4]. What makes it possible for such varied fields of work to be addressed using implications is that they can be mathematically modeled using closure operators, and these operators can be completely characterized by sets of implications [5]. There is a bijective link between lattices, closure operators and sets of implications, and the possible transformations between these objects give rise to relevant tools for data analysis [6].

In the areas where implications appear, many problems can be solved by deciding whether a given implication logically follows from a set of other implications. Relational Databases (where implications are called *functional dependencies*) is the research field where the earliest use of a logic to implication approach can be found [7]. The logic of implications can be considered a fragment of propositional logic that maintains a perfect balance between expressiveness and efficiency. Several problems that are known to be hard in propositional logic become tractable as, for instance, the entailment problem of implications, which is decidable in linear time [8], or the minimality problem of sets of implications, which is decidable in polynomial time [9]. More recently, Simplification Logic [10] has been introduced, which allows automatic reasoning to be addressed and these problems to be solved, at least as efficiently, with algorithms directly based on the inference rules [11–13].

Different sets of implications can characterize the same closure operator. In such a case, they are said to be equivalent sets of implications and, if two sets are so, the implications of one of them can be logically inferred from those of the other. In the literature, there are many papers that study sets of implications, called bases, which fulfill a certain property of minimality (in terms of cardinality, size, redundant information, etc.) among the sets equivalent to it [14–17]. The most

* Corresponding author.
E-mail addresses: manuojeda@uma.es (M. Ojeda-Hernández), ipcabrera@uma.es (I.P. Cabrera), pcordero@uma.es (P. Cordero).

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popular base, introduced in [18], is the so-called canonical base, Duquenne–Guigues base or stem base, which is not only non-redundant but also minimal in terms of their cardinality. In the variety of applications mentioned above, bases of implications are used as a knowledge basis that several other algorithms work with. Thus, the smallest a basis or the less redundant a basis is, without loss of information, the more efficient an algorithm will be. This is key for the efficiency of the algorithms built on this theory.

The canonical basis is characterized by having pseudo-closed sets, also known as pseudo-intents [19], as premises of the implications. A study of computational aspects over pseudo-closed sets can be found in [20]. There are two equivalent ways of defining the concept of pseudo-closedness: one is recursive and the other one is based on the notion of quasi-closedness [21]. It is in the generalization of this latter notion that we focus our current work on.

What has been described so far has been widely studied in the literature. Nevertheless, many of these issues remain as open problems in a graduated or fuzzy framework. A wide variety of generalizations to the fuzzy framework of the notion of implication (and logics for reasoning about them) can be found in the literature, see for example [22]. In [23], the authors include a general framework for these generalizations. In [24] the author proposes a generalized notion of pseudo-closedness which ensures that the constructed bases are complete, but are redundant, in general. In [25], it is proved that, in the finite case, it is possible to select a system of pseudo-closed elements in order to obtain a non-redundant basis. All these results on pseudo-closed elements for the fuzzy case have been obtained by using a recursive definition of pseudo-closedness. As we have already mentioned, in the classical case, there exists an equivalent definition based on the notion of quasi-closed element. In this paper, we aim to generalize such notion to the fuzzy framework, which, in the short term, may provide with an alternative definition of pseudo-closed element, as a starting point for a new approach in the study of bases in a fuzzy environment.

As stated in [25], "clearly, in the graded setting, the topics related to non-redundancy and minimality of bases are considerably more involved than in the classic setting and further investigation focused on theory, algorithms, and experiments is needed".

The paper is organized as follows. Section 2 presents the basic notions needed to be able to follow the rest of the paper. Specifically, basic definitions and results are presented on residuated lattices, fuzzy posets, complete fuzzy lattices and closure operators on these structures. In Section 3 we present a discussion of four possible generalizations of the definition of quasi-closed element and select the most general of them. In Section 4, we define it in the graded style by defining the degree to which an element is quasi-closed and we show the good properties of this definition when the underlying structure is a complete residuated lattice.

2. Preliminaries

In this section we present the framework to which we are going to generalize the notion of quasi-closed element, which has been chosen with the idea of being as general as possible and thus have a wider range of possible applications. Specifically, we introduce complete residuated lattices [26,27], the notions of fuzzy poset and fuzzy complete lattice, and some basics results that will be needed throughout the paper [28,29].

Throughout this paper, let $\mathbb{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a complete residuated lattice, which is an algebra where

- $(L, \wedge, \vee, 0, 1)$ is a complete lattice with 0 and 1 being the least and the greatest elements of L, respectively,
- $(L, \otimes, 1)$ is a commutative monoid (i.e., $\otimes$ is commutative, associative, and 1 is neutral with respect to $\otimes$), and
- $\otimes$ and $\rightarrow$ satisfy the so-called *adjointness property*: for all $a, b, c \in L$, we have that $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

This structure is utilized in mathematical fuzzy logics and their applications as structures of truth degrees with $\otimes$ and $\rightarrow$ used as truth functions of *fuzzy conjunction* and *fuzzy implication*, respectively [27]. The unit interval with the Łukasiewicz, Gödel and Goguen pairs of t-norms and implications are examples of residuated complete lattices.

Working with residuated lattices it is usual to consider a *negation* in $\mathbb{L}$ as the antitone mapping $\neg : L \rightarrow L$ defined by $\neg a = a \rightarrow 0$.

Among the properties of complete residuated lattices, we summarize only those that will be used:

$$a \leq b \text{ implies } a \otimes c \leq b \otimes c, \quad \text{for all } a, b, c \in L \tag{1}$$

$$a \leq b \text{ implies } c \rightarrow a \leq c \rightarrow b, \quad \text{for all } a, b, c \in L \tag{2}$$

$$a \leq b \text{ implies } b \rightarrow c \leq a \rightarrow c, \quad \text{for all } a, b, c \in L \tag{3}$$

$$a \rightarrow b = 1 \text{ iff } a \leq b, \quad \text{for all } a, b \in L \tag{4}$$

$$a \otimes \bigvee_{b \in B} b = \bigvee_{b \in B} (a \otimes b), \quad \text{for all } a \in L \text{ and } B \subseteq L \tag{5}$$

$$\Big(\bigvee_{b \in B} b\Big) \rightarrow a = \bigwedge_{b \in B} (b \rightarrow a), \quad \text{for all } a \in L \text{ and } B \subseteq L \tag{6}$$

$$a \otimes \neg a = 0, \quad \text{for all } a \in L \tag{7}$$

Let L^U be the set of $\mathbb{L}$-sets, or fuzzy sets, on the universe U. Operations with $\mathbb{L}$-sets are defined element-wise. For instance, $A \cup B \in L^U$ is defined as $(A \cup B)(u) = A(u) \vee B(u)$ for all $u \in U$. In addition, given $\alpha \in L$ the α-cut of an $\mathbb{L}$-set A is defined as $A^\alpha = \{u \in U : A(u) \geq \alpha\}$.

Binary $\mathbb{L}$-relations (binary fuzzy relations) on a set U can be thought of as $\mathbb{L}$-sets on the universe $U \times U$. That is, a binary $\mathbb{L}$-relation on U is a mapping $\rho \in L^{U\times U}$ assigning to each $x, y \in U$ a truth degree $\rho(x, y) \in L$ (a degree to which x and y are related by ρ). The composition of binary $\mathbb{L}$-relations $\rho_1, \rho_2 \in L^{U\times U}$ is a binary $\mathbb{L}$-relation $\rho_1 \circ \rho_2$ defined by $(\rho_1 \circ \rho_2)(x, z) = \bigvee_{y\in U}(\rho_1(x, y) \otimes \rho_2(y, z))$.

For ρ being a binary $\mathbb{L}$-relation in U, we say that

- ρ is *reflexive* if $\rho(x, x) = 1$ for all $x \in U$.
- ρ is *symmetric* if $\rho(x, y) = \rho(y, x)$ for all $x, y \in U$.
- ρ is *antisymmetric* if $\rho(x, y) \otimes \rho(y, x) = 1$ implies $x = y$ for all $x, y \in U$.
- ρ is *transitive* if $\rho(x, y) \otimes \rho(y, z) \leq \rho(x, z)$ for all $x, y, z \in U$.

Definition 1. Given a non-empty set A and a binary $\mathbb{L}$-relation ρ on A, the pair $\mathbb{A} = (A, \rho)$ is said to be a *fuzzy poset* if ρ is a *fuzzy order*, i.e. if ρ is reflexive, antisymmetric and transitive.

As in the crisp case, any order $\mathbb{L}$-relation implicitly defines an equivalence $\mathbb{L}$-relation that is called *symmetric kernel relation*. In the fuzzy case, this equivalence $\mathbb{L}$-relation usually replaces the notion of equality in the fuzzy poset.

Definition 2. Given a fuzzy poset $\mathbb{A} = (A, \rho)$, the *symmetric kernel relation* is defined as $\approx : A \times A \to L$ where $(a \approx b) = \rho(a, b) \wedge \rho(b, a)$ for all $a, b \in A$.

Proposition 3. *Given a fuzzy poset $\mathbb{A} = (A, \rho)$, the symmetric kernel relation $\approx$ is a fuzzy equivalence relation, that is, it is a reflexive, symmetric and transitive fuzzy relation.*

A usual way to define fuzzy algebras is to consider as an underlying structure a pair which consists of a set and a tolerance or equivalence relation on it.Thus, an alternative definition of fuzzy poset that can be found in the literature [30] is given by a tuple $(A, \approx, \rho)$ where $\approx$ is a fuzzy equivalence relation on A and ρ is a fuzzy order that is compatible with $\approx$. In [31] it is shown that both definitions of fuzzy poset are equivalent.

To present the notion of fuzzy lattice we need to generalize those of upper (lower) bound and supremum (infimum).

Definition 4. Given a fuzzy poset $\mathbb{A} = (A, \rho)$ and a fuzzy set $X \in L^A$, we define the *up-cone* X and the *down-cone* of X, respectively, as the fuzzy sets $X^\rho, X_\rho \in L^A$ where, for all $a \in A$,

$$X^\rho(a) = \bigwedge_{x\in A}\big(X(x) \to \rho(x, a)\big) \qquad \text{and} \qquad X_\rho(a) = \bigwedge_{x\in A}\big(X(x) \to \rho(a, x)\big)$$

Thus, $X^\rho(a)$ and $X_\rho(a)$ can be seen as the degree to which a is an upper bound and lower bound of X, respectively.

Definition 5. Let $\mathbb{A} = (A, \rho)$ be a fuzzy poset and $X \in L^A$. An element $a \in A$ is said to be *supremum* (resp. *infimum*) of X if the following conditions hold:

1. $X^\rho(a) = 1$ (resp. $X_\rho(a) = 1$).
2. For all $x \in A$, $X^\rho(x) \leq \rho(a, x)$ (resp. $X_\rho(x) \leq \rho(x, a)$).

Theorem 6. *Let $\mathbb{A} = (A, \rho)$ be a fuzzy poset and $X \in L^A$. An element $a \in A$ is supremum (resp. infimum) of X if and only if*

$$\rho(a, x) = X^\rho(x) \qquad \text{(resp. } \rho(x, a) = X_\rho(x)\text{).}$$

Proof. If a is supremum of X, Condition 2 in the definition ensures $X^\rho(x) \leq \rho(a, x)$. By Condition 1, it holds that $1 = \bigwedge_{x\in A}\big(X(x) \to \rho(x, a)\big)$. Therefore, for all $y \in A$, we have $X(y) \to \rho(y, a) = 1$, i.e. $X(y) \leq \rho(y, a)$. By (1) and transitivity, $X(y) \otimes \rho(a, x) \leq \rho(y, a) \otimes \rho(a, x) \leq \rho(y, x)$. Finally, by the adjointness property, $\rho(a, x) \leq \bigwedge_{y\in A}\big(X(y) \to \rho(y, x)\big) = X^\rho(x)$.

The reciprocal is straightforward. Analogously, the characterization for the infimum is proved. □

It is not difficult to see that, if a supremum (resp. infimum) of X exists, it is unique. We will denote it by $\bigsqcup X$ (resp. $\bigsqcap X$).

Definition 7 (*[28]*)**.** We say that a fuzzy poset (A, ρ) is a complete fuzzy lattice if every fuzzy subset $X \in L^A$ has supremum and infimum.

Notice that, if (A, ρ) is a fuzzy poset (resp. complete fuzzy lattice) and ρ^1 is the 1-cut of ρ, then (A, ρ^1) is a poset (resp. complete lattice).

Corollary 8. *Let (A, ρ) be a complete fuzzy lattice. For all $a, b, c \in A$,*

$$\rho(a \sqcup b, c) = \rho(a, c) \wedge \rho(b, c) \quad \text{and} \quad \rho(a, b \sqcap c) = \rho(a, b) \wedge \rho(a, c).$$

We conclude this section with the usual definition of closure operator on a fuzzy poset.

Definition 9. Given a fuzzy poset $\mathbb{A} = (A, \rho)$, a mapping $\mathrm{c}: A \to A$ is said to be a *closure operator* on $\mathbb{A}$ if the following conditions hold:

1. $\rho(a, b) \leq \rho(\mathrm{c}(a), \mathrm{c}(b))$, for all $a, b \in A$ (isotony)
2. $\rho(a, \mathrm{c}(a)) = 1$, for all $a \in A$ (inflationarity)
3. $\rho(\mathrm{c}(\mathrm{c}(a)), \mathrm{c}(a)) = 1$, for all $a \in A$ (idempotency)

An element $q \in A$ is said to be *closed* for c if $\rho(\mathrm{c}(q), q) = 1$.

Notice that, if q is a closed element then, $\big(q \approx \mathrm{c}(q)\big) = \rho(q, \mathrm{c}(q)) \wedge \rho(\mathrm{c}(q), q) = 1$, and, for all $a \in A$, the element $\mathrm{c}(a)$ is closed.

Definition 10. Let $\mathrm{c}: A \to A$ be a closure operator on a fuzzy poset (A, ρ) and X be an $\mathbb{L}$-subset of A. The closure of X wrt c is the $\mathbb{L}$-set defined by

$$\mathrm{c}(X)(a) = \bigvee_{x \in \mathrm{c}^{-1}(a)} X(x), \quad \text{for all } a \in A.$$

3. Generalizing the notion of quasi-closed element to fuzzy posets

The aim of this section is to analyze possible generalizations to fuzzy posets of the classical notion of quasi-closed element. We begin by recalling the definition in the case of crisp posets [32].

Definition 11. Let $\mathbb{A} = (A, \leq)$ be a (crisp) poset and $\mathrm{c}: A \to A$ be a closure operator on $\mathbb{A}$. An element $q \in A$ is said to be *quasi-closed* (with respect to c) whenever the following condition holds:

(i) for any $a \leq q$, we have $\mathrm{c}(a) \leq q$ or $\mathrm{c}(a) = \mathrm{c}(q)$.

It is straightforward that condition **(i)** of the definition of quasi-closedness can be replaced by any of the following, which are equivalent.

(ii) For any $a < q$, we have $\mathrm{c}(a) \leq q$ or $\mathrm{c}(q) \leq \mathrm{c}(a)$.

(iii) For any $a < q$ such that $\mathrm{c}(a) \neq a$, we have $\mathrm{c}(a) \leq q$ or $\mathrm{c}(q) \leq \mathrm{c}(a)$.

(iv) For any $a < q$ such that $\mathrm{c}(a) < \mathrm{c}(q)$, we have $\mathrm{c}(a) \leq q$.

The generalization for each one these statements to a closure operator c on a fuzzy poset $\mathbb{A} = (A, \rho)$ are given below.

(I) $\rho(a, q) \leq \rho(\mathrm{c}(a), q) \vee (\mathrm{c}(a) \approx \mathrm{c}(q))$, for all $a \in A$.

(II) $\rho(a, q) \leq \rho(\mathrm{c}(a), q) \vee \rho(\mathrm{c}(q), \mathrm{c}(a))$, for all $a \in A$.

(III) $\rho(a, q) \otimes \neg\rho(\mathrm{c}(q), \mathrm{c}(a)) \leq \rho(\mathrm{c}(a), q)$, for all $a \in A$.

(IV) $\rho(a, q) \otimes \neg\rho(q, a) \otimes \rho(\mathrm{c}(a), \mathrm{c}(q)) \otimes \neg\rho(\mathrm{c}(q), \mathrm{c}(a)) \leq \rho(\mathrm{c}(a), q)$, for all $a \in A$.

The question we address in this section is whether these statements are still equivalent or not and, in the latter case, which one of them would be a suitable definition for fuzzy quasi-closed elements.

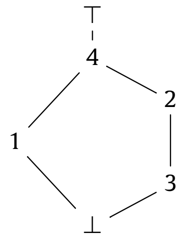
Throughout what remains of this section $\mathbb{A} = (A, \rho)$ is a fuzzy poset and c is a closure operator on $\mathbb{A}$.

Proposition 12. **(I)** *implies* **(II)**.

Proof. It is trivial due to $(\mathrm{c}(a) \approx \mathrm{c}(q)) \leq \rho(\mathrm{c}(q), \mathrm{c}(a))$ and the isotony of $\vee$. □

However, unlike in the classic case, **(I)** and **(II)** are not (in general) equivalent, as the following counterexample shows.

Example 1. Let $L = (\{\bot, \top, 1, 2, 3, 4\}, \wedge, \vee, \otimes, \to, \top, \bot)$ be a residuated lattice with the following Hasse diagram



and product and residuum defined by

$\otimes$	$\bot$	1	2	3	4	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
1	$\bot$	1	$\bot$	$\bot$	1	1
2	$\bot$	$\bot$	3	3	3	2
3	$\bot$	$\bot$	3	3	3	3
4	$\bot$	1	3	3	4	4
$\top$	$\bot$	1	2	3	4	$\top$

$\to$	$\bot$	1	2	3	4	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
1	2	$\top$	2	2	$\top$	$\top$
2	1	1	$\top$	4	$\top$	$\top$
3	1	1	$\top$	$\top$	$\top$	$\top$
4	$\bot$	1	2	2	$\top$	$\top$
$\top$	$\bot$	1	2	3	4	$\top$

Consider the set $A = \{p, q, r, s\}$, endowed with an order relation ρ given by the matrix

ρ	p	q	r	s
p	$\top$	2	$\top$	2
q	$\bot$	$\top$	1	$\top$
r	$\bot$	3	$\top$	2
s	$\bot$	$\bot$	1	$\top$

This relation is obviously reflexive and antisymmetric; let us check for transitivity,

$$\begin{pmatrix} \top & 2 & \top & 2 \\ \bot & \top & 1 & \top \\ \bot & 3 & \top & 2 \\ \bot & \bot & 1 & \top \end{pmatrix} \begin{pmatrix} \top & 2 & \top & 2 \\ \bot & \top & 1 & \top \\ \bot & 3 & \top & 2 \\ \bot & \bot & 1 & \top \end{pmatrix} = \begin{pmatrix} \top & 2 & \top & 2 \\ \bot & \top & 1 & \top \\ \bot & 3 & \top & 2 \\ \bot & \bot & 1 & \top \end{pmatrix}$$

Since the product of the matrix times itself is less than the original one (coordinatewise) the relation is transitive.

Let us define now a closure operator c on (A, ρ) such that $\mathrm{c}(\bot) = \bot$, $\mathrm{c}(\top) = \top$, $\mathrm{c}(p) = \mathrm{c}(r) = r$ and $\mathrm{c}(q) = \mathrm{c}(s) = s$. It is only a matter of calculation to check that the element q meets condition **(II)** and yet does not meet condition **(I)** because

$$\rho(p, q) = 2 \not\leq \rho(r, q) \vee (\rho(s, r) \wedge \rho(r, s)) = 3 \vee (2 \wedge 1) = 3 \vee \bot = 3.$$

The following theorem ensures that **(I)** and **(II)** are equivalent whenever the underlying complete residuated lattice is distributive, for instance, in the case of the unit interval. Notice that we have used a non-distributive lattice structure to provide the above counterexample.

Theorem 13. *If $\mathbb{L}$ is distributive,* **(I)** *and* **(II)** *are equivalent.*

Proof. Proposition 12 ensures that **(I)** implies **(II)**. Let us check the converse.

Assume **(II)**, i.e. $\rho(a, q) \leq \rho(\mathrm{c}(a), q) \vee \rho(\mathrm{c}(q), \mathrm{c}(a))$, for all $a \in A$. By isotony of c we get

$$\rho(a, q) \leq \rho(\mathrm{c}(a), \mathrm{c}(q)) \leq \rho(\mathrm{c}(a), q) \vee \rho(\mathrm{c}(a), \mathrm{c}(q)) \tag{8}$$

Finally, from **(II)** and (8), and applying distributivity, we have that

$$\begin{aligned} \rho(a, q) &\leq \left[\rho(\mathrm{c}(a), q) \vee \rho(\mathrm{c}(q), \mathrm{c}(a))\right] \wedge \left[\rho(\mathrm{c}(a), q) \vee \rho(\mathrm{c}(a), \mathrm{c}(q))\right] \\ &= \rho(\mathrm{c}(a), q) \vee (\mathrm{c}(a) \approx \mathrm{c}(q)). \quad \square \end{aligned}$$

Proposition 14. **(II)** *implies* **(III)**

Proof. Assume **(II)**, i.e. $\rho(a, q) \leq \rho(\mathrm{c}(a), q) \vee \rho(\mathrm{c}(q), \mathrm{c}(a))$, for all $a \in A$.

Multiplying by $\neg\rho(\mathrm{c}(q), \mathrm{c}(a))$ on both sides we get,

$$\rho(a, q) \otimes \neg\rho(\mathrm{c}(q), \mathrm{c}(a)) \leq (\rho(\mathrm{c}(a), q) \vee \rho(\mathrm{c}(q), \mathrm{c}(a))) \otimes \neg\rho(\mathrm{c}(q), \mathrm{c}(a))$$

Applying the distributivity of $\otimes$ over $\vee$, (5)

$$\rho(a,q)\otimes\neg\rho(\mathrm{c}(q),\mathrm{c}(a))\leq[\rho(\mathrm{c}(a),q)\otimes\neg\rho(\mathrm{c}(q),\mathrm{c}(a))]\vee[\rho(\mathrm{c}(q),\mathrm{c}(a))\otimes\neg\rho(\mathrm{c}(q),\mathrm{c}(a))]$$

Finally, by using (7), we can simplify the expression to

$$\rho(a,q)\otimes\neg\rho(\mathrm{c}(q),\mathrm{c}(a))\leq\rho(\mathrm{c}(a),q)\otimes\neg\rho(\mathrm{c}(q),\mathrm{c}(a))\leq\rho(\mathrm{c}(a),q). \quad \square$$

The reciprocal does not hold as the following example shows.

Example 2. Let $\mathbb{L}$ be the unit interval $[0,1]$ with the Łukasiewicz product. Consider the set $A=\{p,q,r,s\}$, and the fuzzy order ρ defined by the following matrix:

ρ	p	q	r	s
p	1	0.5	1	0.5
q	0	1	0.4	1
r	0	0.3	1	0.5
s	0	0	0.4	1

It is clearly antisymmetric and reflexive. Transitivity is shown below.

$$\begin{pmatrix}1&0.5&1&0.5\\0&1&0.4&1\\0&0.3&1&0.5\\0&0&0.4&1\end{pmatrix}\begin{pmatrix}1&0.5&1&0.5\\0&1&0.4&1\\0&0.3&1&0.5\\0&0&0.4&1\end{pmatrix}=\begin{pmatrix}1&0.5&1&0.5\\0&1&0.4&1\\0&0.3&1&0.5\\0&0&0.4&1\end{pmatrix}$$

Now, consider the mapping $\mathrm{c}\colon A\to A$ defined as $\mathrm{c}(p)=r,\ \mathrm{c}(q)=s,\ \mathrm{c}(r)=r,\ \mathrm{c}(s)=s$, which is a closure operator on (A,ρ).

It is easy now to check that q satisfies Condition **(III)**, i.e. for all $x\in A$, we have $\rho(x,q)\otimes\neg\rho(\mathrm{c}(q),\mathrm{c}(x))\leq\rho(\mathrm{c}(x),q)$, whilst Condition **(II)** does not hold:

$$0.5=\rho(p,q)\not\leq\rho(\mathrm{c}(p),q)\vee\rho(\mathrm{c}(q),\mathrm{c}(p))=\rho(r,q)\vee\rho(s,r)=0.3\vee0.4=0.4.$$

Proposition 15. **(III)** *implies* **(IV)**.

Proof. This is fairly straightforward. Assume **(III)** holds, then,

$$\begin{aligned}\rho(a,q)\otimes\neg\rho(q,a)\otimes\rho(\mathrm{c}(a),\mathrm{c}(q))\otimes\neg\rho(\mathrm{c}(q),\mathrm{c}(a))&\leq\rho(a,q)\otimes\neg\rho(\mathrm{c}(q),\mathrm{c}(a))\\&\leq\rho(\mathrm{c}(a),q)\end{aligned}$$

simply using $x\otimes y\leq x$ for every $x,y\in L$. Again, this argument is valid for any $a\in A$. $\square$

However, the converse does not hold as the following example shows.

Example 3. Consider for this example the set $A=\{p,q,r,s\}$, the closure operator $\mathrm{c}\colon A\to A$ defined as $\mathrm{c}(p)=r,\ \mathrm{c}(q)=s,\ \mathrm{c}(r)=r,\ \mathrm{c}(s)=s$ and let the lattice structure be $[0,1]$ with the Łukasiewicz product same as in the previous example. The order ρ will now be defined as the following matrix,

ρ	p	q	r	s
p	1	1	1	1
q	0.5	1	0.6	1
r	0	0	1	1
s	0	0	0.5	1

This indeed defines a fuzzy order since it is antisymmetric and reflexive. Transitivity is shown below.

$$\begin{pmatrix}1&1&1&1\\0.5&1&0.6&1\\0&0&1&1\\0&0&0.5&1\end{pmatrix}\begin{pmatrix}1&1&1&1\\0.5&1&0.6&1\\0&0&1&1\\0&0&0.5&1\end{pmatrix}=\begin{pmatrix}1&1&1&1\\0.5&1&0.6&1\\0&0&1&1\\0&0&0.5&1\end{pmatrix}$$

It is easy now to check **(IV)** holds, but **(III)** is not because

$$\begin{aligned}0.5=1\otimes\neg0.5=\rho(p,q)\otimes\neg\rho(s,r)=\rho(p,q)\otimes\neg\rho(\mathrm{c}(q),\mathrm{c}(p))\\\not\leq\rho(\mathrm{c}(p),q)=\rho(r,q)=0.\end{aligned}$$

Proposition 16. *Given a closure operator* c *on a (crisp) poset, any quasi-closed element with respect to* c *satisfies condition* **(IV)**.

Proof. Let c be a closure operator on a (crisp) poset $(A, \leq)$ and consider the fuzzy order ρ defined by the characteristic mapping of $\leq$, i.e. $\rho(a, b) = 1$ iff $a \leq b$ and $\rho(a, b) = 0$ otherwise. The unique case in which

$$\rho(a, q) \otimes \neg\rho(q, a) \otimes \rho(\mathrm{c}(a), \mathrm{c}(q)) \otimes \neg\rho(\mathrm{c}(q), \mathrm{c}(a)) \neq 0$$

is when $a < q$ and $\mathrm{c}(a) < \mathrm{c}(q)$. In this case, if q is quasi-closed with respect to c, we have that $\mathrm{c}(a) \leq q$ and, therefore, $\rho(\mathrm{c}(a), q) = 1$. $\square$

Once we have established the relations among the properties described in this section, we will adopt the most general of them as the definition of quasi-closed element wrt a closure operator on a fuzzy poset.

Definition 17. Given a closure operator c on a fuzzy poset (A, ρ), an element $q \in A$ is said to be *quasi-closed* (with respect to c) if, for all $a \in A$,

$$\rho(a, q) \otimes \neg\rho(q, a) \otimes \rho(\mathrm{c}(a), \mathrm{c}(q)) \otimes \neg\rho(\mathrm{c}(q), \mathrm{c}(a)) \leq \rho(\mathrm{c}(a), q).$$

Theorem 18.

1. *Every closed element is quasi-closed.*
2. *There exist quasi-closed elements which are not closed.*

Proof. Given a closure operator c on fuzzy poset (A, ρ), an element $q \in A$ is closed if $\rho(\mathrm{c}(q), q) = 1$. Since c is isotone, we have $\rho(a, q) \leq \rho(\mathrm{c}(a), \mathrm{c}(q))$ and, by transitivity of ρ and the properties of the supremum, we obtain that

$$\rho(a, q) \leq \rho(\mathrm{c}(a), \mathrm{c}(q)) \otimes \rho(\mathrm{c}(q), q) \leq \rho(\mathrm{c}(a), q) \leq \rho(\mathrm{c}(a), q) \vee (\mathrm{c}(a) \approx \mathrm{c}(q)).$$

This argument is valid for any $a \in A$, hence q satisfies condition **(I)**. Since **(I)** implies **(IV)**, every closed element is quasi-closed.

In order to find a quasi-closed element which is not closed consider the set $A = \{p, q, r\}$ with the crisp order relation $\rho : A \times A \to \{0, 1\}$

ρ	p	q	r
p	1	1	1
q	0	1	1
r	0	0	1

and the closure operator $\mathrm{c} : A \to A$ defined by $\mathrm{c}(p) = p$ and $\mathrm{c}(q) = \mathrm{c}(r) = r$. Clearly, the element q is not closed since $\rho(\mathrm{c}(q), q) = \rho(r, q) = 0$. Moreover, checking $\rho(x, q) \leq \rho(\mathrm{c}(x), q) \vee \rho(\mathrm{c}(q), \mathrm{c}(x))$ for all x we get the following,

$$1 = \rho(p, q) \leq \rho(\mathrm{c}(p), q) \vee \rho(\mathrm{c}(q), \mathrm{c}(p)) = \rho(p, q) \vee \rho(r, p) = 1 \vee 0 = 1$$
$$1 = \rho(q, q) \leq \rho(\mathrm{c}(q), q) \vee \rho(\mathrm{c}(q), \mathrm{c}(q)) = \rho(r, q) \vee \rho(r, r) = 0 \vee 1 = 1$$
$$0 = \rho(r, q) \leq \rho(\mathrm{c}(r), q) \vee \rho(\mathrm{c}(q), \mathrm{c}(r)) = \rho(r, q) \vee \rho(r, r) = 0 \vee 1 = 1.$$

Hence, q satisfies **(II)**, which implies that q is a quasi-closed element. $\square$

4. Quasi-closed elements in graded setting

In the previous section, quasi-closed elements have been defined in the crisp style in that each element in a given fuzzy poset either is quasi-closed with respect to a closure operator or not. In this section, we define it in the graded style by defining the degree to which an element is quasi-closed.

First, to ease the reading of the definitions and properties, we introduce the following notation.

Notation 1. *Given a closure operator* c *on a fuzzy poset* (A, ρ) *and* $q \in A$*, we use* X_q *to denote the* $\mathbb{L}$*-set with membership function defined as follows:*

$$X_q(a) = \rho(a, q) \otimes \neg\rho(q, a) \otimes \rho(\mathrm{c}(a), \mathrm{c}(q)) \otimes \neg\rho(\mathrm{c}(q), \mathrm{c}(a)), \text{ for each } a \in A.$$

With this notation, an element q is quasi-closed iff $X_q(a) \leq \rho(\mathrm{c}(a), q)$, for all $a \in A$.

Definition 19. Given a closure operator c on a fuzzy poset (A, ρ), for any $q \in A$, we define the *degree in which* q *is quasi-closed* as follows

$$QC(q) = \bigwedge_{x \in A} \left[X_q(x) \to \rho(\mathrm{c}(x), q)\right].$$

The following theorem ensures that both definitions of quasi-closedness, the crisp one introduced in the previous section and the graded one just presented, fit. Specifically, if we consider QC as a fuzzy set, its 1-cut is precisely the set of the quasi-closed elements.

Theorem 20. *Let* c *be a closure operator on a fuzzy poset* (A, ρ) *and* $q \in A$*. Then,* $QC(q) = 1$ *if and only if* q *is quasi-closed.*

Proof. $QC(q) = 1$ if and only if $1 = \bigwedge_{x\in A}\left[X_q(x) \to \rho(\mathsf{c}(x), q)\right]$ which is equivalent to $1 = X_q(x) \to \rho(\mathsf{c}(x), q)$ for all $x \in A$, and by (4) this is equivalent to $X_q(x) \leq \rho(\mathsf{c}(x), q)$ for all $x \in A$, which is the definition of q being quasi-closed. □

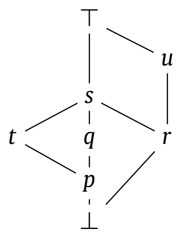
In the classical setting, there is an if-and-only-if condition for a set to be quasi-closed based on an operator that is usually denoted by $^\circ$. We can do an analogous characterization here, even in the graded setting. The definition of this operator depends on the existence of suprema so we need (A, ρ) to be a complete fuzzy lattice.

Definition 21. Let c be a closure operator on a complete fuzzy lattice (A, ρ) and $q \in A$. We define the element q° as follows:

$$q^\circ = q \sqcup \bigsqcup \mathsf{c}(X_q).$$

This is not a quasi-closed element in general, not even in the crisp case.

Example 4. Consider the set $A = \{\bot, p, q, r, s, t, u, \top\}$ with the lattice structure such as the described in the next Hasse diagram and closure operator $\mathsf{c}\colon A \to A$ defined as $\mathsf{c}(\bot) = \bot$, $\mathsf{c}(p) = \mathsf{c}(t) = t$, $\mathsf{c}(q) = \mathsf{c}(s) = \mathsf{c}(\top) = \top$ and $\mathsf{c}(r) = \mathsf{c}(u) = u$.



It is easy to see that $q^\circ = q \sqcup \mathsf{c}(p) = q \sqcup t = s$, but s is not a quasi-closed element since $r \leq s$ and $\mathsf{c}(r) = u \nleq s$ and $\mathsf{c}(r) = u \neq \top = \mathsf{c}(s)$.

Theorem 22. *Let* c *be a closure operator on a complete fuzzy lattice* (A, ρ)*. Then,* $QC(q) = \rho(q^\circ, q)$*, for all* $q \in A$*.*

Proof. From Corollary 8, Theorem 6 and (6),

$$\begin{aligned}\rho(q^\circ, q) &= \rho\left(q \sqcup \bigsqcup \mathsf{c}(X_q), q\right) = \rho(q, q) \wedge \rho\left(\bigsqcup \mathsf{c}(X_q), q\right)\\ &= 1 \wedge \mathsf{c}(X_q)^\rho(q) = \bigwedge_{a\in A}(\mathsf{c}(X_q)(a) \to \rho(a, q))\\ &= \bigwedge_{a\in A}\left(\left(\bigvee_{x\in \mathsf{c}^{-1}(a)} X_q(x)\right) \to \rho(a, q)\right) = \bigwedge_{a\in A}\ \bigwedge_{x\in \mathsf{c}^{-1}(a)}\left(X_q(x) \to \rho(a, q)\right).\end{aligned}$$

If a is not a closed element, then $\mathsf{c}^{-1}(a) = \varnothing$ and $\bigwedge \varnothing = 1$, hence

$$\bigwedge_{a\in A}\ \bigwedge_{x\in \mathsf{c}^{-1}(a)}\left(X_q(x) \to \rho(a, q)\right) = \bigwedge_{s\in \mathsf{c}(A)}\ \bigwedge_{x\in \mathsf{c}^{-1}(s)}\left(X_q(x) \to \rho(s, q)\right).$$

Finally, since $\mathsf{c}^{-1}(\mathsf{c}(A)) = A$, we have that

$$\bigwedge_{s\in \mathsf{c}(A)}\ \bigwedge_{x\in \mathsf{c}^{-1}(s)}\left(X_q(x) \to \rho(\mathsf{c}(x), q)\right) = \bigwedge_{x\in A}\left(X_q(x) \to \rho(\mathsf{c}(x), q)\right) = QC(q). \quad \square$$

Corollary 23. *Let* c *be a closure operator on a complete fuzzy lattice* (A, ρ)*. An element* $q \in A$ *is quasi-closed wrt* c *if and only if* $q^\circ = q$*.*

Proof. If $q^\circ = q$, by reflexivity, $QC(q) = \rho(q^\circ, q) = 1$ and, by Theorem 20, we have that q is a quasi-closed element.

Conversely, if q is a quasi-closed element then $QC(q) = 1$, so $\rho(q^\circ, q) = 1$, hence, since $\rho(q, q^\circ) = 1$, by antisymmetry, we have $q^\circ = q$. □

5. Conclusions and further work

We have analyzed four possible generalizations of the classical notion of the quasi-closed element to the frame of fuzzy posets, analyzed their properties and selected the most general of them. On the other hand, we have extended the definition to graded setting. Finally, we have checked that, when the underlying structure is a complete fuzzy lattice, it extends the classical results that are necessary for its effective use in the search for bases of implications or if-then rules in the fuzzy frame.

In a next step, we will study aspects related to the computability of quasi-closed elements looking for necessary and sufficient conditions to ensure that they can be calculated efficiently. As further work, based on the results obtained here, we will also generalize the notion of pseudo-closed element and compare the definition obtained with the recursive one proposed in [25]. In addition, we will study whether the bases of implications with pseudo-closed premises have the desired properties of completeness, non-redundancy and minimality. We will also consider whether this work can be extended to the multi-adjoint concept lattice framework [33].

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