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Tesis Doctoral

Spaces of analytic functions and operators between them

(Espacios de funciones analíticas y operadores entre ellos)

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HACE CONSTAR:

Que **Noel Merchán** ha realizado bajo su dirección el trabajo de investigación correspondiente a su Tesis Doctoral titulada Espacios de funciones analíticas y operadores entre ellos (Spaces of analytic functions and operators between them).

Revisado el presente trabajo, estimo que puede ser presentado al tribunal que ha de juzgarlo.

Para que así conste a los efectos oportunos, autorizo la presentación de esta Tesis Doctoral en la Universidad de Málaga.

En Málaga, 25 de junio de 2018

Fdo.: Daniel Girela Álvarez.

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Resumen

Esta tesis está dedicada al estudio de ciertos operadores actuando en espacios de funciones analíticas en el disco unidad.

Sea $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ el disco unidad. Sea también $\mathcal{H}ol(\mathbb{D})$ el espacio de todas las funciones analíticas en $\mathbb{D}$ dotado de la topología de la convergencia uniforme en compactos.

Un subespacio X de $\mathcal{H}ol(\mathbb{D})$ puede ser visto como un espacio de sucesiones identificando a una función $f \in X$ con la sucesión de sus coeficientes de Taylor:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \leftrightarrow \quad \{a_n\}_{n=0}^{\infty}.$$

Sea $\mathcal{H}$ la matriz de Hilbert,

$$\mathcal{H} = \left(\frac{1}{n+k+1}\right)_{n,k\geq 0} = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & 1/5 & \cdots \\ 1/3 & 1/4 & 1/5 & 1/6 & \cdots \\ 1/4 & 1/5 & 1/6 & 1/7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

La matriz de Hilbert puede ser considerada como un operador entre espacios de sucesiones. Formalmente, se define su acción como

$$\mathcal{H}\left(\{a_n\}_{n=0}^{\infty}\right) = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & 1/5 & \cdots \\ 1/3 & 1/4 & 1/5 & 1/6 & \cdots \\ 1/4 & 1/5 & 1/6 & 1/7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix},$$

$$\{a_n\}_{n=0}^{\infty} \mapsto \left\{\sum_{k=0}^{\infty} \frac{a_k}{n+k+1}\right\}_{n=0}^{\infty}.$$

De la misma forma, $\mathcal{H}$ puede ser considerado como un operador (al que llamamos el operador de Hilbert) entre espacios de funciones analíticas identificando cada función analítica con la sucesión de sus coeficientes de Taylor.

Si $f(z) = \sum_{n=0}^{\infty} a_n z^n$ entonces

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n,$$

cuando el segundo miembro tenga sentido.

El operador de Hilbert está bien definido en H^1, es acotado en H^p para $1 < p < \infty$ pero no lo es en H^1 o H^∞ [34]. En un artículo reciente [65] Lanucha, Nowak y Pavlović consideran la cuestión de encontrar subespacios de H^1 cuya imagen por el operador $\mathcal{H}$ está contenida en H^1. Dostanić, Jevtić y Vukotić [37] encontraron la norma exacta de $\mathcal{H}$ como operador de H^p en H^p $(1 < p < \infty)$.

Sea μ una medida de Borel positiva en $[0, 1)$ y sea $\{\mu_n\}_{n=0}^{\infty}$ la sucesión de sus momentos: $\mu_n = \int_{[0,1)} t^n \, d\mu(t)$. La matriz de Hilbert puede generalizarse considerando la matriz de Hankel $\mathcal{H}_\mu$ con entradas $(\mu_{n+k})_{n,k\geq 0}$,

$$\mathcal{H}_\mu = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Al igual que anteriormente, la matriz $\mathcal{H}_\mu$ induce formalmente el operador de Hilbert generalizado $\mathcal{H}_\mu$ en espacios de funciones analíticas:

Si $f(z) = \sum_{n=0}^{\infty} a_n z^n$ entonces

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_k \right) z^n,$$

cuando el segundo miembro tenga sentido.

Widom [99, Theorem 3. 1] y Power [89, Theorem 3] (véase también Peller [83, p. 42, Theorem 7. 2]) probaron que $\mathcal{H}_\mu$ es un operador bien definido y acotado actuando de H^2 en sí mismo si y sólo si μ es una medida de Carleson, $\mu([t, 1)) \leq C(1-t)$, $0 < t < 1$.

Galanopoulos y Peláez [48] estudiaron la acción del operador $\mathcal{H}_\mu$ en H^1. Por su parte, Chatzifountas, Girela y Peláez [28] estudiaron $\mathcal{H}_\mu$ como operador de H^p en H^q $(0 < p, q < \infty)$.

Si los pasos tomados a continuación fuesen correctos tendríamos lo siguiente:

Para $f(z) = \sum_{n=0}^{\infty} a_n z^n$,

$$\begin{aligned}
\mathcal{H}_\mu(f)(z) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_k \right) z^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k \int_{[0,1)} t^{n+k} d\mu(t) \right) z^n \\
&= \sum_{k=0}^{\infty} a_k \left(\sum_{n=0}^{\infty} \int_{[0,1)} t^{n+k} z^n d\mu(t) \right) \\
&= \sum_{k=0}^{\infty} a_k \int_{[0,1)} \frac{t^k}{1-tz} d\mu(t) = \int_{[0,1)} \frac{f(t)}{1-tz} d\mu(t).
\end{aligned}$$

Para μ una medida de Borel finita y positiva en $[0,1)$ y $f \in \mathcal{H}ol(\mathbb{D})$ se define

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz} d\mu(t), \quad z \in \mathbb{D},$$

cuando el segundo miembro tenga sentido para todo $z \in \mathbb{D}$ y defina una función analítica en $\mathbb{D}$.

De esto se deduce que los operadores $\mathcal{H}_\mu$ e I_μ están estrechamente relacionados, si f es suficientemente buena $\mathcal{H}_\mu(f)$ e $I_\mu(f)$ están bien definidos y coinciden. En [48] Galanopoulos y Peláez prueban lo siguiente.

Sea μ una medida de Borel positiva en $[0,1)$. Entonces:

(i) El operador I_μ está bien definido en H^1 si y sólo si μ es una medida de Carleson.

(ii) Si μ es una medida de Carleson, entonces el operador $\mathcal{H}_\mu$ está también bien definido en H^1 y además,

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \text{para toda } f \in H^1.$$

(iii) El operador I_μ es acotado de H^1 en sí mismo si y sólo si μ es una medida 1-logarítmica 1-Carleson.

Más tarde en [28] Chatzifountas, Girela y Peláez probaron lo siguiente.

Supongamos que $1 < p < \infty$ y sea μ una medida de Borel positiva en $[0,1)$. Entonces:

(i) El operador I_μ está bien definido en H^p si y sólo si μ es una medida 1-Carleson para H^p.

(ii) Si μ es una medida 1-Carleson para H^p, entonces el operador $\mathcal{H}_\mu$ está también bien definido en H^p y además,

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \text{para toda } f \in H^p.$$

(iii) El operador I_μ es acotado de H^p en sí mismo si y sólo si μ es una medida de Carleson.

El Capítulo 2 está dedicado al estudio de los operadores $\mathcal{H}_\mu$ e I_μ en distintos espacios de funciones analíticas. Empezamos extendiendo los resultados anteriores a algunos espacios conformemente invariantes como el espacio de Bloch, $BMOA$, los espacios de Besov o las clases Q_s. Todos estos resultados se encuentran en el trabajo conjunto con Girela [54].

En el primer resultado caracterizamos las medidas μ para las que el operador I_μ está bien definido o está acotado en $BMOA$ y en el espacio de Bloch.

Para una medida μ de Borel positiva en $[0,1)$ tenemos que el operador I_μ está bien definido en cualquiera de estos dos espacios si y sólo si

$$\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty,$$

y si esto ocurre entonces las siguientes tres condiciones son equivalentes:

(i) La medida μ es una medida 1-logarítmica 1-Carleson.

(ii) El operador I_μ es acotado de $\mathcal{B}$ en $BMOA$.

(iii) El operador I_μ es acotado de $BMOA$ en sí mismo.

Además, si se satisface (i) entonces el operador $\mathcal{H}_\mu$ está bien definido en el espacio de Bloch y

$$\mathcal{H}_\mu(f) \,=\, I_\mu(f), \quad \text{para toda } f \in \mathcal{B},$$

por lo que el operador $\mathcal{H}_\mu$ es acotado de $\mathcal{B}$ en $BMOA$.

También tenemos el siguiente resultado sobre compacidad:

Sea μ una medida de Borel positiva en $[0,1)$ con $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) \,<\, \infty$. Si μ es una medida *vanishing* 1-logarítmica 1-Carleson entonces:

(i) El operador I_μ es un operador compacto de $\mathcal{B}$ en $BMOA$.

(ii) El operador I_μ es un operador compacto de $BMOA$ en sí mismo.

Las condiciones que debe cumplir una medida μ para que el operador I_μ esté bien definido o esté acotado en $BMOA$ y en el espacio de Bloch siguen siendo ciertas para todos los espacios Q_s con $s > 0$. Tenemos lo siguiente:

Para cualquier $s \in (0,\infty)$ y para una medida de Borel positiva μ, el operador I_μ está bien definido en Q_s si y sólo si

$$\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) \,<\, \infty,$$

y si esto ocurre entonces las siguientes condiciones son equivalentes:

(i) La medida μ es una medida 1-logarítmica 1-Carleson.

(ii) Para cualquier $s \in (0,\infty)$, el operador I_μ es acotado de Q_s en $BMOA$.

Además, si se satisface (i), se tiene que para cualquier $s \in (0,\infty)$ el operador $\mathcal{H}_\mu$ coincide con I_μ en Q_s y por tanto, también es acotado de Q_s en $BMOA$.

Hemos estudiado también el operador I_μ actuando en los espacios de Besov. Como es usual, para $1 < p < \infty$, p' denotará al exponente conjugado de p, es decir, $\frac{1}{p} + \frac{1}{p'} = 1$. Hemos probado los siguientes resultados.

Sea $1 < p < \infty$ y sea μ una medida de Borel positiva en $[0,1)$. Se tiene que:

(i) Si $\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{1/p'} d\mu(t) < \infty$, entonces el operador I_μ está bien definido en B^p.

(ii) Si el operador I_μ está bien definido en B^p, entonces $\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{\gamma} d\mu(t) < \infty$ para todo $\gamma < \frac{1}{p'}$.

(iii) Si μ es una medida $1/p'$-logarítmica 1-Carleson entonces el operador I_μ es acotado de B^p en $BMOA$.

(iv) Si μ es una medida *vanishing* $1/p'$-logarítmica 1-Carleson entonces el operador I_μ es compacto de B^p en $BMOA$.

Trabajando directamente con el operador $\mathcal{H}_\mu$ hemos obtenido lo siguiente:

Si μ es una medida de Borel finita y positiva en $[0,1)$ entonces:

(i) Si $1 < p \leq 2$ y $\sum_{k=1}^{\infty} \frac{\mu_k^{p'}}{k} < \infty$, entonces el operador $\mathcal{H}_\mu$ está bien definido en B^p.

(ii) Si $2 < p < \infty$ y $\sum_{k=1}^{\infty} \frac{\mu_k^{p'}}{k^{p'/p}} < \infty$, entonces el operador $\mathcal{H}_\mu$ está bien definido en B^p.

En [16], Bao y Wulan probaron que existen medidas de Borel positivas μ en $[0,1)$ que son medidas de Carleson pero para las que ocurre que $\mathcal{H}_\mu(B^2) \not\subset B^2$. También probaron que si $\mathcal{H}_\mu$ es un operador acotado de B^2 en sí mismo entonces μ es una medida de Carleson. Estos resultados los mejoramos y los extendemos para todos los espacios B^p con $1 < p < \infty$.

Si $1 < p < \infty$ entonces:

(i) Si $0 < \beta \leq \frac{1}{p}$ entonces existe una medida de Borel positiva μ en $[0,1)$ que es β-logarítmica 1-Carleson pero tal que $\mathcal{H}_\mu(B^p) \not\subset B^p$.

(ii) Si μ es una medida de Borel positiva en $[0,1)$ tal que el operador $\mathcal{H}_\mu$ es acotado de B^p en sí mismo entonces μ es una medida $1/p'$-logarítmica 1-Carleson [55].

(iii) Si $\gamma > 1$ y μ es una medida de Borel positiva en $[0,1)$ que es γ-logarítmica 1-Carleson entonces el operador $\mathcal{H}_\mu$ es acotado de B^p en sí mismo.

Más tarde centramos nuestra atención en la acción de $\mathcal{H}_\mu$ en los espacios de Hardy. Los resultados mencionados anteriormente de Galanopoulos y Peláez y de Chatzifountas, Girela y Peláez implican lo siguiente.

(i) Si μ es una medida de Carleson, entonces el operador $\mathcal{H}_\mu$ es acotado de H^1 en sí mismo si y sólo si μ es una medida 1-logarítmica 1-Carleson.

(ii) Si $1 < p < \infty$ y μ es una medida 1-Carleson para H^p, entonces el operador $\mathcal{H}_\mu$ es un operador acotado de H^p en sí mismo si y sólo si μ es una medida de Carleson.

Estos resultados no cierran completamente la cuestión sobre la caracterización de las medidas μ para las que $\mathcal{H}_\mu$ es un operador acotado de H^p en sí mismo. En efecto, en estos trabajos los autores sólo consideran medidas 1-Carleson para H^p. En principio, podría existir una medida μ que no fuera 1-Carleson para H^p pero para la que el operador $\mathcal{H}_\mu$ estuviera bien definido y fuese acotado en H^p. Hemos probado que éste no es el caso. De hecho, se ha probado el siguiente resultado:

Sea μ una medida de Borel positiva en $[0, 1)$.

(i) El operador $\mathcal{H}_\mu$ es acotado de H^1 en sí mismo si y sólo si μ es una medida 1-logarítmica 1-Carleson.

(ii) Si $1 < p < \infty$ entonces el operador $\mathcal{H}_\mu$ es acotado de H^p en sí mismo si y sólo si μ es una medida de Carleson.

En [28] el parámetro p se tomaba finito. También damos un resultado para el caso $p = \infty$.

Sea μ una medida de Borel positiva en $[0, 1)$. Las siguientes condiciones son equivalentes.

(i) $\int_{[0,1)} \frac{d\mu(t)}{1-t} < \infty$.

(ii) $\sum_{n=0}^{\infty} \mu_n < \infty$.

(iii) El operador I_μ es acotado de H^∞ en sí mismo.

(iv) El operador $\mathcal{H}_\mu$ es acotado de H^∞ en sí mismo.

Estos resultados sobre la acción de $\mathcal{H}_\mu$ en los espacios de Hardy han sido publicados en [55] y están contenidos en la Sección 2.1 de esta tesis.

En la Sección 2.2 mencionamos el siguiente resultado de Galanopoulos y Peláez.

Sea μ una medida Borel positiva en $[0,1)$. Si μ es una medida de Carleson entonces $\mathcal{H}_\mu(H^1) \subset \mathscr{C}$, donde $\mathscr{C}$ es el espacio de las funciones holomorfas en el disco que son la integral de Cauchy de una medida de Borel compleja en $\partial\mathbb{D}$.

Llegados a este punto nos preguntamos qué puede decirse acerca de la imagen $\mathcal{H}_\mu(H^1)$ de H^1 bajo la acción del operador $\mathcal{H}_\mu$ si la medida μ es 1-logarítmica 1-Carleson en $[0,1)$.

Con respecto a esta cuestión, observamos que es fácil de ver que el espacio de tipo Dirichlet $\mathcal{D}_0^1$ está incluido en H^1. Probaremos que si μ es una medida 1-logarítmica 1-Carleson en $[0,1)$ entonces $\mathcal{H}_\mu(H^1)$ está contenido en el espacio $\mathcal{D}_0^1$. De hecho, hemos probado un resultado más potente.

Sea μ una medida positiva de Borel en $[0,1)$. Entonces, las siguientes condiciones son equivalentes:

(i) μ es una medida 1-logarítmica 1-Carleson.

(ii) $\mathcal{H}_\mu$ es un operador acotado de H^1 en sí mismo.

(iii) $\mathcal{H}_\mu$ es un operador acotado de H^1 en $\mathcal{D}_0^1$.

(iv) $\mathcal{H}_\mu$ es un operador acotado de $\mathcal{D}_0^1$ en $\mathcal{D}_0^1$.

Hay un salto entre las condiciones de los dos últimos resultados, por lo que es natural estudiar el rango de H^1 bajo la acción de $\mathcal{H}_\mu$ cuando μ es una medida α-logarítmica 1-Carleson con $0 < \alpha < 1$. Probaremos el siguiente resultado.

Sea μ una medida positiva de Borel en $[0,1)$. Supongamos que $0 < \alpha < 1$ y que μ es una medida α-logarítmica 1-Carleson. Entonces $\mathcal{H}_\mu$ aplica el espacio H^1 en el espacio $\mathcal{D}^1(\log^{\alpha-1})$ definido como:

$$\mathcal{D}^1(\log^{\alpha-1}) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)| \left(\log \frac{2}{1-|z|} \right)^{\alpha-1} dA(z) < \infty \right\}.$$

Todos estos resultados se encuentran en un trabajo conjunto con Girela [56].

Enunciamos anteriormente un resultado acerca de la acotación del operador $\mathcal{H}_\mu$ actuando de Q_s (con $0 < s < \infty$) en $BMOA$. Es natural buscar una caracterización para las medidas μ tales que I_μ y/o $\mathcal{H}_\mu$ es acotado de $\mathcal{B}$ en sí mismo o más generalmente de Q_s en sí mismo para cualquier $s > 0$. Tenemos el siguiente resultado.

Sea μ una medida de Borel positiva en $[0, 1)$. Las siguientes condiciones son equivalentes.

(i) El operador I_μ es acotado de Q_s en sí mismo para algún $s > 0$.

(ii) El operador I_μ es acotado de Q_s en sí mismo para todo $s > 0$.

(iii) El operador $\mathcal{H}_\mu$ es acotado de Q_s en sí mismo para algún $s > 0$.

(iv) El operador $\mathcal{H}_\mu$ es acotado de Q_s en sí mismo para todo $s > 0$.

(v) La medida μ es 1-logarítmica 1-Carleson.

De hecho somos capaces de probar un resultado más fuerte que no hace distinciones entre diferentes espacios Q_s.

Sea μ una medida de Borel positiva en $[0, 1)$ y sean $0 < s_1, s_2 < \infty$. Las siguientes condiciones son equivalentes:

(i) El operador I_μ está bien definido en Q_{s_1} y, además, es acotado de Q_{s_1} en Q_{s_2}.

(ii) El operador $\mathcal{H}_\mu$ está bien definido en Q_{s_1} y, además, es acotado de Q_{s_1} en Q_{s_2}.

(iii) La medida μ es 1-logarítmica 1-Carleson.

Este resultado se deduce de un teorema que hemos probado más general en el que aparece el espacio de Lipschitz en media $\Lambda^2_{1/2}$.

Sea μ una medida de Borel positiva en $[0, 1)$ y sea X un espacio de Banach de funciones analíticas en $\mathbb{D}$ con $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. Las siguientes condiciones son equivalentes:

(i) El operador I_μ está bien definido en X y, además, es acotado de X en $\Lambda^2_{1/2}$.

(ii) El operador $\mathcal{H}_\mu$ está bien definido en X y, además, es acotado de X en $\Lambda^2_{1/2}$.

(iii) La medida μ es 1-logarítmica 1-Carleson.

(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) \; = \; \mathrm{O}\left(\frac{1}{n}\right)$.

Todos estos resultados se encuentran publicados en [55] y están incluidos en la Sección 2.3 de la tesis.

La Sección 2.4 está dedicada a extender el resultado anterior a una clase más general de espacios de Lipschitz en media. Los resultados de esta sección se encuentran en [72].

En primer lugar, mejoraremos el último resultado cambiando $\Lambda^2_{1/2}$ por $\Lambda^p_{1/p}$ para cualquier $p > 1$.

Supongamos que $1 < p < \infty$. Sea μ una medida de Borel positiva en $[0, 1)$ y sea X un espacio de Banach de funciones analíticas en $\mathbb{D}$ con $\Lambda^p_{1/p} \subset X \subset \mathcal{B}$. Las siguientes condiciones son equivalentes.

(i) El operador $\mathcal{H}_\mu$ está bien definido en X y, además, es acotado de X en el espacio de Bloch $\mathcal{B}$.

(ii) El operador $\mathcal{H}_\mu$ está bien definido en X y, además, es acotado de X en $\Lambda^p_{1/p}$.

(iii) La medida μ es 1-logarítmica 1-Carleson.

(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) \; = \; \mathrm{O}\left(\frac{1}{n}\right)$.

Los espacios $\Lambda^p_{1/p}$ están contenidos en $BMOA$. El siguiente paso es estudiar el operador $\mathcal{H}_\mu$ actuando en espacios de Lipschitz en media generalizados no contenidos en $BMOA$. Trabajamos con los espacios $\Lambda(p, \omega)$ definidos como

$$\Lambda(p,\omega) = \left\{ f \text{ analítica en } \mathbb{D} : \; M_p(r, f') = \mathrm{O}\left(\frac{\omega(1-r)}{1-r}\right), \text{ as } r \to 1 \right\},$$

donde $1 < p < \infty$ y ω es un peso admisible $\omega : [0, \pi] \to [0, \infty)$ en el sentido de Blasco y de Souza [22, 23]. Hemos probado lo siguiente.

Sea $1 < p < \infty$ y sea ω un peso admisible con $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$ cuando $\delta \searrow 0$ (condición que implica que $\Lambda(p,\omega)$ no está contenido en $BMOA$ ni en el espacio de Bloch). Las siguientes condiciones son equivalentes.

(i) El operador $\mathcal{H}_\mu$ está bien definido en $\Lambda(p,\omega)$ y además es acotado de $\Lambda(p,\omega)$ en sí mismo.

(ii) La medida μ es de Carleson.

En el comienzo de nuestra investigación empezamos a estudiar espacios conformemente invariantes. $BMOA$ tiene un papel muy importante entre estos espacios. Con el objetivo de continuar nuestro trabajo nos hemos concentrado en los espacios de Morrey, una generalización de $BMOA$. Para $0 < \lambda \leq 1$ el espacio de Morrey $\mathcal{L}^{2,\lambda}$ se define como

$$\mathcal{L}^{2,\lambda} = \left\{ f \in H^2 : \|f\|_{\lambda,*} = \sup_{\substack{I \subset \mathbb{T} \\ I \text{ intervalo}}} \left(\frac{1}{|I|^\lambda} \int_I \left| f(e^{i\theta}) - f_I \right|^2 d\theta \right)^{1/2} < \infty \right\}.$$

Es claro que para $\lambda = 1$ el espacio de Morrey $\mathcal{L}^{2,1}$ coincide con $BMOA$. Para $\lambda \in (0,1)$, el espacio de Morrey $\mathcal{L}^{2,\lambda}$ es un espacio propio entre $BMOA$ y el espacio de Hardy H^2.

El Capítulo 3 está dedicado a esta clase de espacios. Se ha dividido el estudio en dos secciones. La Sección 3.1 trata sobre la estructura de estos espacios caracterizando para algunas clases típicas de funciones analíticas $\mathcal{C}$ cuáles son las funciones de $\mathcal{C}$ que residen en los espacios de Morrey, prestando atención a las diferencias y similitudes con los espacios de Hardy y $BMOA$. La Sección 3.2 está dedicada a la acción de semigrupos de operadores de composición en los espacios de Morrey.

En la Sección 3.1 presentamos algunos resultados conocidos para los espacios de Morrey tales como el crecimiento de sus funciones, sus series de potencias lagunares, así como una caracterización de ciertas series de potencias aleatorias en $\mathcal{L}^{2,\lambda}$. También damos una caracterización de las funciones en los espacios de Morrey mediante sus coeficientes de Taylor.

Sea $0 < \lambda \leq 1$ y sea $f(z) = \sum_{n=0}^{\infty} a_n z^n$ una función analítica, tenemos que $f \in \mathcal{L}^{2,\lambda}$ si y sólo si

$$\sup_{w \in \mathbb{D}} \sum_{n=0}^{\infty} \frac{(1-|w|^2)^{2-\lambda}}{(n+1)^2} \left| \sum_{k=0}^{n} (k+1) a_{k+1} \overline{w}^{n-k} \right|^2 < \infty.$$

Si nos restringimos al caso en el que los coeficientes de Taylor de la función f son no negativos tenemos lo siguiente.

Sea $0 < \lambda \leq 1$ y sea $f(z) = \sum_{n=0}^{\infty} a_n z^n$ una función analítica con $a_n \geq 0$ para todo $n \geq 0$, se tiene que $f \in \mathcal{L}^{2,\lambda}$ si y sólo si

$$\sup_{n\geq 1} \frac{1}{n^{1-\lambda}} \sum_{k=0}^{\infty} \left(\sum_{j=kn}^{(k+1)n-1} a_j \right)^2 < \infty.$$

También damos una sencilla caracterización de las funciones en los espacios de Morrey que tienen coeficientes de Taylor no negativos y no crecientes.

Sea $0 < \lambda < 1$ y sea $f(z) = \sum_{n=0}^{\infty} a_n z^n$ una función analítica con $a_n \geq 0$ para todo $n \geq 0$ y $\{a_n\}$ no creciente, tenemos que

$$f \in \mathcal{L}^{2,\lambda} \Leftrightarrow a_n \lesssim n^{-\frac{1+\lambda}{2}}.$$

Gracias a este resultado probamos que los espacios de Morrey contienen funciones con el máximo crecimiento posible en estos espacios y que las funciones con coeficientes de Taylor no negativos y no crecientes que pertenecen a $\mathcal{L}^{2,\lambda}$ pertenecen también a todos los espacios de Hardy H^p con $p < \frac{2}{1-\lambda}$, esto es:

Para $0 < \lambda < 1$ tenemos que

$$\mathcal{L}^{2,\lambda} \cap \mathcal{P} \subset \bigcap_{p<\frac{2}{1-\lambda}} H^p,$$

siendo $\mathcal{P}$ la clase de funciones analíticas en el disco con coeficientes de Taylor no negativos y no crecientes,

$$\mathcal{P} = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}ol(\mathbb{D}) : a_n \geq 0 \text{ y } \{a_n\} \text{ no creciente} \right\}.$$

Al igual que con las funciones con coeficientes de Taylor no negativos y no crecientes probamos que la intersección del espacio de Morrey $\mathcal{L}^{2,\lambda}$ con la clase de funciones univalentes está contenida en todos los espacios de Hardy H^p con $p < \frac{2}{1-\lambda}$, es decir, tenemos que:

Para $0 < \lambda < 1$ se tiene que

$$\mathcal{L}^{2,\lambda} \cap \mathcal{U} \subset \bigcap_{p<\frac{2}{1-\lambda}} H^p.$$

No sabemos si estos dos resultados se pueden extender a todo el espacio de Morrey. Dejamos esta cuestión como conjetura.

Sea $0 < \lambda < 1$. ¿Es cierto que

$$\mathcal{L}^{2,\lambda} \subset \bigcap_{p<\frac{2}{1-\lambda}} H^p \quad ?$$

Como dijimos anteriormente, la Sección 3.2 está dedicada al estudio de semigrupos de operadores de composición en los espacios de Morrey. Este estudio aparece en el trabajo [47] realizado en colaboración con P. Galanopoulos y A. Siskakis.

Un semigrupo (uniparamétrico) de funciones analíticas es un homomorfismo continuo $\Phi : t \mapsto \Phi(t) = \varphi_t$ del semigrupo aditivo de los números reales no negativos al semigrupo de composición de todas las funciones analíticas que llevan $\mathbb{D}$ en $\mathbb{D}$.
En otras palabras, $\Phi = (\varphi_t)$ consiste en funciones analíticas en $\mathbb{D}$ con $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ y para las que se satisfacen las siguientes tres condiciones:

(i) φ_0 es la identidad en $\mathbb{D}$,

(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$, para todo $t, s \geq 0$,

(iii) $\varphi_t \to \varphi_0$, cuando $t \to 0$, uniformemente en subconjuntos compactos de $\mathbb{D}$.

Cada semigrupo de funciones analíticas da lugar a un semigrupo (C_t) de operadores de composición en $\mathcal{H}ol(\mathbb{D})$,

$$C_t(f) \stackrel{\text{def}}{=} f \circ \varphi_t, \quad f \in \mathcal{H}ol(\mathbb{D}).$$

Existe un buen número de trabajos acerca de semigrupos de operadores de composición centrados en la restricción de (C_t) a ciertos subespacios lineales de $\mathcal{H}ol(\mathbb{D})$. Dado un espacio de Banach X de funciones de $\mathcal{H}ol(\mathbb{D})$ y un semigrupo (φ_t), decimos que (φ_t) genera un semigrupo de operadores de composición en X si (C_t) es un semigrupo de operadores acotados en X bien definido y fuertemente continuo. Esto significa exactamente que para toda $f \in X$, se tiene que $C_t(f) \in X$ para todo $t \geq 0$ y

$$\lim_{t \to 0^+} \|C_t(f) - f\|_X = 0.$$

A continuación presentamos algunos resultados conocidos sobre este tema en espacios clásicos de funciones analíticas.

(i) Cada semigrupo de funciones analíticas genera un semigrupo de operadores en los espacios de Hardy H^p $(1 \leq p < \infty)$ [17], los espacios de Bergman A^p $(1 \leq p < \infty)$ [92], el espacio de Dirichlet [93], y en los espacios $VMOA$ y el espacio pequeño de Bloch $\mathcal{B}_0$ [100].

(ii) Ningún semigrupo no trivial genera un semigrupo de operadores en el espacio H^∞ de funciones analíticas acotadas [5, 19].

(iii) Existen bastantes semigrupos (pero no todos) que generan un semigrupo de operadores en el álgebra del disco. De hecho, estos pueden ser caracterizados de varias formas [31].

Recientemente, se ha descubierto [5, 19, 18] que $BMOA$ y el espacio de Bloch son del segundo tipo. Nuestro trabajo aquí es probar que para $0 < \lambda < 1$ los espacios de Morrey $\mathcal{L}^{2,\lambda}$ son también del mismo tipo.

Introduzcamos un poco notación y propiedades básicas de semigrupos.

Dado un semigrupo (φ_t) y un espacio de Banach X, notaremos como $[\varphi_t, X]$ al máximo subespacio lineal cerrado de X tal que (φ_t) genera un semigrupo de operadores en él.

Otra herramienta importante en el estudio de semigrupos es el generador infinitesimal. Éste se define de la forma

$$G(z) \stackrel{\text{def}}{=} \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t}, \; z \in \mathbb{D}.$$

Esta convergencia es uniforme en compactos de $\mathbb{D}$ así que $G \in \mathcal{H}ol(\mathbb{D})$. Es más, G tiene una única representación

$$G(z) = (\bar{b}z - 1)(z - b)P(z),\ z \in \mathbb{D},$$

donde $b \in \overline{\mathbb{D}}$ y $P \in \mathcal{H}ol(\mathbb{D})$ con Re $P(z) \geq 0$ para todo $z \in \mathbb{D}$. Si G no es idénticamente nula, esto es, si (φ_t) no es trivial, el par (b, P) está únicamente determinado por (φ_t) y al punto b se le llama el punto de Denjoy-Wolff del semigrupo.

Probamos un resultado acerca de la existencia del subespacio maximal referido anteriormente para todo semigrupo (φ_t) y también una caracterización del subespacio maximal mediante el generador infinitesimal.

Supongamos que $0 < \lambda < 1$ y sea (φ_t) un semigrupo de funciones analíticas. Existe un subespacio cerrado $Y \subset \mathcal{L}^{2,\lambda}$ tal que (φ_t) genera un semigrupo de operadores en Y y tal que cualquier otro subespacio de $\mathcal{L}^{2,\lambda}$ con esta propiedad está contenido en Y. En nuestra notación, $Y = [\varphi_t, \mathcal{L}^{2,\lambda}]$.

Además, si G es el operador infinitesimal del semigrupo (φ_t) entonces

$$[\varphi_t, \mathcal{L}^{2,\lambda}] = \overline{\{f \in \mathcal{L}^{2,\lambda} : Gf' \in \mathcal{L}^{2,\lambda}\}}.$$

También probamos el siguiente resultado para los espacios pequeños de Morrey.

Para $0 < \lambda < 1$, todo semigrupo (φ_t) genera un semigrupo de operadores en $\mathcal{L}_0^{2,\lambda}$.

Particularmente, en nuestra notación esto es,

$$\mathcal{L}_0^{2,\lambda} \subset [\varphi_t, \mathcal{L}^{2,\lambda}] \subset \mathcal{L}^{2,\lambda},$$

para todo $0 < \lambda < 1$ y todo semigrupo (φ_t).

Podemos probar que para las contracciones y las rotaciones, la primera contención es una igualdad. Esto es,

$$\mathcal{L}_0^{2,\lambda} = [e^{it}z, \mathcal{L}^{2,\lambda}] = [e^{-t}z, \mathcal{L}^{2,\lambda}], \qquad \text{para } 0 < \lambda < 1.$$

Aunque en general la primera inclusión en esta cadena de contenciones puede ser estricta, hemos obtenido una condición suficiente para la igualdad y también una condición necesaria para semigrupos con punto de Denjoy-Wolff interior.

Sea (φ_t) un semigrupo con operador infinitesimal G y sea $0 < \lambda < 1$.

(i) Si
$$\lim_{|I|\to 0} \frac{1}{|I|} \int_{S(I)} \frac{1-|z|}{|G(z)|^2}\, dA(z) = 0$$
entonces $\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$.

(ii) Si $\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$ y el punto de Denjoy-Wolff $b \in \mathbb{D}$, entonces
$$\lim_{|z|\to 1} \frac{(1-|z|)^{\frac{3-\lambda}{2}}}{G(z)} = 0.$$

Finalmente, cerramos este capítulo con un resultado acerca de la posibilidad de tener igualdad en la contención
$$[\varphi_t, \mathcal{L}^{2,\lambda}] \subset \mathcal{L}^{2,\lambda}.$$

Sea X un espacio de Banach de funciones analíticas. Sea $0 < \lambda < 1$ y supongamos que $\mathcal{L}^{2,\lambda} \subset X \subset \mathcal{B}^{\frac{3-\lambda}{2}}$ y sea (φ_t) un semigrupo de funciones analíticas no trivial. Entonces $[\varphi_t, X] \subsetneq X$.
En particular no existen semigrupos no triviales tal que $[\varphi_t, \mathcal{L}^{2,\lambda}] = \mathcal{L}^{2,\lambda}$.

El Capítulo 4 está dedicado a explorar una clase de espacios de funciones analíticas que comparte propiedades con los espacios de Dirichlet y los de Morrey. La mayor parte de los resultados en esta línea aparecen en [46].

Sean λ, $p \in [0,1]$. Decimos que $f \in \mathcal{H}ol(\mathbb{D})$ pertenece al espacio de Dirichlet-Morrey $\mathcal{D}_p^\lambda$ si
$$\|f\|_{\mathcal{D}_p^\lambda} = |f(0)| + \sup_{a\in\mathbb{D}} (1-|a|^2)^{\frac{p}{2}(1-\lambda)} \|f\circ\varphi_a - f(a)\|_{\mathcal{D}_p} < \infty.$$

Primero presentamos algunos resultados sobre la estructura de estos espacios en la Sección 4.1 y más tarde estudiamos también los multiplicadores puntuales en estos espacios en la Sección 4.2.

Los espacios Dirichlet-Morrey pueden ser caracterizados usando medidas de Carleson.

Sean $0 < p, \lambda < 1$ y sea $f \in \mathcal{H}ol(\mathbb{D})$. Se tiene que $f \in \mathcal{D}_p^\lambda$ si y sólo si

$$\|f\|_{p,\lambda,*} = \sup_{\substack{I \subset \mathbb{T} \\ I \text{ intervalo}}} \left(\frac{1}{|I|^{p\lambda}} \int_{S(I)} |f'(z)|^2 (1-|z|^2)^p \, dA(z) \right) < \infty,$$

y la norma $\|f\|_{\mathcal{D}_p^\lambda}$ es comparable a $|f(0)| + \|f\|_{p,\lambda,*}$.

También damos un resultado sobre el crecimiento radial de funciones en los espacios Dirichlet-Morrey y probamos que esta condición no se puede mejorar.

Sea $0 < p, \lambda < 1$. Se tiene que:

(i) Existe una constante $C = C(p, \lambda)$ tal que cualquier $f \in \mathcal{D}_p^\lambda$ satisface

$$|f(z)| \leq \frac{C\|f\|_{\mathcal{D}_p^\lambda}}{(1-|z|)^{\frac{p}{2}(1-\lambda)}}, \quad z \in \mathbb{D}.$$

(ii) La función $f_{p,\lambda}(z) = (1-z)^{-\frac{p}{2}(1-\lambda)}$ pertenece a $\mathcal{D}_p^\lambda$.

Observemos que ambas partes de la proposición anterior son válidas cuando $p = 1$ para $0 < \lambda < 1$.

En el siguiente resultado establecemos una condición necesaria y suficiente para que un espacio Dirichlet-Morrey esté contenido en otro.

Sean $\lambda_1, p_1, \lambda_2, p_2 \in (0, 1)$. Se tiene que

$$\mathcal{D}_{p_1}^{\lambda_1} \subseteq \mathcal{D}_{p_2}^{\lambda_2} \quad \Longleftrightarrow \quad p_1 \leq p_2 \quad \text{y} \quad p_1(1-\lambda_1) \leq p_2(1-\lambda_2).$$

Para finalizar esta sección, estudiamos la caracterización de las funciones de los espacios Dirichlet-Morrey en términos de los valores frontera.

Supongamos que $f \in H^2$ y $0 < p, \lambda < 1$. Entonces $f \in \mathcal{D}_p^\lambda$ si y sólo si

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^{p\lambda}} \int_I \int_I \frac{|f(u) - f(v)|^2}{|u - v|^{2-p}} \, |du| \, |dv| < \infty.$$

Sea X un espacio de Banach de funciones analíticas en $\mathbb{D}$. Se dice que una función $g \in \mathcal{H}ol(\mathbb{D})$ es un multiplicador de X si el operador de multiplicación

$$M_g(f)(z) = g(z)f(z), \quad f \in X$$

es un operador acotado en X. Para esto generalmente basta comprobar que $M_g(X) \subset X$ y aplicar el teorema del grafo cerrado. Denotamos al espacio de todos los multiplicadores de X como $M(X)$. Los operadores de multiplicación están estrechamente relacionados con los operadores de integración J_g e I_g. Éstos están inducidos por el símbolo $g \in \mathcal{H}ol(\mathbb{D})$ como sigue

$$J_g(f)(z) = \int_0^z f(w)g'(w) \, dw, \quad z \in \mathbb{D},$$

$$I_g(f)(z) = \int_0^z f'(w)g(w) \, dw, \quad z \in \mathbb{D},$$

y actúan en funciones $f \in \mathcal{H}ol(\mathbb{D})$. Su relación con M_g viene de la fórmula de integración por partes

$$J_g(f)(z) = M_g(f)(z) - f(0)g(0) - I_g(f)(z).$$

Tenemos una caracterización completa para que el operador I_g sea acotado en los espacios $\mathcal{D}_p^\lambda$.

Sea $0 < p, \lambda < 1$ y $g \in \mathcal{H}ol(\mathbb{D})$. Se tiene que $I_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ es acotado si y sólo si $g \in H^\infty$.

Con respecto a la acción de J_g en $\mathcal{D}_p^\lambda$ tenemos la siguiente condición necesaria.

Sea $0 < p, \lambda < 1$ y $g \in \mathcal{H}ol(\mathbb{D})$. Si $J_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ es acotado entonces $g \in Q_p$.

También hemos obtenido condiciones en g suficientes para que J_g sea acotado en $\mathcal{D}_p^\lambda$.

Supongamos que $0 < p < 1$.

(i) Si $0 < q < p$ y $g \in Q_q$ entonces $J_g : \mathcal{D}_p^{q/p} \to \mathcal{D}_p^{q/p}$ es acotado.

(ii) Si $0 < \lambda < 1$ y $g \in \mathcal{W}_p$ entonces $J_g : \mathcal{D}_p^{\lambda} \to \mathcal{D}_p^{\lambda}$ es acotado.

Donde $\mathcal{W}_p$ es el espacio de funciones $g \in \mathcal{H}ol(\mathbb{D})$ tal que la medida

$$d\mu_g(z) = |g'(z)|^2(1-|z|^2)^p \, dA(z)$$

es una medida $\mathcal{D}_p$-Carleson, esto es, existe una constante $C = C(g)$ tal que

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu_g(z) \leq C\|f\|^2_{\mathcal{D}_p}, \qquad f \in \mathcal{D}_p.$$

Los teoremas anteriores en combinación con la relación entre los operadores M_g, I_g y J_g dan lugar al siguiente corolario sobre multiplicadores de $\mathcal{D}_p^{\lambda}$.

Supongamos que $0 < p, \lambda < 1$ y $g \in \mathcal{H}ol(\mathbb{D})$. Se tiene que

(i) Si $g \in \mathcal{W}_p \cap H^{\infty}$ entonces $M_g : \mathcal{D}_p^{\lambda} \to \mathcal{D}_p^{\lambda}$ es acotado.

(ii) Si $g \in Q_{p\lambda} \cap H^{\infty}$ entonces $M_g : \mathcal{D}_p^{\lambda} \to \mathcal{D}_p^{\lambda}$ es acotado.

(iii) Si $M_g : \mathcal{D}_p^{\lambda} \to \mathcal{D}_p^{\lambda}$ es acotado entonces $g \in Q_p \cap H^{\infty}$.

La descripción completa del espacio de multiplicadores $M(\mathcal{D}_p^{\lambda})$ y de los símbolos g para las que J_g es acotada en $\mathcal{D}_p^{\lambda}$ parece ser un problema complicado.

Introduction

This thesis is devoted to study certain operators acting on spaces of analytic functions in the unit disc.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc. We shall also let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets.

A subspace X of $\mathcal{H}ol(\mathbb{D})$ can be seen as a sequence space by identifying a function $f \in X$ with its sequence of Taylor coefficients:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \leftrightarrow \quad \{a_n\}_{n=0}^{\infty}.$$

Let $\mathcal{H}$ be the Hilbert matrix,

$$\mathcal{H} = \left(\frac{1}{n+k+1}\right)_{n,k\geq 0} = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & 1/5 & \cdots \\ 1/3 & 1/4 & 1/5 & 1/6 & \cdots \\ 1/4 & 1/5 & 1/6 & 1/7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Hilbert matrix can be viewed as an operator between spaces of sequences. Formally, we define its action as

$$\mathcal{H}\left(\{a_n\}_{n=0}^{\infty}\right) = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & 1/5 & \cdots \\ 1/3 & 1/4 & 1/5 & 1/6 & \cdots \\ 1/4 & 1/5 & 1/6 & 1/7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix},$$

$$\{a_n\}_{n=0}^{\infty} \mapsto \left\{\sum_{k=0}^{\infty} \frac{a_k}{n+k+1}\right\}_{n=0}^{\infty}.$$

In the same way, $\mathcal{H}$ can be seen as an operator (called the Hilbert operator) between spaces of analytic functions identifying every analytic function with its sequence of Taylor coefficients.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n,$$

when the right hand side has sense.

The Hilbert operator is well defined in H^1, it is bounded on H^p for $1 < p < \infty$, but it is not bounded on H^1 or H^∞ [34]. In a recent paper [65] Lanucha, Nowak, and Pavlović have considered the question of finding subspaces of H^1 which are mapped by $\mathcal{H}$ into H^1. Dostanić, Jevtić and Vukotić [37] found the exact norm of $\mathcal{H}$ as an operator from H^p to H^p $(1 < p < \infty)$.

Let μ be a finite positive Borel measure on $[0, 1)$ and let $\{\mu_n\}_{n=0}^{\infty}$ be its sequence of moments: $\mu_n = \int_{[0,1)} t^n \, d\mu(t)$. The Hilbert matrix can be generalized considering the Hankel matrix $\mathcal{H}_\mu$ with entries $(\mu_{n+k})_{n,k\geq 0}$,

$$\mathcal{H}_\mu = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

As before, the matrix $\mathcal{H}_\mu$ formally induces the generalized Hilbert operator $\mathcal{H}_\mu$ on spaces of analytic functions:

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_k \right) z^n,$$

when the right hand side has sense.

Widom [99, Theorem 3. 1] and Power [89, Theorem 3] (see also Peller [83, p. 42, Theorem 7. 2]) proved that $\mathcal{H}_\mu$ is a well defined bounded operator from H^2 into itself if and only if μ is a Carleson measure, $\mu([t, 1)) \leq C(1 - t)$, $0 < t < 1$.

Galanopoulos and Peláez [48] studied the action of $\mathcal{H}_\mu$ on H^1. Chatzifountas, Girela and Peláez [28] studied $\mathcal{H}_\mu$ as an operator from H^p into H^q $(0 < p, q < \infty)$.

If everything we wished were OK, we would have:
For $f(z) = \sum_{n=0}^{\infty} a_n z^n$,

$$\begin{aligned}
\mathcal{H}_\mu(f)(z) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_k \right) z^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k \int_{[0,1)} t^{n+k} d\mu(t) \right) z^n \\
&= \sum_{k=0}^{\infty} a_k \left(\sum_{n=0}^{\infty} \int_{[0,1)} t^{n+k} z^n d\mu(t) \right) \\
&= \sum_{k=0}^{\infty} a_k \int_{[0,1)} \frac{t^k}{1-tz} d\mu(t) = \int_{[0,1)} \frac{f(t)}{1-tz} d\mu(t).
\end{aligned}$$

For μ a finite positive Borel measure on $[0,1)$ and $f \in \mathcal{H}ol(\mathbb{D})$ we define

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz} d\mu(t), \quad z \in \mathbb{D},$$

whenever the right hand side makes sense for all $z \in \mathbb{D}$ and defines an analytic function in $\mathbb{D}$.

It turns out that the operators $\mathcal{H}_\mu$ and I_μ are closely related. If f is good enough $\mathcal{H}_\mu(f)$ and $I_\mu(f)$ are well defined and coincide. In [48] Galanopoulos and Peláez proved the following.

Let μ be a positive Borel measure on $[0,1)$. Then:

(i) The operator I_μ is well defined on H^1 if and only if μ is a Carleson measure.

(ii) If μ is a Carleson measure, then the operator $\mathcal{H}_\mu$ is also well defined on H^1 and, furthermore,

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \text{for every } f \in H^1.$$

(iii) The operator I_μ is a bounded operator from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.

Later in [28] Chatzifountas, Girela and Peláez proved the following.

Suppose that $1 < p < \infty$ and let μ be a positive Borel measure on $[0,1)$. Then:

(i) The operator I_μ is well defined on H^p if and only if μ is a 1-Carleson measure for H^p.

(ii) If μ is a 1-Carleson measure for H^p, then the operator $\mathcal{H}_\mu$ is also well defined on H^p and, furthermore,

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \text{for every } f \in H^p.$$

(iii) The operator I_μ is a bounded operator from H^p into itself if and only if μ is a Carleson measure.

Chapter 2 is devoted to study the operators $\mathcal{H}_\mu$ and I_μ on several spaces of analytic functions. We started extending the above results to some conformally invariant spaces as the Bloch space, $BMOA$, Besov spaces or the Q_s classes. All these results can be found in a joint work with Girela [54].

In the first result we characterize those measures μ for which the operator I_μ is well defined or bounded in $BMOA$ and in the Bloch space.

For μ a positive Borel measure on $[0,1)$ we have that the operator I_μ is well defined in any of these spaces if and only if

$$\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty,$$

and if this holds, then the following three conditions are equivalent:

(i) The measure μ is a 1-logarithmic 1-Carleson measure.

(ii) The operator I_μ is bounded from $\mathcal{B}$ into $BMOA$.

(iii) The operator I_μ is bounded from $BMOA$ into itself.

Moreover, if (i) holds, then the operator $\mathcal{H}_\mu$ is also well defined on the Bloch space and

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \text{for all } f \in \mathcal{B},$$

and hence the operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}$ into $BMOA$.

We have also the following result regarding compactness:

Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$. If μ is a vanishing 1-logarithmic 1-Carleson measure then:

(i) The operator I_μ is a compact operator from $\mathcal{B}$ into $BMOA$.

(ii) The operator I_μ is a compact operator from $BMOA$ into itself.

The results concerning the well definition and boundedness of I_μ in $BMOA$ and in the Bloch space remain true for all the Q_s spaces with $s > 0$. That is, we have:

For any given $s \in (0, \infty)$ and for a positive Borel measure μ, the operator I_μ is well defined in Q_s if and only if

$$\int_{[0,1)} \log \frac{2}{1-t} \, d\mu(t) \, < \, \infty,$$

and if this holds, then the following condition are equivalent:

(i) The measure μ is a 1-logarithmic 1-Carleson measure.

(ii) For any given $s \in (0, \infty)$, the operator I_μ is bounded from Q_s into $BMOA$.

Moreover, if (i) holds, then for any given $s \in (0, \infty)$ the operator $\mathcal{H}_\mu$ coincide with I_μ in Q_s, and, hence, it is also bounded from Q_s into $BMOA$.

We have also studied the operator I_μ acting on Besov spaces. As usual, for $1 < p < \infty$, p' will denote the exponent conjugate to p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. We have proved the following results:

Let $1 < p < \infty$ and let μ be a positive Borel measure on $[0, 1)$. We have:

(i) If $\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{1/p'} d\mu(t) \, < \, \infty$, then the operator I_μ is well defined in B^p.

(ii) If the operator I_μ is well defined in B^p, then $\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{\gamma} d\mu(t) \, < \, \infty$ for all $\gamma \, < \, \frac{1}{p'}$.

(iii) If μ is a $1/p'$-logarithmic 1-Carleson measure then the operator I_μ is bounded from B^p into $BMOA$.

(iv) If μ is a vanishing $1/p'$-logarithmic 1-Carleson measure then the operator I_μ is compact from B^p into $BMOA$.

Working directly with the operator $\mathcal{H}_\mu$ we have obtained that:

If μ is a finite positive Borel measure on $[0, 1)$ then:

(i) If $1 < p \leq 2$ and $\sum_{k=1}^{\infty} \frac{\mu_k^{p'}}{k} < \infty$, then the operator $\mathcal{H}_\mu$ is well defined in B^p.

(ii) If $2 < p < \infty$ and $\sum_{k=1}^{\infty} \frac{\mu_k^{p'}}{k^{p'/p}} < \infty$, then the operator $\mathcal{H}_\mu$ is well defined in B^p.

In [16], Bao and Wulan proved that there exists a positive Borel measure μ on $[0,1)$ which is a Carleson measure but such that $\mathcal{H}_\mu(B^2) \not\subset B^2$. They also proved that if $\mathcal{H}_\mu$ is a bounded operator from B^2 into itself then μ is a Carleson measure. We improve these results and extend them to all B^p spaces with $1 < p < \infty$.

If $1 < p < \infty$ then:

(i) If $0 < \beta \leq \frac{1}{p}$ then there exists a positive Borel measure μ on $[0,1)$ which is a β-logarithmic 1-Carleson measure but such that $\mathcal{H}_\mu(B^p) \not\subset B^p$.

(ii) If μ is a positive Borel measure on $[0,1)$ such that the operator $\mathcal{H}_\mu$ is bounded from B^p into itself. Then μ is a $1/p'$-logarithmic 1-Carleson measure [55].

(iii) If $\gamma > 1$ and μ is a positive Borel measure on $[0,1)$ which is a γ-logarithmic 1-Carleson measure. Then the operator $\mathcal{H}_\mu$ is a bounded operator from B^p into itself.

Next, we turned our attention to the action of $\mathcal{H}_\mu$ on Hardy spaces. The above mentioned results of Galanopoulos and Peláez and Chatzifountas, Girela and Peláez imply the following.

(i) If μ is a Carleson measure, then the operator $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.

(ii) If $1 < p < \infty$ and μ is a 1-Carleson measure for H^p, then the operator $\mathcal{H}_\mu$ is a bounded operator from H^p into itself if and only if μ is a Carleson measure.

These results do not close completely the question of characterizing the measures μ for which $\mathcal{H}_\mu$ is a bounded operator from H^p into itself. Indeed, in these works the authors only consider 1-Carleson measures for H^p. In principle, there could exist a measure μ which is not a 1-Carleson measures for H^p but so that the operator $\mathcal{H}_\mu$ is

well defined and bounded on H^p. We have proved that this is not the case. Indeed, we have proved the following result.

Let μ be a positive Borel measure on $[0,1)$.

(i) The operator $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.

(ii) If $1 < p < \infty$ then the operator $\mathcal{H}_\mu$ is a bounded operator from H^p into itself if and only if μ is a Carleson measure.

In [28] the parameter p was only taken to be finite. We also give a result for the case $p = \infty$.

Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent.

(i) $\int_{[0,1)} \frac{d\mu(t)}{1-t} < \infty$.

(ii) $\sum_{n=0}^{\infty} \mu_n < \infty$.

(iii) The operator I_μ is a bounded operator from H^∞ into itself.

(iv) The operator $\mathcal{H}_\mu$ is a bounded operator from H^∞ into itself.

These results about the action of $\mathcal{H}_\mu$ on Hardy spaces have been published in [55] and they are contained in Section 2.1 of the thesis.

In Section 2.2 we recall the following result of Galanopoulos and Peláez.

Let μ be a positive Borel measure on $[0,1)$. If μ is a Carleson measure then $\mathcal{H}_\mu(H^1) \subset \mathscr{C}$, where $\mathscr{C}$ is the space of those analytic functions in the disc which are the Cauchy transform of a complex Borel measure on $\partial\mathbb{D}$.

At this point we ask ourselves what can we say about image $\mathcal{H}_\mu(H^1)$ of H^1 under the action of the operator $\mathcal{H}_\mu$ if the measure μ is a 1-logarithmic 1-Carleson measure on $[0,1)$.

Regarding this question, let us notice that it is easy to see that the space of Dirichlet type $\mathcal{D}_0^1$ is included in H^1. We shall prove that if μ is a 1-logarithmic

1-Carleson measure on $[0,1)$ then $\mathcal{H}_\mu(H^1)$ is contained in the space $\mathcal{D}_0^1$. Actually, we have the following stronger result.

Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent.

(i) μ is a 1-logarithmic 1-Carleson measure.

(ii) $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself.

(iii) $\mathcal{H}_\mu$ is a bounded operator from H^1 into $\mathcal{D}_0^1$.

(iv) $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}_0^1$ into $\mathcal{D}_0^1$.

There is a gap between the last two results above and so it is natural to discuss the range of H^1 under the action of $\mathcal{H}_\mu$ when μ is an α-logarithmic 1-Carleson measure with $0<\alpha<1$. We shall prove the following result.

Let μ be a positive Borel measure on $[0,1)$. Suppose that $0<\alpha<1$ and that μ is an α-logarithmic 1-Carleson measure. Then $\mathcal{H}_\mu$ maps H^1 into the space $\mathcal{D}^1(\log^{\alpha-1})$ defined as follows:

$$\mathcal{D}^1(\log^{\alpha-1}) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)| \left(\log \frac{2}{1-|z|}\right)^{\alpha-1} dA(z) < \infty \right\}.$$

All these results can be found in a joint work with Girela [56].

We gave before a result about the boundedness of the operator $\mathcal{H}_\mu$ acting from Q_s spaces (with $0<s<\infty$) into $BMOA$. It is natural to look for a characterization of those μ for which I_μ and/or $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{B}$ into itself or, more generally, from Q_s into itself for any $s>0$. We have the following result.

Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent.

(i) The operator I_μ is bounded from Q_s into itself for some $s>0$.

(ii) The operator I_μ is bounded from Q_s into itself for all $s>0$.

(iii) The operator $\mathcal{H}_\mu$ is bounded from Q_s into itself for some $s > 0$.

(iv) The operator $\mathcal{H}_\mu$ is bounded from Q_s into itself for all $s > 0$.

(v) The measure μ is a 1-logarithmic 1-Carleson measure.

In fact, we are able to prove a stronger result which does not distinguish between different Q_s spaces.

Let μ be a positive Borel measure on $[0, 1)$ and let $0 < s_1, s_2 < \infty$. Then the following conditions are equivalent.

(i) The operator I_μ is well defined in Q_{s_1} and, furthermore, it is a bounded operator from Q_{s_1} into Q_{s_2}.

(ii) The operator $\mathcal{H}_\mu$ is well defined in Q_{s_1} and, furthermore, it is a bounded operator from Q_{s_1} into Q_{s_2}.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

This result follows from a more general theorem which we have proved where the mean Lipschitz space $\Lambda^2_{1/2}$ shows up.

Let μ be a positive Borel measure on $[0, 1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.

(i) The operator I_μ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.

(ii) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) = \mathrm{O}\left(\frac{1}{n}\right)$.

All these results are published in [55] and they are included in Section 2.3 of the thesis.

Section 2.4 is devoted to extend the above result to a more general class of mean Lipschitz spaces. The results in this section can be found in [72].

First of all, we improve the last result changing $\Lambda^2_{1/2}$ by $\Lambda^p_{1/p}$ for any $p > 1$.

Suppose that $1 < p < \infty$. Let μ be a positive Borel measure on $[0, 1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^p_{1/p} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.

(i) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into the Bloch space $\mathcal{B}$.

(ii) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^p_{1/p}$.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) \; = \; \mathrm{O}\left(\frac{1}{n}\right)$.

The spaces $\Lambda^p_{1/p}$ are included in $BMOA$. Our next step is to study the operator $\mathcal{H}_\mu$ acting in generalized mean Lipschitz spaces not included in $BMOA$. We work with the spaces $\Lambda(p, \omega)$ defined as

$$\Lambda(p, \omega) = \left\{ f \text{ analytic in } \mathbb{D} : \; M_p(r, f') = \mathrm{O}\left(\frac{\omega(1-r)}{1-r}\right), \text{ as } r \to 1 \right\},$$

where $1 < p < \infty$ and ω is an admissible weight $\omega : [0, \pi] \to [0, \infty)$ in the sense of Blasco and de Souza [22, 23]. We have proved the following.

Let $1 < p < \infty$ and let ω an admissible weight with $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$ when $\delta \searrow 0$ (this condition implies that $\Lambda(p, \omega)$ is not included in the Bloch space). The following conditions are equivalent:

(i) The operator $\mathcal{H}_\mu$ is well defined in $\Lambda(p, \omega)$ and, furthermore, it is a bounded operator from $\Lambda(p, \omega)$ into itself.

(ii) The measure μ is a Carleson measure.

In the beginning of our research we started to study conformally invariant spaces. $BMOA$ has a very important role among these spaces. In order to continue our work, we have focused in Morrey spaces, a generalization of $BMOA$. For $0 < \lambda \leq 1$ the Morrey space $\mathcal{L}^{2,\lambda}$ is defined as

$$\mathcal{L}^{2,\lambda} = \left\{ f \in H^2 : \|f\|_{\lambda,*} = \sup_{\substack{I \subset \mathbb{T} \\ I \text{ interval}}} \left(\frac{1}{|I|^\lambda} \int_I \left| f(e^{i\theta}) - f_I \right|^2 d\theta \right)^{1/2} < \infty \right\}.$$

It is clear that for $\lambda = 1$ the Morrey space $\mathcal{L}^{2,1}$ coincides with $BMOA$. For $\lambda \in (0,1)$, the Morrey space $\mathcal{L}^{2,\lambda}$ is a proper space between $BMOA$ and the Hardy space H^2.

Chapter 3 is devoted to this class of spaces. We have divided the study in two sections. In Section 3.1 we speak about the structure of these spaces characterizing for some typical classes of analytic functions $\mathcal{C}$ those functions in $\mathcal{C}$ which lie in the Morrey spaces, and paying attention to the differences and similarities with Hardy spaces and $BMOA$. Section 3.2 is devoted to the action of semigroups of composition operators on Morrey spaces.

In Section 3.1 we present some known results for Morrey spaces such as the growth of functions, their power series with Hadamard gaps, or a characterization of certain random power series in $\mathcal{L}^{2,\lambda}$. We also give a characterization of the functions in Morrey spaces in term of its Taylor coefficients.

For $0 < \lambda \leq 1$ and for an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have that $f \in \mathcal{L}^{2,\lambda}$ if and only if

$$\sup_{w \in \mathbb{D}} \sum_{n=0}^{\infty} \frac{(1-|w|^2)^{2-\lambda}}{(n+1)^2} \left| \sum_{k=0}^{n} (k+1) a_{k+1} \overline{w}^{n-k} \right|^2 < \infty.$$

If we restrict to the case that the Taylor coefficients of the function f are non-negative, we have the following.

For $0 < \lambda \leq 1$ and for an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \geq 0$ for every $n \geq 0$, we have that $f \in \mathcal{L}^{2,\lambda}$ if and only if

$$\sup_{n \geq 1} \frac{1}{n^{1-\lambda}} \sum_{k=0}^{\infty} \left(\sum_{j=kn}^{(k+1)n-1} a_j \right)^2 < \infty.$$

We give also an easy characterization of functions in Morrey spaces with non-negative and non-increasing Taylor coefficients.

For $0 < \lambda < 1$ and for an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \geq 0$ for every $n \geq 0$ and $\{a_n\}$ non-increasing, we have that

$$f \in \mathcal{L}^{2,\lambda} \Leftrightarrow a_n \lesssim n^{-\frac{1+\lambda}{2}}.$$

Thanks to this result, we also prove that Morrey spaces contain functions with the maximum possible growth and that the functions with non-negative and non-increasing Taylor coefficients which belong to $\mathcal{L}^{2,\lambda}$ belong also to all Hardy spaces H^p with $p < \frac{2}{1-\lambda}$, that is:

Let $0 < \lambda < 1$. We define $\mathcal{P}$ as the class of analytic functions in the disc with non-negative and non-increasing Taylor coefficients,

$$\mathcal{P} = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}ol(\mathbb{D}) : a_n \geq 0 \text{ and } \{a_n\} \text{ non-increasing} \right\}.$$

Then

$$\mathcal{L}^{2,\lambda} \cap \mathcal{P} \subset \bigcap_{p < \frac{2}{1-\lambda}} H^p.$$

In the same way as what happens with functions with non-negative and non-increasing Taylor coefficients, we prove that the intersection of the Morrey space $\mathcal{L}^{2,\lambda}$ with the class of univalent functions is contained in all Hardy spaces H^p with $p < \frac{2}{1-\lambda}$, that is, we have:

For $0 < \lambda < 1$ we have that

$$\mathcal{L}^{2,\lambda} \cap \mathcal{U} \subset \bigcap_{p < \frac{2}{1-\lambda}} H^p.$$

We do not know if these two results can be extended to the whole Morrey space. We leave this question as a conjecture.

Let $0 < \lambda < 1$. It is true that

$$\mathcal{L}^{2,\lambda} \subset \bigcap_{p<\frac{2}{1-\lambda}} H^p \quad ?$$

As we said before, Section 3.2 is devoted to the study of semigroups of composition operators on Morrey spaces. This appears in [47], a joint work with P. Galanopoulos and A. Siskakis.

A (one-parameter) semigroup of analytic functions is a continuous homomorphism $\Phi : t \mapsto \Phi(t) = \varphi_t$ from the additive semigroup of nonnegative real numbers into the composition semigroup of all analytic functions which map $\mathbb{D}$ into $\mathbb{D}$.
In other words, $\Phi = (\varphi_t)$ consists of analytic functions on $\mathbb{D}$ with $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ and for which the following three conditions hold:

(i) φ_0 is the identity in $\mathbb{D}$,

(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,

(iii) $\varphi_t \to \varphi_0$, as $t \to 0$, uniformly on compact subsets of $\mathbb{D}$.

Each such semigroup gives rise to a semigroup (C_t) consisting of composition operators on $\mathcal{H}ol(\mathbb{D})$,

$$C_t(f) \stackrel{\text{def}}{=} f \circ \varphi_t, \quad f \in \mathcal{H}ol(\mathbb{D}).$$

There is a good number of works about semigroups of composition operators focused on the restriction of (C_t) to certain linear subspaces of $\mathcal{H}ol(\mathbb{D})$. Given a Banach space X consisting of functions in $\mathcal{H}ol(\mathbb{D})$ and a semigroup (φ_t), we say that (φ_t) generates a semigroup of operators on X if (C_t) is a well-defined strongly continuous semigroup of bounded operators in X. This exactly means that for every $f \in X$, we have $C_t(f) \in X$ for all $t \geq 0$ and

$$\lim_{t\to 0^+} \|C_t(f) - f\|_X = 0.$$

Some known results about this topic in classical spaces of analytic functions are the following:

(i) Every semigroup of analytic functions generates a semigroup of operators on the Hardy spaces H^p $(1 \leq p < \infty)$ [17], the Bergman spaces A^p $(1 \leq p < \infty)$ [92], the Dirichlet space [93], and on the spaces $VMOA$ and the little Bloch space $\mathcal{B}_0$ [100].

(ii) No non-trivial semigroup generates a semigroup of operators in the space H^∞ of bounded analytic functions [5, 19].

(iii) There are plenty of semigroups (but not all) which generate semigroups of operators in the disc algebra. Indeed, they can be well characterized in several analytical terms [31].

Recently, it has been discovered [5, 19, 18] that $BMOA$ and the Bloch space are in the second case. Our work here is to prove that for $0 < \lambda < 1$ Morrey spaces $\mathcal{L}^{2,\lambda}$ are also in the same case.

Let us introduce some notation and basic facts about semigroups.

Given a semigroup (φ_t) and a Banach space X, we will denote by $[\varphi_t, X]$ the maximal closed linear subspace of X such that (φ_t) generates a semigroup of operators on it.

Another important tool in the study of semigroups is the infinitesimal generator. We define it as

$$G(z) \stackrel{\text{def}}{=} \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t}, \; z \in \mathbb{D}.$$

This convergence holds uniformly on compact subsets of $\mathbb{D}$, so $G \in \mathcal{H}ol(\mathbb{D})$. Furthermore, G has a unique representation

$$G(z) = (\bar{b}z - 1)(z - b)P(z), \; z \in \mathbb{D},$$

where $b \in \overline{\mathbb{D}}$ and $P \in \mathcal{H}ol(\mathbb{D})$ with $\operatorname{Re} P(z) \geq 0$ for all $z \in \mathbb{D}$. If G is not identically null, that is, if (φ_t) is not trivial, the couple (b, P) is uniquely determined from (φ_t) and the point b is called the Denjoy-Wolff point of the semigroup.

We prove a result about the existence of the maximal subspace referred before for all semigroup (φ_t) and also a characterization of this maximal subspace via the infinitesimal generator.

Suppose that $0 < \lambda < 1$ and let (φ_t) be a semigroup of analytic functions. Then there exists a closed subspace $Y \subset \mathcal{L}^{2,\lambda}$ such that (φ_t) generates a semigroup

of operators on Y and such that any other subspace of $\mathcal{L}^{2,\lambda}$ with this property is contained in Y. In our notation, $Y = [\varphi_t, \mathcal{L}^{2,\lambda}]$.

Moreover, if G is the infinitesimal generator of the semigroup (φ_t) then

$$[\varphi_t, \mathcal{L}^{2,\lambda}] = \overline{\{f \in \mathcal{L}^{2,\lambda} : Gf' \in \mathcal{L}^{2,\lambda}\}}.$$

We also prove the following result for little Morrey spaces.

For $0 < \lambda < 1$, every semigroup (φ_t) generates a semigroup of operators on $\mathcal{L}_0^{2,\lambda}$. This in particular means that in our notation,

$$\mathcal{L}_0^{2,\lambda} \subset [\varphi_t, \mathcal{L}^{2,\lambda}] \subset \mathcal{L}^{2,\lambda},$$

for every $0 < \lambda < 1$ and every semigroup (φ_t).

We can prove that for dilatations and rotations, the left hand side equality holds. That is,

$$\mathcal{L}_0^{2,\lambda} = [e^{it}z, \mathcal{L}^{2,\lambda}] = [e^{-t}z, \mathcal{L}^{2,\lambda}], \quad \text{for } 0 < \lambda < 1.$$

Although, in general the first inclusion in this chain of contentions can be proper, we have obtained a sufficient condition for the equality in the left hand side and also a necessary condition for semigroups with inner Denjoy-Wolff point.

Let (φ_t) be a semigroup with infinitesimal generator G and let $0 < \lambda < 1$.

(i) If

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_{S(I)} \frac{1-|z|}{|G(z)|^2}\, dA(z) = 0$$

then $\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$.

(ii) If $\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$ and the Denjoy-Wolff point $b \in \mathbb{D}$, then

$$\lim_{|z|\to 1} \frac{(1-|z|)^{\frac{3-\lambda}{2}}}{G(z)} = 0.$$

Finally, we close this chapter with a result about the possibility of having equality in the inclusion

$$[\varphi_t, \mathcal{L}^{2,\lambda}] \subset \mathcal{L}^{2,\lambda}.$$

Let X be a Banach space of analytic functions and $0 < \lambda < 1$. Suppose $\mathcal{L}^{2,\lambda} \subset X \subset \mathcal{B}^{\frac{3-\lambda}{2}}$ and let (φ_t) be a non trivial semigroup of analytic functions. Then $[\varphi_t, X] \subsetneq X$.
In particular there are no non-trivial semigroups such that $[\varphi_t, \mathcal{L}^{2,\lambda}] = \mathcal{L}^{2,\lambda}$.

Chapter 4 is devoted to explore a class of spaces of analytic functions which shares properties with Dirichlet spaces and Morrey spaces. Most of the results in this line are contained in [46].

Let λ, $p \in [0, 1]$. We say that an $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the Dirichlet-Morrey space $\mathcal{D}_p^\lambda$ if

$$\|f\|_{\mathcal{D}_p^\lambda} = |f(0)| + \sup_{a\in\mathbb{D}}(1 - |a|^2)^{\frac{p}{2}(1-\lambda)}\|f \circ \varphi_a - f(a)\|_{\mathcal{D}_p} < \infty.$$

We first give some results on the structure of these spaces in Section 4.1 and then we study the pointwise multipliers on them in Section 4.2.

Dirichlet-Morrey spaces can be characterized using Carleson measures.

Let $0 < p, \lambda < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$. Then $f \in \mathcal{D}_p^\lambda$ if and only if

$$\|f\|_{p,\lambda,*} = \sup_{\substack{I\subset\mathbb{T} \\ I \text{ interval}}} \left(\frac{1}{|I|^{p\lambda}} \int_{S(I)} |f'(z)|^2(1 - |z|^2)^p \, dA(z)\right) < \infty,$$

and the norm $\|f\|_{\mathcal{D}_p^\lambda}$ is comparable to $|f(0)| + \|f\|_{p,\lambda,*}$.

We also give a result about the radial growth of functions in Dirichlet-Morrey spaces and show that this condition is sharp.

Let $0 < p, \lambda < 1$ then,

(i) There is a constant $C = C(p, \lambda)$ such that any $f \in \mathcal{D}_p^\lambda$ satisfies

$$|f(z)| \leq \frac{C\|f\|_{\mathcal{D}_p^\lambda}}{(1 - |z|)^{\frac{p}{2}(1-\lambda)}}, \quad z \in \mathbb{D}.$$

(ii) The function $f_{p,\lambda}(z) = (1-z)^{-\frac{p}{2}(1-\lambda)}$ belongs to $\mathcal{D}_p^\lambda$.

Observe that both parts of the above proposition are also valid when $p = 1$ for $0 < \lambda < 1$.

In the next result we present a necessary and sufficient condition for a Dirichlet-Morrey space to be contained in another one.

Let $\lambda_1, p_1, \lambda_2, p_2 \in (0,1)$. Then

$$\mathcal{D}_{p_1}^{\lambda_1} \subseteq \mathcal{D}_{p_2}^{\lambda_2} \quad \Longleftrightarrow \quad p_1 \leq p_2 \quad \text{and} \quad p_1(1-\lambda_1) \leq p_2(1-\lambda_2).$$

To end this section, we next discuss the boundary values characterization of Dirichlet-Morrey spaces.

Suppose $f \in H^2$ and let $0 < p, \lambda < 1$. Then $f \in \mathcal{D}_p^\lambda$ if and only if

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^{p\lambda}} \int_I \int_I \frac{|f(u) - f(v)|^2}{|u-v|^{2-p}} \, |du| \, |dv| < \infty.$$

Let X be a Banach space of analytic functions on $\mathbb{D}$. A function $g \in \mathcal{H}ol(\mathbb{D})$ is said to be a multiplier of X if the multiplication operator

$$M_g(f)(z) = g(z)f(z), \quad f \in X,$$

is a bounded operator on X. For this it is usually enough to check that $M_g(X) \subset X$ and apply the closed graph theorem. The space of all multipliers of X is denoted by $M(X)$. Multiplication operators are closely related to the integration operators J_g and I_g. These are induced by symbols $g \in \mathcal{H}ol(\mathbb{D})$ as follows

$$J_g(f)(z) = \int_0^z f(w)g'(w)\, dw, \quad z \in \mathbb{D},$$

and

$$I_g(f)(z) = \int_0^z f'(w)g(w)\, dw, \quad z \in \mathbb{D},$$

and act on functions $f \in \mathcal{H}ol(\mathbb{D})$. Their relations with M_g comes from the integration by parts formula

$$J_g(f)(z) = M_g(f)(z) - f(0)g(0) - I_g(f)(z).$$

We have a complete characterization for the operator I_g being bounded on $\mathcal{D}_p^\lambda$ spaces.

Let $0 < p, \lambda < 1$ and $g \in \mathcal{H}ol(\mathbb{D})$. Then $I_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ is bounded if and only if $g \in H^\infty$.

Concerning the action of J_g on $\mathcal{D}_p^\lambda$ we have the following necessary condition.

Let $0 < p, \lambda < 1$ and $g \in \mathcal{H}ol(\mathbb{D})$. If $J_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ is bounded then $g \in Q_p$.

We also have obtained sufficient conditions on g for J_g to be bounded on $\mathcal{D}_p^\lambda$.

Suppose $0 < p < 1$.

(i) If $0 < q < p$ and $g \in Q_q$ then $J_g : \mathcal{D}_p^{q/p} \to \mathcal{D}_p^{q/p}$ is bounded.

(ii) If $0 < \lambda < 1$ and $g \in \mathcal{W}_p$ then $J_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ is bounded.

Where $\mathcal{W}_p$ is the space of functions $g \in \mathcal{H}ol(\mathbb{D})$ such that the measure

$$d\mu_g(z) = |g'(z)|^2(1 - |z|^2)^p \, dA(z)$$

is a $\mathcal{D}_p$-Carleson measure, that is, there is a constant $C = C(g)$ such that

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu_g(z) \leq C\|f\|^2_{\mathcal{D}_p}, \qquad f \in \mathcal{D}_p.$$

The above theorems in combination with the relation between operators M_g, I_g and J_g give the following corollary for multipliers of $\mathcal{D}_p^\lambda$.

Suppose $0 < p, \lambda < 1$ and $g \in \mathcal{H}ol(\mathbb{D})$. Then

(i) If $g \in \mathcal{W}_p \cap H^\infty$ then $M_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ is bounded.

(ii) If $g \in Q_{p\lambda} \cap H^\infty$ then $M_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ is bounded.

(iii) If $M_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ is bounded then $g \in Q_p \cap H^\infty$.

The complete description of the multiplier space $M(\mathcal{D}_p^\lambda)$ and of the symbols g for which J_g is bounded on $\mathcal{D}_p^\lambda$, seems to be a hard problem.

Chapter 1

Preliminaries

This chapter is devoted to present some of the main spaces which will be the object of our work.

We shall let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$ will be the boundary of $\mathbb{D}$. We shall also let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets.

If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p \, dt\right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p \leq \infty$, the Hardy space H^p consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty$$

(see [40, 49] for the theory of H^p-spaces). In particular, it is known that whenever $f \in H^p$, $0 < p \leq \infty$, f has finite non-tangential limits a.e. on $\mathbb{T}$. We shall also denote this function defined on $\mathbb{T}$ by f.

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha+1)\int_{\mathbb{D}} (1-|z|^2)^\alpha |f(z)|^p \, dA(z)\right)^{1/p} < \infty.$$

The unweighted Bergman space A^p_0 is simply denoted by A^p. Here, $dA(z) = \frac{1}{\pi} dx\, dy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We refer to [43, 59, 111] for the theory of these spaces.

The space of Dirichlet type $\mathcal{D}^p_\alpha$ $(0 < p < \infty$ and $\alpha > -1)$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that $f' \in A^p_\alpha$. In other words, a function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to $\mathcal{D}^p_\alpha$ if and only if

$$\|f\|_{\mathcal{D}^p_\alpha} \stackrel{\text{def}}{=} |f(0)| + \left((\alpha+1) \int_{\mathbb{D}} (1-|z|^2)^\alpha |f'(z)|^p \, dA(z) \right)^{1/p} < \infty.$$

We recall that the Bloch space $\mathcal{B}$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1-|z|^2)\, |f'(z)| < \infty.$$

We refer to [6, 111] for the theory of Bloch functions.

We shall write I for an interval of $\mathbb{T}$ and $|I|$ for its length. If $\psi \in L^1(\partial\mathbb{D})$, we let ψ_I denote the mean of f over the interval I, that is,

$$\psi_I \stackrel{\text{def}}{=} \frac{1}{|I|} \int_I \psi(e^{i\theta})\, d\theta.$$

The mean oscillation of ψ over I is

$$|\psi - \psi_I|_I = \frac{1}{|I|} \int_I |\psi(e^{i\theta}) - \psi_I|\, d\theta.$$

We say that ψ has bounded mean oscillation or that $\psi \in BMO(\mathbb{T})$ if

$$\sup_{\substack{I \subset \mathbb{T} \\ I \text{ interval}}} \frac{1}{|I|} \int_I |\psi(e^{i\theta}) - \psi_I|\, d\theta < \infty.$$

We can also consider the small version of this space: We say that ψ has vanishing mean oscillation or that $\psi \in VMO(\mathbb{T})$ if

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_I |\psi(e^{i\theta}) - \psi_I|\, d\theta = 0.$$

We define $BMOA$ as the space of those functions $f \in H^1$ such that the function $e^{i\theta} \mapsto f(e^{i\theta})$ of the boundary values of f belongs to $BMO(\mathbb{T})$ and, in the same way, we define $VMOA$ as the space of those functions $f \in BMOA$ such that the function

of the boundary values of f belongs to VMO. These spaces can be equipped with several different equivalent norms [15, 49, 52]. We often work with the one given in terms of Carleson measures.

If $I \subset \mathbb{T}$ is an interval, the *Carleson square* $S(I)$ is defined as

$$S(I) = \{re^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1\}.$$

Also, for $a \in \mathbb{D}$, the Carleson box $S(a)$ is defined by

$$S(a) = \Big\{z \in \mathbb{D} : 1 - |z| \leq 1 - |a|, \Big|\frac{\arg(a\bar{z})}{2\pi}\Big| \leq \frac{1-|a|}{2}\Big\}.$$

If $s > 0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right) \leq C|I|^s, \quad \text{for any interval } I \subset \partial\mathbb{D},$$

or, equivalently, if there exists $C > 0$ such that

$$\mu\left(S(a)\right) \leq C(1-|a|)^s, \quad \text{for all } a \in \mathbb{D}.$$

If μ satisfies $\lim\limits_{|I|\to 0} \dfrac{\mu\left(S(I)\right)}{|I|^s} = 0$ or, equivalently, $\lim\limits_{|a|\to 1} \dfrac{\mu\left(S(a)\right)}{(1-|a|^2)^s} = 0$, then we say that μ is a *vanishing s-Carleson measure*.

An 1-Carleson measure, respectively, a vanishing 1-Carleson measure, will be simply called a Carleson measure, respectively, a vanishing Carleson measure.

As an important ingredient in his work on interpolation by bounded analytic functions, Carleson [27] (see also Theorem 9.3 of [40]) proved that if $0 < p < \infty$ and μ is a positive Borel measure in $\mathbb{D}$ then $H^p \subset L^p(d\mu)$ if and only if μ is a Carleson measure. This result was extended by Duren [39] (see also [40, Theorem 9.4]) who proved that for $0 < p \leq q < \infty$, $H^p \subset L^q(d\mu)$ if and only if μ is a q/p-Carleson measure.

If X is a subspace of $\mathcal{H}ol(\mathbb{D})$, $0 < q < \infty$, and μ is a positive Borel measure in $\mathbb{D}$, μ is said to be a "*q-Carleson measure for the space X*" or an "*(X,q)-Carleson measure*" if $X \subset L^q(d\mu)$. The q-Carleson measures for the spaces H^p, $0 < p, q < \infty$ are completely characterized. The mentioned results of Carleson and Duren can be stated that if $0 < p \leq q < \infty$ then a positive Borel measure μ in $\mathbb{D}$ is a q-Carleson measure for H^p if and only if μ is a q/p-Carleson measure. Luecking [70]

and Videnskii [97] solved the remaining case $0 < q < p$. We mention [21] for a complete information on Carleson measures for Hardy spaces.

Now we can give a characterization of $BMOA$ and $VMOA$ in terms of Carleson measures: Let $f \in \mathcal{H}ol(\mathbb{D})$, then $f \in BMOA$ (resp. $VMOA$) if and only if the measure $|f'(z)|^2(1-|z|^2)dA(z)$ is a Carleson measure (resp. vanishing Carleson measure), and we equip both spaces with the norm [52],

$$\|f\|^2_{\mathrm{BMOA}} = |f(0)|^2 + \sup_{\substack{I\subset\mathbb{T}\\ I \text{ interval}}} \frac{\int_{S(I)} |f'(z)|^2(1-|z|^2)\,dA(z)}{|I|}.$$

The following chain of embeddings [15, 49, 52] holds

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

For $w \in \mathbb{D}$, we let φ_w denote the Möbius transformation defined by

$$\varphi_w(z) = \frac{w-z}{1-\bar{w}z}.$$

Then φ_w is a conformal mapping from the unit disc onto itself and interchanges the origin with w.

Let us denote by $\mathrm{Aut}(\mathbb{D})$ the group of all conformal mappings from $\mathbb{D}$ onto itself. It is known that

$$\mathrm{Aut}(\mathbb{D}) = \{\lambda\varphi_w : w \in \mathbb{D}, |\lambda| = 1\}.$$

We can give a characterization of $BMOA$ and $VMOA$ in terms of H^2 norms: Let $f \in \mathcal{H}ol\,(\mathbb{D})$, then $f \in BMOA$ if and only if $\{f\circ\varphi_a - f(a)\}_{a\in\mathbb{D}}$ is a bounded family in H^2. The condition $\|f\circ\varphi_a - f(a)\|_{H^2} \to 0$, as $|a| \to 1$, is equivalent to saying that $f \in VMOA$. We equip both spaces with the following norm [52], which is called the Garsia's norm.

$$\|f\|_{\mathrm{BMOA}} = |f(0)| + \sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|_{H^2}.$$

Fefferman's duality theorem [52] gives a very important result about these spaces.

Let $X \subset \mathcal{H}ol(\mathbb{D})$ and let X^* denote the dual space of X, that is, the space of all continuous linear functionals $T : X \to \mathbb{C}$.

There is a bijection between the dual space of H^1 and $BMOA$. If $f \in BMOA$ then, the operator T_f defined by

$$T_f(g) = \lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(e^{i\theta})}g(re^{i\theta})\,d\theta, \quad g \in H^1,$$

belongs to $(H^1)^*$ and $\|T_f\| \asymp \|f\|_{BMOA}$.

Conversely, for every $T \in (H^1)^*$ there exists a unique $f \in BMOA$ such that $T = T_f$.

In a similar way, there is a bijection between H^1 and the dual space of $VMOA$. If $g \in H^1$ then, the operator S_g defined by

$$S_g(f) = \lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(re^{i\theta})}\, d\theta, \quad f \in VMOA,$$

belongs to $VMOA^*$ and $\|S_g\| \asymp \|g\|_{H^1}$.

Conversely, for every $S \in VMOA^*$ there exists a unique $g \in H^1$ such that $S = S_g$.

So we conclude that

$$\left(H^1\right)^* \approx BMOA \quad \text{and} \quad VMOA^* \approx H^1.$$

A space $X \subset \mathcal{H}(\mathbb{D})$ equipped with a seminorm ρ is called *conformally invariant* or *Möbius invariant* if there exists a constant $C > 0$ such that

$$\sup_{\varphi} \rho(g \circ \varphi) \leq C\rho(g), \quad g \in X,$$

where the supremum is taken on all Möbius transformations φ of $\mathbb{D}$ onto itself. BMOA and $\mathcal{B}$ have the important property of being conformally invariant spaces [52].

Other important Möbius invariant spaces are the Q_s-spaces ($s > 0$) and the analytic Besov spaces B^p ($1 < p < \infty$).

If $0 \leq s < \infty$, we say that $f \in Q_s$ if f is analytic in $\mathbb{D}$ and

$$\|f\|_{Q_s} \stackrel{\text{def}}{=} \left(|f(0)|^2 + \rho_{Q_s}(f)^2\right)^{1/2} < \infty,$$

where

$$\rho_{Q_s}(f) \stackrel{\text{def}}{=} \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(z,a)^s\, dA(z)\right)^{1/2}.$$

Here, $g(z,a)$ is the Green's function in $\mathbb{D}$, given by $g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right|$. All Q_s spaces ($0 \leq s < \infty$) are conformally invariant with respect to the semi-norm ρ_{Q_s} [35, 103].

These spaces were introduced by Aulaskari and Lappan in [11] while looking for new characterizations of Bloch functions. They proved that for $s > 1$, Q_s is the Bloch space. Using one of the many characterizations of the space $BMOA$ (see,

e. g., [15, Theorem 5] or [52, Theorem 6. 2]) we see that $Q_1 = BMOA$. In the limit case $s = 0$, Q_s is the classical Dirichlet space $\mathcal{D}$ of those analytic functions f in $\mathbb{D}$ satisfying

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty.$$

It is well known that $\mathcal{D} \subset VMOA$. Aulaskari, Xiao and Zhao proved in [14] that

$$\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq BMOA, \qquad 0 < s_1 < s_2 < 1.$$

We mention the book [103] as an excellent reference for the theory of Q_s-spaces.

For $1 < p < \infty$, the *analytic Besov space* B^p is defined as the set of all functions f analytic in $\mathbb{D}$ such that

$$\|f\|_{B^p} \stackrel{\text{def}}{=} (|f(0)|^p + \rho_p(f)^p)^{1/p} < \infty,$$

where

$$\rho_p(f) = \left(\int_{\mathbb{D}} (1 - |z|^2)^{p-2} |f'(z)|^p \, dA(z) \right)^{1/p}.$$

All B^p spaces ($1 < p < \infty$) are conformally invariant with respect to the semi-norm ρ_p (see [8, p. 112] or [35, p. 46]). We have that $\mathcal{D} = B^2$. A lot of information on Besov spaces can be found in [8, 35, 36, 61, 110, 111]. Let us recall that

$$B^p \subsetneq B^q \subsetneq VMOA, \quad 1 < p < q < \infty.$$

We close this chapter noticing that, as usual, we shall be using the convention that $C = C(p, \alpha, q, \beta, \dots)$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta \dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant C independent of the arguments such that $E_1 \leq CE_2$, respectively $E_1 \geq CE_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that E_1 and E_2 are equivalent and we write $E_1 \asymp E_2$.

Chapter 2

A generalized Hilbert matrix acting on spaces of analytic functions

In this chapter we shall study a class of integral operators associated with certain Hankel matrices acting on different spaces of analytic functions. Most of our results concerning this topic are included in [54], [55], [56] and [72].

If μ is a finite positive Borel measure on $[0,1)$ and $n = 0,1,2,\ldots$, we let μ_n denote the moment of order n of μ, that is,

$$\mu_n = \int_{[0,1)} t^n \, d\mu(t),$$

and we let $\mathcal{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$.

$$\mathcal{H}_\mu = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix $\mathcal{H}_\mu$ can be viewed as an operator on spaces of analytic functions by its action on the Taylor coefficients:

$$\{a_n\}_{n=0}^{\infty} \mapsto \left\{ \sum_{k=0}^{\infty} \mu_{n,k} a_k \right\}_{n=0}^{\infty}.$$

$$\begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} \mu_k a_k \\ \sum_{k=0}^{\infty} \mu_{k+1} a_k \\ \sum_{k=0}^{\infty} \mu_{k+2} a_k \\ \vdots \end{pmatrix}.$$

To be precise, if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbb{D})$ we define

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n,$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

If μ is the Lebesgue measure on $[0,1)$ the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix $\mathcal{H} = ((n+k+1)^{-1})_{n,k\geq 0}$, which induces the classical Hilbert operator $\mathcal{H}$, a prototype of a Hankel operator which has extensively studied recently (see [2, 33, 34, 37, 63, 65]). Other related generalizations of the Hilbert operator have been considered in [45] and [81].

Hardy's inequality [40, page 48] guarantees that $\mathcal{H}(f)$ is a well defined analytic function in $\mathbb{D}$ for every $f \in H^1$. However, the resulting Hilbert operator $\mathcal{H}$ is bounded from H^p to H^p if and only if $1 < p < \infty$ [34]. In a recent paper [65] Lanucha, Nowak, and Pavlović have considered the question of finding subspaces of H^1 which are mapped by $\mathcal{H}$ into H^1.

The question of describing the measures μ for which the operator $\mathcal{H}_\mu$ is well defined and bounded on distinct spaces of analytic functions has been studied in a good number of papers (see [16, 28, 48, 54, 55, 72, 74, 89, 99]). Carleson measures play a basic role in these works.

Galanopoulos and Peláez [48] studied the question of characterizing the measures μ so that the generalized Hilbert operator $\mathcal{H}_\mu$ becomes well defined and bounded on H^1. Indeed, they proved that if μ is a Carleson measure then the operator $\mathcal{H}_\mu$ is well defined in H^1, obtaining en route the following integral representation

$$\mathcal{H}_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t), \quad z \in \mathbb{D}, \quad \text{for all } f \in H^1.$$

For simplicity, we shall write throughout the chapter

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t), \tag{2.0.1}$$

whenever the right hand side makes sense and it defines an analytic function in $\mathbb{D}$.

In [28], Chatzifountas, Girela and Peláez extended the above results studying the operator $\mathcal{H}_\mu$ acting from H^p into H^q, $0 < p, q < \infty$. In these works, an extension of the classical definition of Carleson measures shows up:

Following [109], if μ is a positive Borel measure on $\mathbb{D}$, $0 \leq \alpha < \infty$, and $0 < s < \infty$ we say that μ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that

$$\frac{\mu\left(S(I)\right)\left(\log \frac{2\pi}{|I|}\right)^{\alpha}}{|I|^s} \leq C, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

If $\mu\left(S(I)\right)\left(\log \frac{2\pi}{|I|}\right)^{\alpha} = \mathrm{o}\left(|I|^s\right)$, as $|I| \to 0$, we say that μ is a vanishing α-logarithmic s-Carleson measure.

A positive Borel measure μ on $[0,1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(A) = \mu\left(A \cap [0,1)\right), \quad \text{for any Borel subset } A \text{ of } \mathbb{D}.$$

In this way a positive Borel measure μ on $[0,1)$ is an α-logarithmic s-Carleson measure if and only if there exists a positive constant C such that

$$\mu\left([t,1)\right)\left(\log \frac{2\pi}{1-t}\right)^{\alpha} \leq C(1-t)^s, \quad 0 \leq t < 1,$$

and μ is a vanishing α-logarithmic s-Carleson measure if

$$\mu\left([t,1)\right)\left(\log \frac{2\pi}{1-t}\right)^{\alpha} = \mathrm{o}\left((1-t)^s\right), \quad \text{as } t \to 1.$$

Our main aim in this chapter is to improve the above results about the generalized Hilbert matrix $\mathcal{H}_\mu$ acting on H^p spaces ($1 \leq p \leq \infty$) and the study of this operator in some of the most important conformally invariant spaces as well as in mean Lipschitz spaces. A key tool will be a description of those positive Borel measures μ on $[0,1)$ for which $\mathcal{H}_\mu$ is well defined in these spaces and satisfies that $\mathcal{H}_\mu(f) = I_\mu(f)$ for all f.

2.1 A generalized Hilbert matrix acting on Hardy spaces

Let us start with some of the previous results about $\mathcal{H}_\mu$ and I_μ on Hardy spaces. In 1966, Widom [99, Theorem 3. 1] (see also [89, Theorem 3] and [83, p. 42, Theorem 7. 2]) proved that $\mathcal{H}_\mu$ is a bounded operator from H^2 into itself if and only μ is a Carleson measure. More recently, Galanopoulos and Peláez in [48] and Chatzifountas, Girela and Peláez in [28] have extended these works studying the action of $\mathcal{H}_\mu$ on H^1 and H^p for $0 < p < \infty$ respectively. Some of their results are the following ones:

Theorem A ([48]). *Let μ be a positive Borel measure on $[0, 1)$. Then:*

(i) The operator I_μ is well defined on H^1 if and only if μ is a Carleson measure.

(ii) If μ is a Carleson measure, then the operator $\mathcal{H}_\mu$ is also well defined on H^1 and, furthermore,

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \text{for every } f \in H^1.$$

(iii) The operator I_μ is a bounded operator from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.

Theorem B ([28]). *Suppose that $1 < p < \infty$ and let μ be a positive Borel measure on $[0, 1)$. Then:*

(i) The operator I_μ is well defined on H^p if and only if μ is a 1-Carleson measure for H^p.

(ii) If μ is a 1-Carleson measure for H^p, then the operator $\mathcal{H}_\mu$ is also well defined on H^p and, furthermore,

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \text{for every } f \in H^p.$$

(iii) The operator I_μ is a bounded operator from H^p into itself if and only if μ is a Carleson measure.

Theorem A and Theorem B immediately yield the following.

Theorem C. *Let μ be a positive Borel measure on $[0,1)$.*

(i) If μ is a Carleson measure, then the operator $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.

(ii) If $1 < p < \infty$ and μ is a 1-Carleson measure for H^p, then the operator $\mathcal{H}_\mu$ is a bounded operator from H^p into itself if and only if μ is a Carleson measure.

Theorem C does not close completely the question of characterizing the measures μ for which $\mathcal{H}_\mu$ is a bounded operator from H^p into itself. Indeed, in Theorem C we only consider 1-Carleson measures for H^p. In principle, there could exist a measure μ which is not a 1-Carleson measure for H^p but so that the operator $\mathcal{H}_\mu$ is well defined and bounded on H^p. Our first result in this section asserts that this is not the case.

Theorem 1 ([55])**.** *Let μ be a positive Borel measure on $[0,1)$.*

(i) The operator $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.

(ii) If $1 < p < \infty$ then the operator $\mathcal{H}_\mu$ is a bounded operator from H^p into itself if and only if μ is a Carleson measure.

In [28] the parameter p was only considered to be finite. Here we give a result for the case $p = \infty$.

Theorem 2 ([55])**.** *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent.*

(i) $\int_{[0,1)} \frac{d\mu(t)}{1-t} < \infty$.

(ii) $\sum_{n=0}^{\infty} \mu_n < \infty$.

(iii) The operator I_μ is a bounded operator from H^∞ into itself.

(iv) The operator $\mathcal{H}_\mu$ is a bounded operator from H^∞ into itself.

2.1.1 Proofs

This section is devoted to prove Theorem 1 and Theorem 2.

Proof of Theorem 1 (i). Suppose that $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself. For $0 < b < 1$, set

$$f_b(z) = \frac{1-b^2}{(1-bz)^2}, \quad z \in \mathbb{D}.$$

We have that $f_b \in H^1$ and $\|f_b\|_{H^1} = 1$. Since $\mathcal{H}_\mu$ is bounded on H^1, this implies that

$$1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{H^1}. \tag{2.1.1}$$

We also have,

$$f_b(z) \;=\; \sum_{k=0}^{\infty} a_{k,b} z^k, \quad \text{with } a_{k,b} = (1-b^2)(k+1)b^k.$$

Using Hardy's inequality, (2.1.1) and the definition of the $a_{k,b}$'s, we obtain

$$\begin{aligned}
1 \;\gtrsim\; \|\mathcal{H}_\mu(f_b)\|_{H^1} \;&\gtrsim\; \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{\infty} a_{k,b} \int_{[0,1)} t^{n+k} \, d\,\mu(t) \right) \\
&\gtrsim\; (1-b^2) \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=1}^{\infty} k b^k \int_{[b,1)} t^{n+k} \, d\mu(t) \right) \\
&\gtrsim\; (1-b^2) \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=1}^{\infty} k b^{n+2k} \, \mu\left([b,1)\right) \right) \\
&=\; (1-b^2) \mu\left([b,1)\right) \sum_{n=1}^{\infty} \frac{b^n}{n} \left(\sum_{k=1}^{\infty} k b^{2k} \right) \\
&=\; (1-b^2) \mu\left([b,1)\right) \left(\log \frac{1}{1-b} \right) \frac{b}{(1-b^2)^2}.
\end{aligned}$$

Then it follows that

$$\mu\left([b,1)\right) \;=\; \mathrm{O}\left(\frac{1-b}{\log \frac{1}{1-b}} \right), \quad \text{as } b \to 1.$$

Hence, μ is a 1-logarithmic 1-Carleson measure.

The converse follows from Theorem C (i). □

Proof of Theorem 1 (ii). Suppose that $1 < p < \infty$ and that μ is a positive Borel measure on $[0,1)$ such that the operator $\mathcal{H}_\mu$ is a bounded operator from H^p into itself.

For $0 < b < 1$, set

$$f_b(z) = \left(\frac{1-b^2}{(1-bz)^2}\right)^{1/p}, \quad z \in \mathbb{D}.$$

We have that $f_b \in H^p$ and $\|f_b\|_{H^p} = 1$. Since $\mathcal{H}_\mu$ is bounded on H^p, this implies that

$$1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{H^p}. \tag{2.1.2}$$

We also have,

$$f_b(z) = \sum_{k=0}^{\infty} a_{k,b} z^k, \quad \text{with } a_{k,b} \approx (1-b^2)^{1/p} k^{\frac{2}{p}-1} b^k.$$

Since the $a_{k,b}$'s are positive, it is clear that the sequence $\{\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b}\}_{n=0}^{\infty}$ of the Taylor coefficients of $\mathcal{H}_\mu(f_b)$ is a decreasing sequence of non-negative real numbers. Using this, Theorem A of [78], (2.1.2), and the definition of the $a_{k,b}$'s, we obtain

$$\begin{aligned}
1 &\gtrsim \|\mathcal{H}_\mu(f_b)\|_{H^p}^p \gtrsim \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b}\right)^p \\
&= \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=0}^{\infty} a_{k,b} \int_{[0,1)} t^{n+k}\, d\mu(t)\right)^p \\
&\gtrsim (1-b^2) \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=1}^{\infty} k^{\frac{2}{p}-1} b^k \int_{[b,1)} t^{n+k}\, d\mu(t)\right)^p \\
&\gtrsim (1-b^2) \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=1}^{\infty} k^{\frac{2}{p}-1} b^{n+2k} \mu([b,1))\right)^p \\
&= (1-b^2)\mu([b,1))^p \sum_{n=1}^{\infty} n^{p-2} b^{np} \left(\sum_{k=1}^{\infty} k^{\frac{2}{p}-1} b^{2k}\right)^p \\
&\asymp (1-b^2)\mu([b,1))^p \frac{1}{(1-b)^2} \sum_{n=1}^{\infty} n^{p-2} b^{np} \\
&\asymp \mu([b,1))^p \frac{1}{(1-b)^p}, \quad \text{as } b \to 1.
\end{aligned}$$

Then it follows that

$$\mu\left([b,1)\right) \;=\; \mathrm{O}\left(1-b\right), \quad \text{as } b \to 1,$$

and, hence, μ is a Carleson measure.

The other implication follows from Theorem C (ii). □

Proof of Theorem 2. The equivalence (i) ⇔ (ii) is clear because

$$\int_{[0,1)} \frac{d\mu(t)}{1-t} \;=\; \int_{[0,1)} \left(\sum_{n=0}^{\infty} t^n\right) d\mu(t) \;=\; \sum_{n=0}^{\infty} \int_{[0,1)} t^n d\mu(t) \;=\; \sum_{n=0}^{\infty} \mu_n.$$

The implication (i) ⇒ (iii) is obvious.

(iii) ⇒ (i): Suppose (iii). Let f be the constant function $f(z) = 1$, for all z. Then (iii) implies that there exists a positive constant C such that

$$\left| \int_{[0,1)} \frac{d\mu(t)}{1-tz} \right| \leq C, \quad z \in \mathbb{D}.$$

Taking $z = r \in (0,1)$ in this inequality, we have

$$\int_{[0,1)} \frac{d\mu(t)}{1-tr} \leq C, \quad r \in (0,1).$$

Letting r tend to 1, (i) follows.

(iii) ⇒ (iv): Suppose (iii). We have seen that then (i) holds, and it is easy to see that (i) implies that μ is a Carleson measure. Using part (ii) of Theorem A, it follows that $\mathcal{H}_\mu$ is well defined in H^∞ and that $\mathcal{H}_\mu(f) = I_\mu(f)$ for all f in H^∞. Then (iii) gives that $\mathcal{H}_\mu$ is bounded from H^∞ into itself.

(iv) ⇒ (iii): Suppose that (iv) is true and, as above, let f be the constant function $f(z) = 1$, for all z. Then $\mathcal{H}_\mu(f) \in H^\infty$. But $\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \mu_n z^n$ and then it is clear that

$$\mathcal{H}_\mu(f) \in H^\infty \;\Leftrightarrow\; \sum_{n=0}^{\infty} \mu_n < \infty.$$

Thus we have seen that (iv) ⇒ (ii). Since (ii) ⇔ (iii), this finishes the proof. □

2.2 Further results on the action of a generalized Hilbert matrix on H^1

Let us recall some results concerning the Hilbert operator $\mathcal{H}$ and the integral operator

$$If(z) = \int_0^1 \frac{f(t)}{1-tz}\,dt, \quad z \in \mathbb{D},$$

which is defined when the right-hand side converge for all $z \in \mathbb{D}$ and the resulting function If is analytic in $\mathbb{D}$.

As we said before, if $f \in H^1$, $f(z) = \sum_{n=0}^{\infty} a_n z^z$ by Hardy's inequality [40, p. 48] we have that

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} \leq \pi \|f\|_{H^1},$$

and then $\mathcal{H}f$ is a well defined analytic function for every $f \in H^1$. By the Fejér-Riesz inequality [40, Theorem 3. 13, p. 46] we also have that

$$\int_0^1 |f(t)|\,dt \leq \pi \|f\|_{H^1},$$

and then If is a well defined analytic function for every $f \in H^1$. Furthermore, $\mathcal{H}f = If$ for every $f \in H^1$.

Diamantopoulos and Siskakis [34] proved that $\mathcal{H}$ is a bounded operator from H^p into itself if $1 < p < \infty$, but this is not true for $p = 1$. In fact, they proved that $\mathcal{H}(H^1) \nsubseteq H^1$. Cima [29] has recently proved the following result.

Theorem D.

(i) The operator $\mathcal{H}$ maps H^1 into the space $\mathscr{C}$ of Cauchy transforms of measures on the unit circle $\partial\mathbb{D}$.

(ii) $\mathcal{H} : H^1 \to \mathscr{C}$ is injective.

We recall that if σ is a finite complex Borel measure on $\partial\mathbb{D}$, the Cauchy transform $C\sigma$ is defined by

$$C\sigma(z) = \int_{\partial\mathbb{D}} \frac{d\sigma(\xi)}{1-\bar{\xi}\,z}, \quad z \in \mathbb{D}.$$

We let $\mathscr{M}$ be the space of all finite complex Borel measure on $\partial\mathbb{D}$. It is a Banach space with the total variation norm. The space of Cauchy transforms is $\mathscr{C} = \{C\sigma : \sigma \in \mathscr{M}\}$. It is a Banach space with the norm $\|C\sigma\| \stackrel{\text{def}}{=} \inf\{\|\tau\| : C\tau = C\sigma\}$. We mention [30] as an excellent reference for the main results about Cauchy transforms. We let $\mathcal{A}$ denote the disc algebra, that is, the space of analytic functions in $\mathbb{D}$ with a continuous extension to the closed unit disc, endowed with the $\|\cdot\|_{H^\infty}$-norm. It turns out [30, Chapter 4] that $\mathcal{A}$ can be identified with the pre-dual of $\mathscr{C}$ via the pairing

$$\langle g, C\sigma\rangle \stackrel{\text{def}}{=} \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\overline{C\sigma(re^{i\theta})}\, d\theta, \quad g \in \mathcal{A}. \tag{2.2.1}$$

This is the basic ingredient used by Cima to prove the inclusion $\mathcal{H}(H^1) \subset \mathscr{C}$.

In [48, Theorem 2. 2] Galanopoulos and Peláez proved the following.

Theorem E. *Let μ be a positive Borel measure on $[0,1)$. If μ is a Carleson measure then $\mathcal{H}_\mu(H^1) \subset \mathscr{C}$.*

This result is stronger than Theorem D(i).

In view of Theorem A and Theorem E, the following question arises naturally.

Question 1. Suppose that μ is a 1-logarithmic 1-Carleson measure on $[0,1)$. What can we say about the image $\mathcal{H}_\mu(H^1)$ of H^1 under the action of the operator $\mathcal{H}_\mu$?

To answer Question 1, let us start noticing that it is easy to see that the space of Dirichlet type $\mathcal{D}_0^1$ is included in H^1. Actually, we have $\mathcal{D}_{p-1}^p \subset H^p$ for $0 < p \le 2$ (see [98, Lemma 1. 4]). The following result of Pavlović [78, Theorem 3.2] implies that for a function $f \in \mathcal{H}ol(\mathbb{D})$ whose sequence of Taylor coefficients is decreasing we have that $f \in \mathcal{D}_0^1 \Leftrightarrow f \in H^1$.

Theorem F. *Let $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty a_n z^n$, and suppose that the sequence $\{a_n\}$ is a decreasing sequence of non-negative real numbers. Then $f \in \mathcal{D}_0^1$ if and only if $\sum_{n=0}^\infty \frac{a_n}{n+1} < \infty$, and we have*

$$\|f\|_{\mathcal{D}_0^1} \asymp \sum_{n=0}^\infty \frac{a_n}{n+1}.$$

We shall prove that if μ is a 1-logarithmic 1-Carleson measure on $[0,1)$ then $\mathcal{H}_\mu(H^1)$ is contained in the space $\mathcal{D}_0^1$. Actually, we have the following stronger result.

Theorem 3 ([56]). *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent.*

(i) μ is a 1-logarithmic 1-Carleson measure.

(ii) $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself.

(iii) $\mathcal{H}_\mu$ is a bounded operator from H^1 into $\mathcal{D}_0^1$.

(iv) $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}_0^1$ into $\mathcal{D}_0^1$.

There is a gap between Theorem E and Theorem 3 and so it is natural to discuss the range of H^1 under the action of $\mathcal{H}_\mu$ when μ is an α-logarithmic 1-Carleson measure with $0<\alpha<1$. We shall prove the following result.

Theorem 4 ([56]). *Let μ be a positive Borel measure on $[0,1)$. Suppose that $0<\alpha<1$ and that μ is an α-logarithmic 1-Carleson measure. Then $\mathcal{H}_\mu$ maps H^1 into the space $\mathcal{D}^1(\log^{\alpha-1})$ defined as follows:*

$$\mathcal{D}^1(\log^{\alpha-1}) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)| \left(\log \frac{2}{1-|z|}\right)^{\alpha-1} dA(z) < \infty \right\}.$$

All these results can be found in a joint work with Girela [56]. In the same work we study the action of the operators $\mathcal{H}_\mu$ on the Bergman spaces A^p_α and on the Dirichlet spaces D^p_α.

2.2.1 Proofs

We include the proof of Theorem E for the sake of completness.

Proof of Theorem E. We shall argue as in the proof of Theorem D in [29]. Suppose that μ is a Carleson measure and $f \in H^1$. Recall that $\mathcal{H}_\mu f = I_\mu f$. Hence, we have to show that $I_\mu f$ defines a bounded linear functional on the disc algebra $\mathcal{A}$ with the duality relation (2.2.1). Take $g \in \mathcal{A}$ and $0<r<1$. Using the definition of I_μ and Fubini's theorem, we obtain

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\,\overline{I_\mu f(re^{i\theta})}\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta}) \left(\int_{[0,1)} \frac{\overline{f(t)}}{1-tre^{-i\theta}}\,d\mu(t)\right) d\theta \\
&= \int_{[0,1)} \overline{f(t)} \left(\frac{1}{2\pi i}\int_0^{2\pi} \frac{g(re^{i\theta})ie^{i\theta}}{e^{i\theta}-tr}\,d\theta\right) d\mu(t) \\
&= \int_{[0,1)} \overline{f(t)} \left(\frac{1}{2\pi i}\int_{|z|=1} \frac{g(rz)}{z-tr}\,dz\right) d\mu(t).
\end{aligned}$$

Then using Cauchy's integral formula it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\,\overline{I_\mu f(re^{i\theta})}\,d\theta \;=\; \int_{[0,1)} \overline{f(t)}g(r^2t)\,d\mu(t). \tag{2.2.2}$$

Since μ is a Carleson measure, we have that $\int_0^1 |f(t)|\,d\mu(t) \lesssim \|f\|_{H^1}$, and then it follows that

$$\int_{[0,1)} |f(t)g(t)|\;d\mu(t) \le \|g\|_{H^\infty}\int_{[0,1)} |f(t)|\,d\mu(t) \lesssim \|g\|_{H^\infty}\|f\|_{H^1}. \tag{2.2.3}$$

Since $g \in \mathcal{A}$ we have that $g(r^2t) \to g(t)$, as $r \to 1$, uniformly on $[0,1)$. Then using (2.2.3) and (2.2.2), we obtain that the limit $\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\,\overline{I_\mu f(re^{i\theta})}\,d\theta$ exists and that

$$g \;\mapsto\; \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\,\overline{I_\mu f(re^{i\theta})}\,d\theta$$

defines a continuous linear functional on $\mathcal{A}$. □

In the proof of Theorem 3 we shall use the following result which can be found in [54, Proposition 2. 5].

It is worth noticing that for μ a positive Borel measure and ν defined as

$$d\nu(t) = \log\frac{2}{1-t}\,d\mu(t),$$

ν being a Carleson measure is equivalent to μ being an 1-logarithmic 1-Carleson measure. Actually, we have the following more general result.

Proposition 1. *Let μ be a positive Borel measure on $[0,1)$, $s > 0$, and $\alpha \ge 0$. Let ν be the Borel measure on $[0,1)$ defined by*

$$d\nu(t) \;=\; \left(\log\frac{2}{1-t}\right)^{\alpha} d\mu(t).$$

Then, the following two conditions are equivalent.

(a) ν is an s-Carleson measure.

(b) μ is an α-logarithmic s-Carleson measure.

Proof.

(a) $\Rightarrow$ (b). Assume (a). Then there exists a positive constant C such that

$$\int_{[t,1)} \left(\log\frac{2}{1-u}\right)^{\alpha} d\mu(u) \;\le\; C(1-t)^s, \quad t \in [0,1).$$

Using this and the fact that the function $u \mapsto \log \frac{2}{1-u}$ is increasing in $[0,1)$, we obtain

$$\left(\log \frac{2}{1-t}\right)^{\alpha} \int_{[t,1)} d\mu(u) \leq \int_{[t,1)} \left(\log \frac{2}{1-u}\right)^{\alpha} d\mu(u) \leq C(1-t)^s, \quad t \in [0,1).$$

This shows that μ is an α-logarithmic s-Carleson measure.

(b) $\Rightarrow$ (a). Assume (b). Then there exists a positive constant C such that

$$\left(\log \frac{2}{1-t}\right)^{\alpha} \mu\left([t,1)\right) \leq C\,(1-t)^s, \quad 0 \leq t < 1. \tag{2.2.4}$$

For $0 \leq u < 1$, set $F(u) = \mu\left([0,u)\right) - \mu\left([0,1)\right) = -\mu\left([u,1)\right)$. Integrating by parts and using (2.2.4), we obtain

$$\begin{aligned}
\nu\left([t,1)\right) &= \int_{[t,1)} \left(\log \frac{2}{1-u}\right)^{\alpha} d\mu(u) \\
&= \left(\log \frac{2}{1-t}\right)^{\alpha} \mu\left([t,1)\right) - \lim_{u\to 1^-} \left(\log \frac{2}{1-u}\right)^{\alpha} \mu\left([u,1)\right) \\
&\quad + \alpha \int_{[t,1)} \mu\left([u,1)\right) \left(\log \frac{2}{1-u}\right)^{\alpha-1} \frac{du}{1-u} \\
&= \left(\log \frac{2}{1-t}\right)^{\alpha} \mu\left([t,1)\right) + \alpha \int_{[t,1)} \mu\left([u,1)\right) \left(\log \frac{2}{1-u}\right)^{\alpha-1} \frac{du}{1-u} \\
&\leq C\,(1-t)^s + C\,\alpha \int_t^1 \frac{(1-u)^{s-1}}{\log \frac{2}{1-u}}\, du \\
&\lesssim (1-t)^s, \quad 0 \leq t < 1.
\end{aligned}$$

Thus, ν is an s-Carleson measure. □

Proof of Theorem 3. We already know that (i) and (ii) are equivalent by Theorem A.

To prove that (i) implies (iii) we shall use some results about the Bloch space. We recall that a function $f \in \mathcal{H}ol(\mathbb{D})$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions will be denoted by $\mathcal{B}$. It is a non-separable Banach space with the norm $\|\cdot\|_{\mathcal{B}}$ just defined. A classical source for the theory of Bloch functions is [6]. The closure of the polynomials in the Bloch norm is the *little Bloch space* $\mathcal{B}_0$ which consists of those $f \in \mathcal{H}ol(\mathbb{D})$ with the property that

$$\lim_{|z|\to 1} (1-|z|^2)|f'(z)| = 0.$$

It is well known that (see [6, p. 13])

$$|f(z)| \lesssim \|f\|_{\mathcal{B}} \log \frac{2}{1-|z|}. \tag{2.2.5}$$

The basic ingredient to prove that (i) implies (iii) is the fact that the dual $(\mathcal{B}_0)^*$ of the little Bloch space can be identified with the Bergman space A^1 via the integral pairing

$$\langle h, f\rangle \;=\; \int_{\mathbb{D}} h(z)\,\overline{f(z)}\,dA(z), \quad h \in \mathcal{B}_0,\; f \in A^1. \tag{2.2.6}$$

(See [111, Theorem 5. 15]).

Let us proceed to prove the implication (i) $\Rightarrow$ (iii). Assume that μ is a 1-logarithmic 1-Carleson measure and take $f \in H^1$. We have to show that $I_\mu f \in \mathcal{D}_0^1$ or, equivalently, that $(I_\mu f)' \in A^1$. Since $\mathcal{B}_0$ is the closure of the polynomials in the Bloch norm, it suffices to show that

$$\left|\int_{\mathbb{D}} h(z)\,\overline{(I_\mu f)'(z)}\,dA(z)\right| \lesssim \|h\|_{\mathcal{B}}\|f\|_{H^1}, \quad \text{for any polynomial } h. \tag{2.2.7}$$

So, let h be a polynomial. We have

$$\begin{aligned}
\int_{\mathbb{D}} h(z)\,\overline{(I_\mu f)'(z)}\,dA(z) \;&=\; \int_{\mathbb{D}} h(z)\,\overline{\left(\int_{[0,1)} \frac{t\,f(t)}{(1-tz)^2}\,d\mu(t)\right)}\,dA(z)\\
&=\; \int_{\mathbb{D}} h(z) \int_{[0,1)} \frac{t\,\overline{f(t)}}{(1-t\,\overline{z})^2}\,d\mu(t)\,dA(z)\\
&=\; \int_{[0,1)} t\,\overline{f(t)} \int_{\mathbb{D}} \frac{h(z)}{(1-t\,\overline{z})^2}\,dA(z)\,d\mu(t).
\end{aligned}$$

Because of the reproducing property of the Bergman kernel [111, Proposition 4. 23], $\int_{\mathbb{D}} \frac{h(z)}{(1-t\,\overline{z})^2}\,dA(z) \;=\; h(t)$. Then it follows that

$$\int_{\mathbb{D}} h(z)\,\overline{(I_\mu f)'(z)}\,dA(z) \;=\; \int_{[0,1)} t\,\overline{f(t)}\,h(t)\,d\mu(t). \tag{2.2.8}$$

Since μ is a 1-logarithmic 1-Carleson measure, the measure ν defined by

$$d\nu(t) \;=\; \log \frac{2}{1-t}\,d\mu(t)$$

is a Carleson measure by Proposition 1. This implies that

$$\int_{[0,1)} |f(t)| \log \frac{2}{1-t}\,d\mu(t) \lesssim \|f\|_{H^1}.$$

This and (2.2.5) yield

$$\int_{[0,1)} \left| t\,\overline{f(t)}\,h(t) \right| \, d\mu(t) \lesssim \|h\|_{\mathcal{B}} \|f\|_{H^1}.$$

Using this and (2.2.8), (2.2.7) follows.

Since $\mathcal{D}_0^1 \subset H^1$, the implication (iii) $\Rightarrow$ (iv) is trivial.

Now we turn to prove the implication (iv) $\Rightarrow$ (i). Assume that $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}_0^1$ into $\mathcal{D}_0^1$. We argue as in the proof of Theorem 1. For $\frac{1}{2} < b < 1$ set

$$f_b(z) = \frac{1-b^2}{(1-bz)^2}, \quad z \in \mathbb{D}.$$

We have $f_b'(z) = \frac{2b(1-b^2)}{(1-bz)^3}$ $(z \in \mathbb{D})$. Then, using Lemma 3. 10 of [111] with $t = 0$ and $c = 1$, we see that

$$\|f_b\|_{\mathcal{D}_0^1} \asymp \int_{\mathbb{D}} \frac{1-b^2}{|1-bz|^3} \, dA(z) \asymp 1.$$

Since $\mathcal{H}_\mu$ is bounded on $\mathcal{D}_0^1$, this implies that

$$1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{\mathcal{D}_0^1}. \tag{2.2.9}$$

We also have,

$$f_b(z) = \sum_{k=0}^{\infty} a_{k,b} z^k, \quad \text{with } a_{k,b} = (1-b^2)(k+1)b^k.$$

Using Hardy's inequality and the fact that $\mathcal{D}_0^1 \subset H^1$ (or, alternatively, Theorem F),

(2.2.9), and the definition of the $a_{k,b}$'s, we obtain

$$\begin{aligned}
1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{\mathcal{D}_0^1} &\gtrsim \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{\infty} a_{k,b} \int_{[0,1)} t^{n+k} \, d\mu(t) \right) \\
&\gtrsim (1-b^2) \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=1}^{\infty} k b^k \int_{[b,1)} t^{n+k} \, d\mu(t) \right) \\
&\gtrsim (1-b^2) \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=1}^{\infty} k b^{n+2k} \, \mu\left([b,1)\right) \right) \\
&= (1-b^2) \mu\left([b,1)\right) \sum_{n=1}^{\infty} \frac{b^n}{n} \left(\sum_{k=1}^{\infty} k b^{2k} \right) \\
&= (1-b^2) \mu\left([b,1)\right) \left(\log \frac{1}{1-b} \right) \frac{b^2}{(1-b^2)^2}.
\end{aligned}$$

Then it follows that

$$\mu\left([b,1)\right) = \mathrm{O}\left(\frac{1-b}{\log \frac{1}{1-b}} \right), \quad \text{as } b \to 1.$$

Hence, μ is a 1-logarithmic 1-Carleson measure. □

Before embarking into the proof of Theorem 4 we have to introduce some notation and results. Following [79], for $\alpha \in \mathbb{R}$ the weighted Bergman space $A^1(\log^\alpha)$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^1(\log^\alpha)} \stackrel{\text{def}}{=} \int_{\mathbb{D}} |f(z)| \left(\log \frac{2}{1-|z|} \right)^\alpha dA(z) < \infty.$$

This is a Banach space with the norm $\|\cdot\|_{A^1(\log^\alpha)}$ just defined and the polynomials are dense in $A^1(\log^\alpha)$. Likewise, we define

$$\mathcal{D}^1(\log^\alpha) = \{f \in \mathcal{H}ol(\mathbb{D}) : f' \in A^1(\log^\alpha)\}.$$

We define also the Bloch-type space $\mathcal{B}(\log^\alpha)$ as the space of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}(\log^\alpha)} \stackrel{\text{def}}{=} |f(0)| + \sup_{z \in \mathbb{D}} (1-|z|^2) \left(\log \frac{2}{1-|z|} \right)^{-\alpha} |f'(z)| < \infty,$$

and

$$\mathcal{B}_0(\log^\alpha) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : |f'(z)| = \mathrm{o}\left(\frac{\left(\log \frac{2}{1-|z|}\right)^\alpha}{1-|z|} \right), \text{ as } |z| \to 1 \right\}.$$

The space $\mathcal{B}(\log^\alpha)$ is a Banach space and $\mathcal{B}_0(\log^\alpha)$ is the closure of the polynomials in $\mathcal{B}(\log^\alpha)$.

We remark that the spaces $\mathcal{D}^1(\log^\alpha)$, $\mathcal{B}(\log^\alpha)$, and $\mathcal{B}_0(\log^\alpha)$ were called $\mathfrak{B}^1_{\log^\alpha}$, $\mathfrak{B}_{\log^\alpha}$, and $\mathfrak{b}_{\log^\alpha}$ in [79]. Pavlović identified in [79, Theorem 2. 4] the dual of the space $\mathcal{B}_0(\log^\alpha)$.

Theorem G. *Let $\alpha \in \mathbb{R}$. Then the dual of $\mathcal{B}_0(\log^\alpha)$ is $A^1(\log^\alpha)$ via the pairing*

$$\langle h, g \rangle = \int_{\mathbb{D}} h(z)\, \overline{g(z)}\, dA(z), \quad h \in \mathcal{B}_0(\log^\alpha), \ g \in A^1(\log^\alpha).$$

Actually, Pavlović formulated the duality theorem in another way but it is a simple exercise to show that his formulation is equivalent to this one which is better suited to our work.

Proof of Theorem 4. Let μ be a positive Borel measure on $[0,1)$ and $0 < \alpha < 1$. Suppose that μ is an α-logarithmic 1-Carleson measure. Take $f \in H^1$. We have to show that $I_\mu f \in \mathcal{D}^1(\log^{\alpha-1})$ or, equivalently, that $(I_\mu f)' \in A^1(\log^{\alpha-1})$. Bearing in mind Theorem G and the fact that $\mathcal{B}_0(\log^{\alpha-1})$ is the closure of the polynomials in $\mathcal{B}(\log^{\alpha-1})$, it suffices to show that

$$\left| \int_{\mathbb{D}} h(z)\, \overline{(I_\mu f)'(z)}\, dA(z) \right| \lesssim \|h\|_{\mathcal{B}(\log^{\alpha-1})} \|f\|_{H^1}, \quad \text{for any polynomial } h. \tag{2.2.10}$$

So, let h be a polynomial. Arguing as in the proof of the implication (i) $\Rightarrow$ (iii) in Theorem 3 we obtain

$$\int_{\mathbb{D}} h(z)\, \overline{(I_\mu f)'(z)}\, dA(z) = \int_{[0,1)} t\, \overline{f(t)}\, h(t)\, d\mu(t). \tag{2.2.11}$$

Now, it is clear that

$$|h(z)| \lesssim \|h\|_{\mathcal{B}(\log^{\alpha-1})} \left(\log \frac{2}{1-|z|} \right)^\alpha,$$

and then it follows that

$$\int_{[0,1)} \left| t\, \overline{f(t)}\, h(t) \right| d\mu(t) \lesssim \|h\|_{\mathcal{B}(\log^{\alpha-1})} \int_{[0,1)} |f(t)| \left(\log \frac{2}{1-t} \right)^\alpha d\mu(t).$$

Using the fact that the measure $\left(\log \frac{2}{1-t}\right)^{\alpha} d\mu(t)$ is a Carleson measure (Proposition 1) this implies that

$$\int_{[0,1)} \left| t\, \overline{f(t)}\, h(t) \right| d\mu(t) \lesssim \|h\|_{\mathcal{B}(\log^{\alpha-1})} \|f\|_{H^1}.$$

This and (2.2.11) give (2.2.10). □

2.3 A generalized Hilbert matrix acting on conformally invariant spaces

We start our study on conformally invariant spaces with $BMOA$ and the Bloch space. Let us recall that

$$H^{\infty} \subsetneq BMOA \subsetneq \bigcap_{0<p<\infty} H^p \quad \text{and } BMOA \subsetneq \mathcal{B}.$$

The Bloch space has a very important role among all conformally invariant spaces. Rubel and Timoney [91] proved that $\mathcal{B}$ is the biggest *natural* conformally invariant space.

Our first result in this section is devoted to characterize those μ for which the operator I_μ is well defined in $BMOA$ and in the Bloch space. It turns out that they coincide.

Theorem 5 ([54]). *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent:*

(i) $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$.

(ii) For any given $f \in \mathcal{B}$, the integral in (2.0.1) converges for all $z \in \mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic in $\mathbb{D}$.

(iii) For any given $f \in BMOA$, the integral in (2.0.1) converges for all $z \in \mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic in $\mathbb{D}$.

The next step is characterizing the measures μ so that I_μ is bounded in $BMOA$ or $\mathcal{B}$ and seeing whether or not I_μ and $\mathcal{H}_\mu$ coincide for such measures. We have the following results.

Theorem 6 ([54]). *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)} \log \frac{2}{1-t} d\mu(t) < \infty$. Then the following three conditions are equivalent:*

(i) The measure μ is a 1-logarithmic 1-Carleson measure.

(ii) The operator I_μ is bounded from $\mathcal{B}$ into $BMOA$.

(iii) The operator I_μ is bounded from $BMOA$ into itself.

Theorem 7 ([54]). *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$. If μ is a 1-logarithmic 1-Carleson measure, then $\mathcal{H}_\mu$ is well defined on the Bloch space and*

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \text{for all } f \in \mathcal{B}.$$

Theorem 6 and Theorem 7 together yield the following.

Theorem 8 ([54]). *Let μ be a positive Borel measure on $[0,1)$ such that is a 1-logarithmic 1-Carleson measure. Then the operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}$ into $BMOA$.*

We have also the following result regarding compactness.

Theorem 9 ([54]). *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$. If μ is a vanishing 1-logarithmic 1-Carleson measure then:*

(i) The operator I_μ is a compact operator from $\mathcal{B}$ into $BMOA$.

(ii) The operator I_μ is a compact operator from $BMOA$ into itself.

As it was said in the preliminaries, the Q_s spaces have the following relation with $BMOA$, the Bloch space and the Dirichlet space:

$$\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq BMOA, \qquad 0 < s_1 < s_2 < 1.$$

In the limit case $s = 1$, Q_s is the space $BMOA$ and for $s > 1$, all the spaces Q_s coincide with the Bloch space.

It is well known that the function $F(z) = \log \frac{2}{1-z}$ belong to Q_s, for all $s > 0$, (in fact, it is proved in [12] that the univalent functions in all Q_s-spaces $(0 < s < \infty)$ are the same). Using this we easily see that Theorem 5, Theorem 6 and Theorem 8 can be improved as follows.

Theorem 10 ([54]). *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent:*

(i) $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$.

(ii) For any given $s \in (0,\infty)$ and any $f \in Q_s$, the integral in (2.0.1) converges for all $z \in \mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic in $\mathbb{D}$.

We remark that condition (ii) with $s \geq 1$ includes the points (ii) and (iii) of Theorem 5.

Theorem 11 ([54]). *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$. Then the following two conditions are equivalent:*

(i) The measure μ is a 1-logarithmic 1-Carleson measure.

(ii) For any given $s \in (0,\infty)$, the operator I_μ is bounded from Q_s into $BMOA$.

We remark that (ii) with $s > 1$ reduces to condition (ii) of Theorem 6, while (ii) with $s = 1$ reduces to condition (iii) of Theorem 6.

Theorem 7 and Theorem 11 together yield the following.

Theorem 12. *Let μ be a positive Borel measure on $[0,1)$ such that is a 1-logarithmic 1-Carleson measure. Then, for any given $s \in (0,\infty)$, the operator $\mathcal{H}_\mu$ is bounded from Q_s into $BMOA$.*
We remark that for $s > 1$ the theorem reduces Theorem 8.

At this point it is natural to look for a characterization of those μ for which I_μ and/or $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{B}$ into itself or, more generally, from Q_s into itself for any $s > 0$. We have the following result.

Theorem 13 ([55]). *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent.*

(i) The operator I_μ is bounded from Q_s into itself for some $s > 0$.

(ii) The operator I_μ is bounded from Q_s into itself for all $s > 0$.

(iii) The operator $\mathcal{H}_\mu$ is bounded from Q_s into itself for some $s > 0$.

(iv) The operator $\mathcal{H}_\mu$ is bounded from Q_s into itself for all $s > 0$.

(v) The measure μ is a 1-logarithmic 1-Carleson measure.

In fact, we are able to prove a stronger result which does not distinguish between different Q_s spaces.

Theorem 14 ([55])**.** *Let μ be a positive Borel measure on $[0,1)$ and let $0 < s_1, s_2 < \infty$. Then following conditions are equivalent.*

(i) The operator I_μ is well defined in Q_{s_1} and, furthermore, it is a bounded operator from Q_{s_1} into Q_{s_2}.

(ii) The operator $\mathcal{H}_\mu$ is well defined in Q_{s_1} and, furthermore, it is a bounded operator from Q_{s_1} into Q_{s_2}.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

These results cannot be extended to the limit case $s = 0$. Indeed, the function $F(z) = \log \frac{2}{1-z}$ does not belong to the Dirichlet space $\mathcal{D}$.

Since the Dirichlet space is one among the analytic Besov spaces, $\mathcal{D} = B^2$, this case will be covered in our study of the operator on these spaces.

From now on, if $1 < p < \infty$ we let p' denote the exponent conjugate to p, that is, p' is defined by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in B^p$ $(1 < p < \infty)$ then, see [61] or [110],

$$|f(z)| = \mathrm{o}\left(\left(\log \frac{2}{1-|z|}\right)^{1/p'}\right), \quad \text{as } |z| \to 1, \tag{2.3.1}$$

and there exists a positive constant $C > 0$ such that

$$|f(z)| \leq C\|f\|_{B^p}\left(\log \frac{2}{1-|z|}\right)^{1/p'}, \quad z \in \mathbb{D}, \quad f \in B^p. \tag{2.3.2}$$

Clearly, (2.3.1) or (2.3.2) implies that the function $F(z) = \log \frac{2}{1-z}$ does not belong to B^p $(1 < p < \infty)$, a fact that we have already mentioned for $p = 2$. Our substitutes of Theorem 5 and Theorem 6 for Besov spaces are the following.

Theorem 15 ([54])**.** *Let $1 < p < \infty$ and let μ be a positive Borel measure on $[0,1)$. We have:*

(i) *If* $\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{1/p'} d\mu(t) < \infty$, *then for any given* $f \in B^p$, *the integral in (2.0.1) converges for all* $z \in \mathbb{D}$ *and the resulting function* $I_\mu(f)$ *is analytic in* $\mathbb{D}$.

(ii) *If for any given* $f \in B^p$, *the integral in (2.0.1) converges for all* $z \in \mathbb{D}$ *and the resulting function* $I_\mu(f)$ *is analytic in* $\mathbb{D}$, *then* $\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{\gamma} d\mu(t) < \infty$ *for all* $\gamma < \frac{1}{p'}$.

Theorem 16 ([54]). *Suppose that* $1 < p < \infty$ *and let* μ *be a positive Borel measure on* $[0, 1)$.

(i) *If* μ *is a* $1/p'$*-logarithmic* 1*-Carleson measure then the operator* I_μ *is bounded from* B^p *into* $BMOA$.

(ii) *If* μ *is a vanishing* $1/p'$*-logarithmic* 1*-Carleson measure then the operator* I_μ *is compact from* B^p *into* $BMOA$.

These results follow using the growth condition (2.3.2), the fact that if $\gamma < \frac{1}{p'}$ then the function $f(z) = \left(\log \frac{2}{1-z}\right)^{\gamma}$ belongs to B^p (see [61, Theorem 1]), and with arguments similar to those used in the proofs of Theorem 5, Theorem 6, and Theorem 9. We shall omit the details.

Let us work next with the operator $\mathcal{H}_\mu$ directly. The first results that we have obtained are sufficient conditions on μ which ensure that $\mathcal{H}_\mu$ is well defined on the Besov spaces.

Theorem 17 ([54]). *Let* μ *be a finite positive Borel measure on* $[0, 1)$.

(i) *If* $1 < p \leq 2$ *and* $\sum_{k=1}^{\infty} \frac{\mu_k^{p'}}{k} < \infty$, *then the operator* $\mathcal{H}_\mu$ *is well defined in* B^p.

(ii) *If* $2 < p < \infty$ *and* $\sum_{k=1}^{\infty} \frac{\mu_k^{p'}}{k^{p'/p}} < \infty$, *then the operator* $\mathcal{H}_\mu$ *is well defined in* B^p.

Let us turn to study when is the operator $\mathcal{H}_\mu$ bounded from B^p into itself. Let us mention that Bao and Wulan [16] considered an operator which is closely related to the operator $\mathcal{H}_\mu$ acting on the Dirichlet spaces $\mathcal{D}_p$ ($p \in \mathbb{R}$) which are defined as follows:

For $p \in \mathbb{R}$, the space $\mathcal{D}_p$ consists of those functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\mathbb{D}$ for which

$$\|f\|_{\mathcal{D}_p} \stackrel{\text{def}}{=} \left(\sum_{n=0}^{\infty} (n+1)^{1-p} |a_n|^2 \right)^{1/2} < \infty.$$

Let us remark that $\mathcal{D}_0$ is the Dirichlet spaces $\mathcal{D} = B^2$, while $\mathcal{D}_1 = H^2$.

Bao and Wulan proved that if μ is a positive Borel measure on $[0,1)$ and $0 < p < 2$, then the operator $\mathcal{H}_\mu$ is bounded from $\mathcal{D}_p$ into itself if and only if μ is a Carleson measure. Let us remark that this does not include the case $p = 0$. In fact, the following results are proved in [16].

Theorem H.

(i) There exists a positive Borel measure μ on $[0,1)$ which is a Carleson measure but such that $\mathcal{H}_\mu(B^2) \not\subset B^2$.

(ii) Let μ be a positive Borel measure on $[0,1)$ such that the operator $\mathcal{H}_\mu$ is a bounded operator from B^2 into itself. Then μ is a Carleson measure.

We can improve these results and, even more, we shall obtain extensions of these improvements to all B^p spaces ($1 < p < \infty$). More precisely we are going to prove the following results.

Theorem 18 ([54])**.** *Suppose that $1 < p < \infty$ and $0 < \beta \leq \frac{1}{p}$. Then there exists a positive Borel measure μ on $[0,1)$ which is a β-logarithmic 1-Carleson measure but such that the operator $\mathcal{H}_\mu$ does not apply B^p into itself.*

Next we prove that μ being a β-logarithmic 1-Carleson measure for a certain β is a necessary condition for $\mathcal{H}_\mu$ being a bounded operator from B^p into itself.

Theorem 19 ([55])**.** *Suppose that $1 < p < \infty$ and let μ be a positive Borel measure on $[0,1)$ such that the operator $\mathcal{H}_\mu$ is bounded from B^p into itself. Then μ is a $1/p'$-logarithmic 1-Carleson measure.*

Finally, we obtain a sufficient condition for the boundedness of $\mathcal{H}_\mu$ from B^p into itself.

Theorem 20 ([54])**.** *Suppose that $1 < p < \infty$, $\gamma > 1$, and let μ be a positive Borel measure on $[0,1)$ which is a γ-logarithmic 1-Carleson measure. Then the operator $\mathcal{H}_\mu$ is a bounded operator from B^p into itself.*

2.3.1 Preliminary results

In this section we shall collect a number of results which will be needed in our work.

The following lemma will be needed in the proof of Theorem 7.

Lemma 1. *Let μ be a positive Borel measure in $[0,1)$. If μ is a 1-logarithmic 1-Carleson measure then the sequence of moments $\{\mu_n\}$ satisfies*

$$\mu_n = \mathrm{O}\left(\frac{1}{n\log n}\right), \quad \text{as } n \to \infty.$$

Actually, we shall prove the following more general result.

Lemma 2. *Suppose that $0 \le \alpha \le \beta$, $s \ge 1$, and let μ be a positive Borel measure on $[0,1)$ which is a β-logarithmic s-Carleson measure. Then*

$$\int_{[0,1)} t^k \left(\log\frac{2}{1-t}\right)^{\alpha} d\mu(t) \;=\; \mathrm{O}\left(\frac{(\log k)^{\alpha-\beta}}{k^s}\right), \quad \text{as } k \to \infty.$$

Lemma 1 follows taking $\alpha = 0$, $\beta = 1$, and $s = 1$ in Lemma 2.

Proof of Lemma 2. Arguing as in the proof of the implication (b) $\Rightarrow$ (a) of Proposition 1, integrating by parts and using the fact that μ is a β-logarithmic s-Carleson measure, we obtain

$$\begin{aligned}
&\int_{[0,1)} t^k \left(\log\frac{2}{1-t}\right)^{\alpha} d\mu(t) \qquad (2.3.3)\\
&= k\int_0^1 \mu\big([t,1)\big)t^{k-1}\left(\log\frac{2}{1-t}\right)^{\alpha} dt \;+\; \alpha\int_0^1 \mu\big([t,1)\big)t^{k}\left(\log\frac{2}{1-t}\right)^{\alpha-1}\frac{dt}{1-t}\\
&\lesssim k\int_0^1 (1-t)^s t^{k-1}\left(\log\frac{2}{1-t}\right)^{\alpha-\beta} dt \;+\; \alpha\int_0^1 (1-t)^{s-1} t^{k}\left(\log\frac{2}{1-t}\right)^{\alpha-\beta-1} dt.
\end{aligned}$$

Now, we notice that the weight functions

$$\omega_1(t) = (1-t)^s\left(\log\frac{2}{1-t}\right)^{\alpha-\beta} \quad \text{and} \quad \omega_2(t) = (1-t)^{s-1}\left(\log\frac{2}{1-t}\right)^{\alpha-\beta-1}$$

are regular in the sense of [82] (see [82, p. 6] and [4, Example 2]). Then, using Lemma 1. 3 of [82] and the fact that the ω_j's are also decreasing, we obtain

$$\begin{aligned}
\int_0^1 (1-t)^s t^{k-1}\left(\log\frac{2}{1-t}\right)^{\alpha-\beta} dt &\lesssim \int_{1-\frac{1}{k}}^1 (1-t)^s t^{k-1}\left(\log\frac{2}{1-t}\right)^{\alpha-\beta} dt\\
&\lesssim \frac{(\log k)^{\alpha-\beta}}{k^{s+1}}
\end{aligned}$$

and

$$\int_0^1 (1-t)^{s-1} t^k \left(\log \frac{2}{1-t}\right)^{\alpha-\beta-1} dt \lesssim \int_{1-\frac{1}{k}}^1 (1-t)^{s-1} t^k \left(\log \frac{2}{1-t}\right)^{\alpha-\beta-1} dt$$
$$\lesssim \frac{(\log k)^{\alpha-\beta-1}}{k^s}.$$

Using these two estimates in (2.3.3) yields

$$\int_{[0,1)} t^k \left(\log \frac{2}{1-t}\right)^{\alpha} d\mu(t) \lesssim \frac{(\log k)^{\alpha-\beta}}{k^s}$$

finishing the proof. □

We shall also use the characterization of the coefficient multipliers from $\mathcal{B}$ into ℓ^1 obtained by Anderson and Shields in [7].

Theorem I. *A sequence $\{\lambda_n\}_{n=0}^{\infty}$ of complex numbers is a coefficient multiplier from $\mathcal{B}$ into ℓ^1 if and only if*

$$\sum_{n=1}^{\infty} \left(\sum_{k=2^n+1}^{2^{n+1}} |\lambda_k|^2 \right)^{1/2} < \infty.$$

Bearing in mind Definition 1 of [7], Theorem I reduces to the case $p = 1$ in Corollary 1 in p. 259 of [7].

We recall that if X is a space of analytic functions in $\mathbb{D}$ and Y is a space of complex sequences, a sequence $\{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{C}$ is said to be a multiplier of X into Y if whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n \in X$ one has that the sequence $\{\lambda_n a_n\}_{n=0}^{\infty}$ belongs to Y. Thus:

By saying that $\{\lambda_n\}_{n=0}^{\infty}$ is a coefficient multiplier from $\mathcal{B}$ into ℓ^1 we mean that

$$\text{If } f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B} \text{ then } \sum_{n=0}^{\infty} |\lambda_n a_n| < \infty.$$

Actually, using the closed graph theorem, we can assert the following:

A complex sequence $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier from $\mathcal{B}$ to ℓ^1 if and only if there exists a positive constant C such that whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$, we have that $\sum_{n=0}^{\infty} |\lambda_n a_n| \leq C \|f\|_{\mathcal{B}}$.

In the proof of Theorem 13 we will use as a basic ingredient a characterization of the functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ whose sequence of Taylor coefficients $\{a_n\}_{n=0}^{\infty}$ is

a decreasing sequence of nonnegative numbers which lie in the Q_s-spaces. This is quite simple for $s > 1$ (recall that $Q_s = \mathcal{B}$ if $s > 1$).

Hwang and Lappan proved in [62, Theorem 1] that if $\{a_n\}$ is a decreasing sequence of nonnegative numbers then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a Bloch function if and only if $a_n = \mathrm{O}\left(\frac{1}{n}\right)$.

Fefferman gave a characterization of the analytic functions having nonnegative Taylor coefficients which belong to $BMOA$, proofs of this criterium can be found in [24, 52, 60, 95]. Characterizations of the analytic functions having nonnegative Taylor coefficients which belong to Q_s $(0 < s < 1)$ were obtained in [13, Theorem 1. 2] and [10, Theorem 2. 3]. Using the mentioned result in [13, Theorem 1. 2], Xiao proved in [103, Corollary 3. 3. 1, p. 29] the following result.

Theorem J. *Let $s \in (0, \infty)$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\{a_n\}$ being a decreasing sequence of nonnegative numbers. Then $f \in Q_s$ if and only if $a_n = \mathrm{O}\left(\frac{1}{n}\right)$.*

Being based on Theorem 1. 2 of [13], Xiao's proof of this result is complicated. We shall give next an alternative simpler proof. It will simply use the validity of the result for the Bloch space and the simple fact that the mean Lipschitz space $\Lambda^2_{1/2}$ is contained in all the Q_s spaces $(0 < s < \infty)$ (see [10, Remark 4, p. 427] or [103, Theorem 4. 2. 1.]).

We recall [40, Chapter 5] that a function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the mean Lipschitz space $\Lambda^2_{1/2}$ if and only if

$$M_2(r, f') = \mathrm{O}\left(\frac{1}{(1-r)^{1/2}}\right).$$

We have the following simple result for the space $\Lambda^2_{1/2}$.

Lemma 3. *If $\{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative numbers and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, then $f \in \Lambda^2_{1/2}$ if and only if $a_n = \mathrm{O}\left(\frac{1}{n}\right)$.*

Proof. If $a_n = \mathrm{O}\left(\frac{1}{n}\right)$, then

$$M_2(r, f')^2 = \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} \lesssim \sum_{n=1}^{\infty} r^{2n-2} \lesssim \frac{1}{1-r},$$

and, hence, $f \in \Lambda^2_{1/2}$.

Suppose now that $\{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative numbers and $f \in \Lambda^2_{1/2}$. Then, for all n

$$\sum_{k=1}^{n} k^2 a_k^2 r^{2k-2} \leq \sum_{k=1}^{\infty} k^2 a_k^2 r^{2k-2} = M_2(r, f')^2 \lesssim \frac{1}{1-r}. \tag{2.3.4}$$

Taking $r = 1 - \frac{1}{n}$ in (2.3.4), we obtain

$$\sum_{k=1}^{n} k^2 a_k^2 \lesssim n. \tag{2.3.5}$$

Since $\{a_n\}$ is decreasing, using (2.3.5) we have

$$a_n^2 \sum_{k=1}^{n} k^2 \lesssim \sum_{k=1}^{n} k^2 a_k^2 \lesssim n$$

and then it follows that $a_n = \mathrm{O}\left(\frac{1}{n}\right)$. □

Now Theorem J follows using the result of Hwang and Lappan for the Bloch space, Lemma 3, and the fact that

$$\Lambda^2_{1/2} \subset Q_s \subset \mathcal{B}, \qquad \text{for all } s. \tag{2.3.6}$$

Using (2.3.6), it is clear that Theorem 14 follows from the following result.

Theorem 21. *Let μ be a positive Borel measure on $[0,1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.*

(i) The operator I_μ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.

(ii) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) = \mathrm{O}\left(\frac{1}{n}\right)$.

Proof. According to Proposition 1 ([54, Proposition 2.5]), μ is a 1-logarithmic 1-Carleson measure if and only if the measure ν defined by $d\nu(t) = \log \frac{1}{1-t} d\mu(t)$ is a Carleson measure and, using Proposition 1 of [28], this is equivalent to (iv). Hence, we have shown that (iii) $\Leftrightarrow$ (iv).

Set $F(z) = \log \frac{1}{1-z}$ $(z \in \mathbb{D})$. We have that $F \in X$.

(i) $\Rightarrow$ (iv): Suppose (i). Then

$$I_\mu(F)(z) = \int_{[0,1)} \frac{\log \frac{1}{1-t}}{1-tz} d\mu(t)$$

is well defined for all $z \in \mathbb{D}$. Taking $z = 0$, we see that $\int_{[0,1)} \log \frac{1}{1-t} d\mu(t) < \infty$. Since $F \in X$ we have also that $I_\mu(F) \in \Lambda^2_{1/2}$, but

$$I_\mu(F)(z) = \int_{[0,1)} \frac{\log \frac{1}{1-t}}{1-tz} d\mu(t) = \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) \right) z^n.$$

Since the sequence $\left\{ \int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) \right\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative numbers, using Lemma 3 we see that (iv) holds.

(iv) $\Rightarrow$ (i): Suppose (iv) and take $f \in X$. Since $X \subset \mathcal{B}$, it is well known that $|f(z)| \lesssim \log \frac{2}{1-|z|}$, see [6, p. 13]. This and (iv) give

$$\int_{[0,1)} t^n |f(t)| d\mu(t) = \mathrm{O}\left(\frac{1}{n}\right). \tag{2.3.7}$$

Then it follows easily that $I_\mu(f)$ is well defined and that

$$I_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t) d\mu(t) \right) z^n.$$

Now (2.3.7) implies that $\int_{[0,1)} t^n f(t) d\mu(t) = \mathrm{O}\left(\frac{1}{n}\right)$ and then it follows that $I_\mu(f) \in \Lambda^2_{1/2}$.

The implication (iv) $\Rightarrow$ (ii) follows using Theorem 7 ([54, Theorem 2.3]) and the already proved equivalences (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

It remains to prove that (ii) $\Rightarrow$ (iv). Suppose (ii) then $\mathcal{H}_\mu(F) \in \Lambda^2_{1/2}$. Now

$$\mathcal{H}_\mu(F)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{\mu_{n+k}}{k} \right) z^n.$$

Notice that the sequence $\{\sum_{k=1}^{\infty} \frac{\mu_{n+k}}{k}\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative numbers. Then, using Lemma 3 and the fact that $\mathcal{H}_\mu(F) \in \Lambda^2_{1/2}$, we deduce that

$$\sum_{k=1}^{\infty} \frac{\mu_{n+k}}{k} = \mathrm{O}\left(\frac{1}{n}\right). \tag{2.3.8}$$

Now

$$\sum_{k=1}^{\infty} \frac{\mu_{n+k}}{k} = \int_{[0,1)} \sum_{k=1}^{\infty} \frac{t^{n+k}}{k} d\mu(t) = \int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t).$$

Then (iv) follows using (2.3.8). □

In order to to prove Theorem 17 we need some results on the Taylor coefficients of functions in B^p. The following result was proved by Holland and Walsh in [61, Theorem 2].

Theorem K.

(i) Suppose that $1 < p \leq 2$. Then there exists a positive constant C_p such that if $f \in B^p$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$ then

$$\sum_{k=1}^{\infty} k^{p-1} |a_k|^p \leq C_p \rho_p(f)^p.$$

(ii) If $2 \leq p < \infty$ then there exists $C_p > 0$ such that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$ with $\sum_{k=1}^{\infty} k^{p-1}|a_k|^p < \infty$ then $f \in B^p$ and

$$\rho_p(f)^p \leq C_p \sum_{k=1}^{\infty} k^{p-1} |a_k|^p.$$

If $p \neq 2$ the converses to (i) and (ii) are false.

Theorem K is the analogue for Besov spaces of results of Hardy and Littlewood for Hardy spaces (Theorem 6. 2 and Theorem 6. 3 of [40]).

In spite of the fact that the converse to (ii) is not true, the membership of f in B^p $(p > 2)$ implies some summability conditions on the Taylor coefficients $\{a_k\}$ of f. Indeed, Pavlović has proved the following result in [80, Theorem 2. 3] (see also [64, Theorem 8.4.1(iv)]).

Theorem L. *Suppose that $2 < p < \infty$. Then there exists a positive constant C_p such that if $f \in B^p$ and $f(z) = \sum_{k=0}^{\infty} a_k\, z^k$ $(z \in \mathbb{D})$ then*

$$\sum_{k=1}^{\infty} k|a_k|^p \le C_p\, \rho_p(f)^p.$$

We shall need a number of results on Besov spaces, as well as some lemmas, to prove Theorem 18 and Theorem 20. First of all we notice that the Besov spaces can be characterized in terms of "dyadic blocks". In order to state this in a precise way we need to introduce some notation.

For a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\mathbb{D}$, define the polynomials $\Delta_j f$ as follows:

$$\Delta_j f(z) = \sum_{k=2^j}^{2^{j+1}-1} a_k z^k, \quad \text{for } j \ge 1,$$

$$\Delta_0 f(z) = a_0 + a_1 z.$$

Mateljević and Pavlović proved in [71, Theorem 2. 1] (see also [78, Theorem C]) the following result.

Theorem M. *Let $1 < p < \infty$ and $\alpha > -1$. For a function f analytic in $\mathbb{D}$ we define*

$$Q_1(f) \stackrel{\text{def}}{=} \int_{\mathbb{D}} |f(z)|^p (1-|z|)^\alpha \, dA(z), \quad Q_2(f) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \|\Delta_n f\|_{H^p}^p.$$

Then, $Q_1(f) \asymp Q_2(f)$.

Theorem M readily implies the following result.

Corollary 1. *Suppose that $1 < p < \infty$ and f is an analytic function in $\mathbb{D}$. Then*

$$f \in B^p \;\Leftrightarrow\; \sum_{n=0}^{\infty} 2^{-n(p-1)} \|\Delta_n f'\|_{H^p}^p < \infty.$$

Furthermore,

$$\rho_p(f)^p \asymp \sum_{n=0}^{\infty} 2^{-n(p-1)} \|\Delta_n f'\|_{H^p}^p.$$

Using Corollary 1 we can prove that the converses of (i) and (ii) in Theorem K hold if the sequence of Taylor coefficients $\{a_n\}$ decreases to 0. This is the analogue for Besov spaces of the result proved in [57] by Hardy and Littlewood for Hardy spaces (see also [77, 7. 5. 9], [78] and [112, Chapter XII, Lemma 6. 6]). Analogous results for the spaces $\mathcal{D}^p_{p-1}$ $(p > 1)$ and for Bergman spaces A^p $(p > 1)$ were proved in [78, Theorem 3.1] and [26, Proposition 2.4] respectively.

Theorem 22. *Suppose that $1 < p < \infty$ and let $\{a_n\}_{n=0}^{\infty}$ be a decreasing sequence of non-negative numbers with $\{a_n\} \to 0$, as $n \to \infty$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. Then*

$$f \in B^p \;\Leftrightarrow\; \sum_{n=1}^{\infty} n^{p-1} a_n^p \;<\; \infty.$$

Furthermore, $\rho_p(f)^p \asymp \sum_{n=1}^{\infty} n^{p-1} a_n^p$.

Proof. For every n, we have

$$z\,(\Delta_n f')\,(z) \;=\; \sum_{k=2^n+1}^{2^{n+1}} k a_k z^k.$$

Since the sequence $\lambda = \{k\}_{k=0}^{\infty}$ is an increasing sequence of non-negative numbers, using Lemma A of [78] we see that

$$\|z\,(\Delta_n f')\,\|_{H^p}^p \;\asymp\; 2^{np} \|\Delta_n f\|_{H^p}^p. \tag{2.3.9}$$

Now, set $h(z) = \sum_{n=0}^{\infty} z^n$ $(z \in \mathbb{D})$. Since the sequence $\tilde{\lambda} = \{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of non-negative numbers, using the second part of Lemma A of [78], we see that

$$a_{2^n}^p \|\Delta_n h\|_{H^p}^p \;\lesssim\; \|\Delta_n f\|_{H^p}^p \;\lesssim\; a_{2^{n-1}}^p \|\Delta_n h\|_{H^p}^p. \tag{2.3.10}$$

Notice that $h(z) = \frac{1}{1-z}$ $(z \in \mathbb{D})$. Then it is well known that $M_p(r, h) \asymp (1-r)^{\frac{1}{p}-1}$ (recall that $1 < p < \infty$). Following the notation of [71], this can be written as $h \in H\left(p, \infty, 1 - \frac{1}{p}\right)$. Then using Theorem 2. 1 of [71] (see also [77, p. 120]), we deduce that $\|\Delta_n f\|_{H^p}^p \asymp 2^{n(p-1)}$. Using this and (2.3.10), it follows that

$$2^{n(p-1)} a_{2^n}^p \;\lesssim\; \|\Delta_n f\|_{H^p}^p \;\lesssim\; 2^{n(p-1)} a_{2^{n-1}}^p. \tag{2.3.11}$$

Using Corollary 1, (2.3.9), and (2.3.11), we see that

$$\rho_p(f)^p \;\asymp\; \sum_{n=0}^{\infty} 2^{-n(p-1)} \|z\,\Delta_n f'\|_{H^p}^p \;\asymp\; \sum_{n=0}^{\infty} 2^n \|\Delta_n f\|_{H^p}^p \;\asymp\; \sum_{n=0}^{\infty} 2^{np} a_{2^n}^p.$$

Now, the fact that $\{a_n\}$ is decreasing implies that $\sum_{n=0}^{\infty} 2^{np} a_{2^n}^p \asymp \sum_{n=1}^{\infty} n^{p-1} a_n^p$ and, then it follows that $\rho_p(f)^p \asymp \sum_{n=1}^{\infty} n^{p-1} a_n^p$. $\square$

Remark 1. If f is an analytic function in $\mathbb{D}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, and $1 < p < \infty$ then any of the two conditions $f \in B^p$ and $\sum_{n=1}^{\infty} n^{p-1} |a_n|^p < \infty$ implies that $\{a_n\} \to 0$. Consequently, the condition $\{a_n\} \to 0$ can be omitted in the hypotheses of Theorem 22.

Suppose that $\beta \geq 0$, $s \geq 1$, $1 < p < \infty$, and μ is a positive Borel measure on $[0,1)$ which is a β-logarithmic s-Carleson measure. Using Lemma 2 and Theorem 17, it follows that $\mathcal{H}_\mu$ is well defined on B^p. Also, it is easy to see that $\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{1/p'} d\mu(t) < \infty$, a fact that, using Theorem 15 (i), shows that I_μ is also well defined in B^p. Using then standard arguments it follows that I_μ and $\mathcal{H}_\mu$ coincide in B^p. Let us state this as a lemma.

Lemma 4. *Suppose that $\beta \geq 0$, $s \geq 1$, $1 < p < \infty$, and μ is a positive Borel measure on $[0,1)$ which is a β-logarithmic s-Carleson measure. Then the operators $\mathcal{H}_\mu$ and I_μ are well defined in B^p and $\mathcal{H}_\mu(f) = I_\mu(f)$, for all $f \in B^p$.*

The following lemma will be used to prove Theorem 20. It is an adaptation of [45, Lemma 7] to our setting. The proof is very similar to that of the latter but we include it for the sake of completeness.

Lemma 5. *Let p, γ, and μ be as in Theorem 20. Then, there exists a constant $C = C(p,\gamma,\mu) > 0$ such that if $f \in B^p$, $g(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathcal{H}ol(\mathbb{D})$, and we set*

$$h(z) = \sum_{k=0}^{\infty} c_k \left(\int_0^1 t^{k+1} f(t)\, d\mu(t) \right) z^k,$$

then

$$\|\Delta_n h\|_{H^p} \leq C \left(\int_0^1 t^{2^{n-2}+1} |f(t)|\, d\mu(t) \right) \|\Delta_n g\|_{H^p}, \quad n \geq 3.$$

Proof. For each $n = 1, 2, \ldots$, define

$$\Upsilon_n(s) = \int_0^1 t^{2^n s+1} f(t)\, d\mu(t), \quad s \geq 0.$$

Clearly, Υ_n is a $C^\infty(0,\infty)$-function and

$$|\Upsilon_n(s)| \leq \int_0^1 t^{2^{n-2}+1} |f(t)|\, d\mu(t), \quad s \geq \frac{1}{2}. \tag{2.3.12}$$

Furthermore, since $\sup_{0<x<1}\left(\log\frac{1}{x}\right)^2 x^{1/2} = C(2) < \infty$, we have

$$\begin{aligned}|\Upsilon_n''(s)| &\le \int_0^1 \left[\left(\log\frac{1}{t^{2^n}}\right)^2 t^{2^{n-1}}\right] t^{2^n s+1-2^{n-1}}|f(t)|\,d\mu(t)\\ &\le C(2)\int_0^1 t^{2^n s+1-2^{n-1}}|f(t)|\,d\mu(t) \le C(2)\int_0^1 t^{2^{n-2}+1}|f(t)|\,d\mu(t), \quad s\ge \tfrac{3}{4}.\end{aligned} \tag{2.3.13}$$

Then, using (2.3.12) and (2.3.13), for each $n = 1, 2, \dots$, we can take a function $\Phi_n \in C^\infty(\mathbb{R})$ with $\operatorname{supp}(\Phi_n) \in \left(\frac{3}{4}, 4\right)$, and such that

$$\Phi_n(s) = \Upsilon_n(s), \quad s \in [1, 2],$$

and

$$A_{\Phi_n} = \max_{s\in\mathbb{R}}|\Phi_n(s)| + \max_{s\in\mathbb{R}}|\Phi_n''(s)| \le C\int_0^1 t^{2^{n-2}+1}|f(t)|\,d\mu(t).$$

Following the notation used in [45, p. 236], we can then write

$$\begin{aligned}\Delta_n h(z) &= \sum_{k=2^n}^{2^{n+1}-1} c_k\left(\int_0^1 t^{k+1}f(t)\,d\mu(t)\right) z^k\\ &= \sum_{k=2^n}^{2^{n+1}-1} c_k\Phi_n\left(\frac{k}{2^n}\right) z^k = W_{2^n}^{\Phi_n} * \Delta_n g(z).\end{aligned}$$

So by using part (iii) of Theorem B of [45], we have

$$\begin{aligned}\|\Delta_n h\|_{H^p} &= \|W_{2^n}^{\Phi_n} * \Delta_n g\|_{H^p} \le C_p A_{\Phi_n}\|\Delta_n g\|_{H^p}\\ &\le C\left(\int_0^1 t^{2^{n-2}+1}|f(t)|\,d\mu(t)\right)\|\Delta_n g\|_{H^p}.\end{aligned}$$

□

2.3.2 Proofs

Proof of Theorem 5.

(i) ⇒ (ii). It is well known that there exists a positive constant C such that

$$|f(z)| \le C\|f\|_{\mathcal{B}}\log\frac{2}{1-|z|}, \quad (z\in\mathbb{D}), \text{ for every } f\in\mathcal{B}, \tag{2.3.14}$$

(see [6, p. 13]). Assume (i) and set $A = \int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t)$. Using (2.3.14) we see that

$$\int_{[0,1)} |f(t)|\, d\mu(t) \le C\|f\|_{\mathcal{B}} \int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) = AC\|f\|_{\mathcal{B}}, \quad f \in \mathcal{B}. \tag{2.3.15}$$

This implies that

$$\int_{[0,1)} \frac{|f(t)|}{|1-tz|}\, d\mu(t) \le \frac{AC\|f\|_{\mathcal{B}}}{1-|z|}, \quad (z \in \mathbb{D}), \quad f \in \mathcal{B}. \tag{2.3.16}$$

Using (2.3.15), (2.3.16), and Fubini's theorem we see that if $f \in \mathcal{B}$ then:

- For every $n \in \mathbb{N}$, the integral $\int_{[0,1)} t^n f(t)\, d\mu(t)$ converges absolutely and

$$\sup_{n\ge 0} \left| \int_{[0,1)} t^n f(t)\, d\mu(t) \right| < \infty.$$

- The integral $\int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t)$ converges absolutely, and

$$\int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t) = \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t)\, d\mu(t) \right) z^n, \quad z \in \mathbb{D}.$$

Thus, if $f \in \mathcal{B}$ then $I_\mu(f)$ is a well defined analytic function in $\mathbb{D}$ and

$$I_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t)\, d\mu(t) \right) z^n, \quad z \in \mathbb{D}.$$

The implication (ii) $\Rightarrow$ (iii) is clear because $BMOA \subset \mathcal{B}$.

(iii) $\Rightarrow$ (i). Suppose (iii). Since the function $F(z) = \log \frac{2}{1-z}$ belongs to $BMOA$, $I_\mu(F)(z)$ is well defined for every $z \in \mathbb{D}$. In particular

$$I_\mu(F)(0) = \int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t)$$

is a complex number. Since μ is a positive measure and $\log \frac{2}{1-t} > 0$ for all $t \in [0,1)$, (i) follows. □

Proof of Theorem 6. Since $\int_{[0,1)} \log \frac{2}{1-t} d\mu(t) < \infty$, (2.3.14) implies that

$$\int_{[0,1)} |f(t)|\, d\mu(t) < \infty, \quad \text{for all } f \in \mathcal{B}$$

and this implies that

$$\int_0^{2\pi}\int_{[0,1)}\left|\frac{f(t)g(e^{i\theta})}{1-re^{i\theta}t}\right|d\mu(t)d\theta<\infty,\quad 0\le r<1,\ f\in\mathcal{B},\ g\in H^1.$$

Using this, Fubini's theorem and Cauchy's integral representation of H^1-functions [40, Theorem 3. 6], we deduce that whenever $f\in\mathcal{B}$ and $g\in H^1$ we have

$$\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta \;=\; \int_0^{2\pi}\left(\int_{[0,1)}\frac{f(t)d\mu(t)}{1-re^{i\theta}t}\right)\overline{g(e^{i\theta})}\,d\theta \tag{2.3.17}$$

$$=\int_{[0,1)} f(t)\left(\int_0^{2\pi}\frac{\overline{g(e^{i\theta})}d\theta}{1-re^{i\theta}t}\right)d\mu(t)\;=\;2\pi\int_{[0,1)} f(t)\overline{g(rt)}\,d\mu(t),\quad 0\le r<1.$$

(i) $\Rightarrow$ (ii). Using Proposition 1 we assume that ν is a Carleson measure and take $f\in\mathcal{B}$ and $g\in H^1$. Using (2.3.17) and (2.3.14), we obtain

$$\left|\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta\right| \;=\; \left|\int_{[0,1)} f(t)\overline{g(rt)}\,d\mu(t)\right|$$

$$\lesssim\ \|f\|_{\mathcal{B}}\int_{[0,1)}|g(rt)|\log\frac{2}{1-t}\,d\mu(t)\;=\;\|f\|_{\mathcal{B}}\int_{[0,1)}|g(rt)|\;d\nu(t).$$

Since ν is a Carleson measure

$$\int_{[0,1)}|g(rt)|\;d\nu(t)\;\lesssim\;\|g_r\|_{H^1}\;\le\;\|g\|_{H^1}.$$

Here, as usual, g_r is the function defined by $g_r(z)=g(rz)$ $(z\in\mathbb{D})$.

Thus, we have proved that

$$\left|\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta\right|\;\lesssim\;\|f\|_{\mathcal{B}}\|g\|_{H^1},\quad f\in\mathcal{B},\ g\in H^1.$$

Using Fefferman's duality Theorem (see [52, Theorem 7. 1]) we deduce that if $f\in\mathcal{B}$ then $I_\mu(f)\in BMOA$ and

$$\|I_\mu(f)\|_{BMOA}\lesssim\|f\|_{\mathcal{B}}.$$

The implication (ii) $\Rightarrow$ (iii) is trivial because $BMOA\subset\mathcal{B}$.

(iii) $\Rightarrow$ (i). Assume (iii). Then there exists a positive constant A such that $\|I_\mu(f)\|_{BMOA}\le A\|f\|_{BMOA}$, for all $f\in BMOA$. Set

$$F(z)=\log\frac{2}{1-z},\quad z\in\mathbb{D}.$$

It is well known that $F \in BMOA$. Then $I_\mu(F) \in BMOA$ and

$$\|I_\mu(F)\|_{BMOA} \leq A\|F\|_{BMOA}.$$

Then using again Fefferman's duality theorem we obtain that

$$\left|\int_0^{2\pi} I_\mu(F)(re^{i\theta})\,\overline{g(e^{i\theta})}\,d\theta\right| \lesssim \|g\|_{H^1}, \quad g \in H^1.$$

Using (2.3.17) and the definition of F, this implies

$$\left|\int_{[0,1)]} \overline{g(rt)}\,\log\frac{2}{1-t}\,d\mu(t)\right| \lesssim \|g\|_{H^1}, \quad g \in H^1. \tag{2.3.18}$$

Take $g \in H^1$. Using Proposition 2 of [28] we know that there exists a function $G \in H^1$ with $\|G\|_{H^1} = \|g\|_{H^1}$ and such that

$$|g(s)| \leq G(s), \quad \text{for all } s \in [0,1).$$

Using these properties and (2.3.18) for G, we obtain

$$\begin{aligned}\int_{[0,1)} |g(rt)|\log\frac{2}{1-t}\,d\mu(t) &\leq \int_{[0,1)} G(rt)\log\frac{2}{1-t}\,d\mu(t)\\ &\leq C\|G_r\|_{H^1} \leq C\|G\|_{H^1} = C\|g\|_{H^1}\end{aligned}$$

for a certain constant $C > 0$, independent of g. Letting r tend to 1, it follows that

$$\int_{[0,1)} |g(t)|\log\frac{2}{1-t}\,d\mu(t) \lesssim \|g\|_{H^1}, \quad g \in H^1.$$

This is equivalent to saying that ν is a Carleson measure so, by Proposition 1, μ is a 1-logarithmic 1-Carleson measure. □

Proof of Theorem 7. Suppose that μ is a 1-logarithmic 1-Carleson measure. Then, using Lemma 1, we see that there exists $C > 0$ such that

$$|\mu_n| \leq \frac{C}{n\log n}, \quad n \geq 2. \tag{2.3.19}$$

It is clear that

$$k^2\log^2 k \geq 2^{2n}n^2(\log 2)^2, \quad \text{if } 2^n + 1 \leq k \leq 2^{n+1} \text{ for all } n.$$

Then it follows that

$$\sum_{n=1}^{\infty}\left(\sum_{k=2^n+1}^{2^{n+1}}\frac{1}{k^2\log^2 k}\right)^{1/2}\lesssim\sum_{n=1}^{\infty}\left(\frac{2^n}{n^2 2^{2n}}\right)^{1/2}=\sum_{n=1}^{\infty}\frac{1}{n2^{n/2}}<\infty.$$

Using this, (2.3.19) and Theorem I, we obtain:

$$\text{The sequence of moments } \{\mu_n\}_{n=0}^{\infty} \text{ is a multiplier from } \mathcal{B} \text{ to } \ell^1. \tag{2.3.20}$$

Take now $f\in\mathcal{B}$, $f(z)=\sum_{n=0}^{\infty}a_n z^n$ $(z\in\mathbb{D})$. Using the simple fact that the sequence $\{\mu_n\}_{n=0}^{\infty}$ is a decreasing sequence of positive numbers and (2.3.20), we see that there exists $C>0$ such that

$$\sum_{k=0}^{\infty}|\mu_{n+k}\,a_k|\le\sum_{k=0}^{\infty}|\mu_k\,a_k|\le C\|f\|_{\mathcal{B}},\quad n=0,1,2,\dots. \tag{2.3.21}$$

This implies that $\mathcal{H}_\mu(f)(z)$ is well defined for all $z\in\mathbb{D}$ and that, in fact, $\mathcal{H}_\mu(f)$ is an analytic function in $\mathbb{D}$. Furthermore, since (2.3.21) also implies that we can interchange the order of summation in the expression defining $\mathcal{H}_\mu(f)(z)$, we have

$$\begin{aligned}\mathcal{H}_\mu(f)(z)&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n+k}a_k\right)z^n=\sum_{k=0}^{\infty}a_k\left(\sum_{n=0}^{\infty}\mu_{n+k}z^n\right)\\&=\sum_{k=0}^{\infty}a_k\left(\sum_{n=0}^{\infty}\int_{[0,1)}t^{n+k}z^n\,d\mu(t)\right)=\sum_{k=0}^{\infty}\int_{[0,1)}\frac{a_k t^k}{1-tz}\,d\mu(t)\\&=\int_{[0,1)}\frac{f(t)}{1-tz}\,d\mu(t)=I_\mu(f)(z),\quad z\in\mathbb{D}.\end{aligned}$$

□

Before embarking into the proof of Theorem 9 it is convenient to recall some facts about Carleson measures and to fix some notation.

If μ is a Carleson measure on $\mathbb{D}$, we define the Carleson-norm of μ, denoted $\mathcal{N}(\mu)$, as

$$\mathcal{N}(\mu)=\sup_{\substack{I\subset\mathbb{T}\\ I\text{ interval}}}\frac{\mu\left(S(I)\right)}{|I|}.$$

We let also $\mathcal{E}(\mu)$ denote the norm of the inclusion operator $i:H^1\to L^1(d\mu)$. It turns out that these quantities are equivalent: There exist two positive constants A_1,A_2 such that

$$A_1\mathcal{N}(\mu)\le\mathcal{E}(\mu)\le A_2\mathcal{N}(\mu),\quad\text{for every Carleson measure } \mu \text{ on } \mathbb{D}.$$

For a Carleson measure μ on $\mathbb{D}$ and $0 < r < 1$, we let μ_r be the measure on $\mathbb{D}$ defined by

$$d\mu_r(z) \;=\; \chi_{\{r<|z|<1\}} d\mu(z).$$

We have that μ is a vanishing Carleson measure if and only if

$$\mathcal{N}(\mu_r) \to 0, \quad \text{as } r \to 1.$$

Proof of Theorem 9. Since $BMOA$ is continuously contained in the Bloch space, it suffices to prove (i).

Suppose that μ is a vanishing 1-logarithmic 1-Carleson measure. By Proposition 1, ν is a vanishing Carleson measure. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Bloch functions with $\sup_{n\geq 1} \|f_n\|_{\mathcal{B}} < \infty$ and such that $\{f_n\} \to 0$, uniformly on compact subsets of $\mathbb{D}$. We have to prove that $I_\mu(f_n) \to 0$ in $BMOA$.

The condition $\sup_{n\geq 1} \|f_n\|_{\mathcal{B}} < \infty$ implies that there exists a positive constant M such that

$$|f_n(z)| \;\leq M \log \frac{2}{1-|z|}, \quad z \in \mathbb{D}, \quad n \geq 1. \tag{2.3.22}$$

Recall that for $0 < r < 1$, ν_r is the measure defined by

$$d\nu_r(t) \;=\; \chi_{\{r<t<1\}}\, d\nu(t).$$

Since ν is a vanishing Carleson measure, we have that $\mathcal{N}(\nu_r) \to 0$, as $r \to 1$, or, equivalently,

$$\mathcal{E}(\nu_r) \;\to\; 0, \quad \text{as } t \;\to\; 1. \tag{2.3.23}$$

Take $g \in H^1$ and $r \in [0,1)$. Using (2.3.22) we have

$$\begin{aligned}
\int_{[0,1)} |f_n(t)||g(t)|\, d\mu(t) \;&=\; \int_{[0,r)} |f_n(t)||g(t)|\, d\mu(t) \;+\; \int_{[r,1)} |f_n(t)||g(t)|\, d\mu(t) \\
&\leq \int_{[0,r)} |f_n(t)||g(t)|\, d\mu(t) \;+\; M \int_{[r,1)} \log \frac{2}{1-t} |g(t)|\, d\mu(t) \\
&= \int_{[0,r)} |f_n(t)||g(t)|\, d\mu(t) \;+\; M \int_{[0,1)} |g(t)|\, d\nu_r(t) \\
&\leq \int_{[0,r)} |f_n(t)||g(t)|\, d\mu(t) \;+\; M\mathcal{E}(\nu_r)\|g\|_{H^1}.
\end{aligned}$$

Using (2.3.23) and the fact that $\{f_n\} \to 0$, uniformly on compact subsets of $\mathbb{D}$, it follows that

$$\lim_{n\to\infty} \int_{[0,1)} |f_n(t)||g(t)|\, d\mu(t) \;=\; 0, \quad \text{for all } g \in H^1.$$

Bearing in mind (2.3.17), this yields

$$\lim_{n\to\infty}\left(\lim_{r\to 1}\left|\int_0^{2\pi} I_\mu(f_n)(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta\right|\right) = 0, \quad \text{for all } g\in H^1.$$

By the duality relation $(H^1)^\star = BMOA$, this is equivalent to saying that $I_\mu(f_n) \to 0$ in $BMOA$. □

Proof of Theorem 14. It is clear that Theorem 21 actually implies the result. □

Let us to prove the theorems about Besov spaces.

Proof of Theorem 17. Suppose that $1 < p < \infty$ and $f \in B^p$, $f(z) = \sum_{k=0}^\infty a_k z^k$ $(z \in \mathbb{D})$. Since the sequence of moments $\{\mu_n\}_{n=0}^\infty$ is clearly decreasing we have

$$\sum_{k=1}^\infty |\mu_{n+k}||a_k| \le \sum_{k=1}^\infty |\mu_k||a_k|, \quad \text{for all } n\ge 0.$$

Consequently, we have:

(i) If $1 < p \le 2$ and $f \in B^p$, $f(z) = \sum_{k=0}^\infty a_k z^k$ $(z \in \mathbb{D})$, then

$$\sum_{k=1}^\infty |\mu_{n+k}a_k| \le \sum_{k=1}^\infty |\mu_k||a_k| = \sum_{k=1}^\infty k^{1-\frac{1}{p}}|a_k|\frac{\mu_k}{k^{1/p'}}, \quad n\ge 0.$$

Then using Hölder inequality and Theorem K (i), we obtain

$$\begin{aligned}\sum_{k=1}^\infty |\mu_{n+k}a_k| &\le \left(\sum_{k=1}^\infty k^{p-1}|a_k|^p\right)^{1/p}\left(\sum_{k=1}^\infty \frac{|\mu_k|^{p'}}{k}\right)^{1/p'}\\ &\le C\,\rho_p(f)\left(\sum_{k=1}^\infty \frac{|\mu_k|^{p'}}{k}\right)^{1/p'}, \quad n\ge 0.\end{aligned}$$

Then it is clear that the condition $\sum_{k=1}^\infty \frac{|\mu_k|^{p'}}{k} < \infty$ implies that the power series appearing in the definition of $\mathcal{H}_\mu(f)$ defines an analytic function in $\mathbb{D}$.

(ii) If $2 < p < \infty$ and $f \in B^p$, $f(z) = \sum_{k=0}^\infty a_k z^k$ $(z \in \mathbb{D})$, then

$$\sum_{k=1}^\infty |\mu_{n+k}a_k| \le \sum_{k=1}^\infty |\mu_k||a_k| = \sum_{k=1}^\infty k^{\frac{1}{p}}|a_k|\frac{\mu_k}{k^{1/p}}, \quad n\ge 0.$$

Then using Hölder inequality and Theorem L, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}|\mu_{n+k}a_k| &\leq \left(\sum_{k=1}^{\infty}k|a_k|^p\right)^{1/p}\left(\sum_{k=1}^{\infty}\frac{|\mu_k|^{p'}}{k^{p'/p}}\right)^{1/p'}\\ &\leq C\,\rho_p(f)\left(\sum_{k=1}^{\infty}\frac{|\mu_k|^{p'}}{k^{p'/p}}\right)^{1/p'}, \quad n\geq 0.\end{aligned}$$

Then we see that the condition $\sum_{k=1}^{\infty}\frac{|\mu_k|^{p'}}{k^{p'/p}} < \infty$ implies that the power series appearing in the definition of $\mathcal{H}_\mu(f)$ defines an analytic function in $\mathbb{D}$.

□

Proof of Theorem 18. Let μ be the Borel measure on $[0,1)$ defined by

$$d\mu(t) = \left(\log\frac{2}{1-t}\right)^{-\beta}dt.$$

Since the function $x\mapsto\left(\log\frac{2}{1-x}\right)^{-\beta}$ is decreasing in $[0,1)$, we have

$$\mu\left([t,1)\right) = \int_t^1\left(\log\frac{2}{1-x}\right)^{-\beta}dx \leq (1-t)\left(\log\frac{2}{1-t}\right)^{-\beta}, \quad 0\leq t<1.$$

Hence, μ is a β-logarithmic 1-Carleson measure. Then, taking $\alpha=0$ in Lemma 2, we see that

$$\mu_k = \mathrm{O}\left(\frac{1}{k(\log k)^\beta}\right).$$

On the other hand,

$$\mu_k \geq \int_0^{1-\frac{1}{k}}t^k\left(\log\frac{2}{1-t}\right)^{-\beta}dt \gtrsim \frac{1}{(\log k)^\beta}\int_0^{1-\frac{1}{k}}t^k\,dt \gtrsim \frac{1}{k(\log k)^\beta}.$$

Thus, we have seen that μ is a β-logarithmic 1-Carleson measure which satisfies

$$\mu_n \asymp \frac{1}{n(\log n)^\beta}. \tag{2.3.24}$$

Take $p\in(1,\infty)$ and $\alpha>\frac{1}{p}$ and set

$$a_n = \frac{1}{(n+1)\left(\log(n+2)\right)^\alpha}, \quad n = 0,\,1,\,2,\,\ldots,$$

and

$$g(z) = \sum_{n=0}^{\infty} a_n\, z^n, \quad z \in \mathbb{D}.$$

Notice that $\{a_n\} \downarrow 0$ and that $\sum_{n=0}^{\infty} n^{p-1}|a_n|^p < \infty$. Hence, by Theorem 22, $g \in B^p$.

We are going to prove that $\mathcal{H}_\mu(g) \notin B^p$. This implies that $\mathcal{H}_\mu(B^p) \not\subset B^p$, proving the theorem.

We have $\mathcal{H}_\mu(g)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_k\right) z^n$. Notice that $a_k \geq 0$ for all k and that the sequence of moments $\{\mu_n\}$ is a decreasing sequence of non-negative numbers. Then it follows that the sequence $\{\sum_{k=0}^{\infty} \mu_{n+k} a_k\}_{n=0}^{\infty}$ of the Taylor coefficients of $\mathcal{H}_\mu(g)$ is decreasing. Consequently, using again Theorem 22, we have that

$$\mathcal{H}_\mu(g) \in B^p \;\Leftrightarrow\; \sum_{n=1}^{\infty} n^{p-1} \left|\sum_{k=0}^{\infty} \mu_{n+k} a_k\right|^p < \infty. \tag{2.3.25}$$

Using the definition of the sequence $\{a_k\}$, (2.3.24) and the simple inequalities $\frac{k}{n+k} \geq \frac{1}{n+1}$ and $\log(n+k) \leq (\log n)(\log k)$ which hold whenever $k, n \geq 10$, say, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{p-1} \left|\sum_{k=0}^{\infty} \mu_{n+k} a_k\right|^p &\geq \sum_{n=10}^{\infty} n^{p-1} \left|\sum_{k=10}^{\infty} \mu_{n+k} a_k\right|^p \\
&\gtrsim \sum_{n=10}^{\infty} n^{p-1} \left(\sum_{k=10}^{\infty} \left[\frac{1}{(n+k)\,(\log(n+k))^{\beta}}\, \frac{1}{k\,(\log k)^{\alpha}}\right]\right)^p \\
&\gtrsim \sum_{n=10}^{\infty} \frac{1}{n(\log n)^{p\beta}} \left(\sum_{k=10}^{\infty} \frac{1}{k^2\,(\log k)^{\alpha+\beta}}\right)^p = \infty.
\end{aligned}$$

Bearing in mind (2.3.25), this implies that $\mathcal{H}_\mu(g) \notin B^p$ as desired. □

Proof of Theorem 19. Suppose that $1 < p < \infty$ and let μ be a positive Borel measure on $[0, 1)$ such that the operator $\mathcal{H}_\mu$ is bounded from B^p into itself. For $\frac{1}{2} < b < 1$, set

$$g_b(z) = \left(\log \frac{1}{1-b^2}\right)^{-1/p} \log \frac{1}{1-bz}, \quad z \in \mathbb{D}.$$

We have,

$$g_b'(z) = \left(\log \frac{1}{1-b^2}\right)^{-1/p} \frac{b}{1-bz}, \quad z \in \mathbb{D}$$

and then, using Lemma 3. 10 of [111] with $t = p-2$ and $c = 0$, we have

$$\int_{\mathbb{D}} (1-|z|^2)^{p-2} |g_b'(z)|^p \, dA(z) \asymp \left(\log \frac{1}{1-b^2}\right)^{-1} \int_{\mathbb{D}} \frac{(1-|z|^2)^{p-2}}{|1-bz|^p}\, dA(z) \asymp 1.$$

In other words, we have that

$$g_b \in B^p \quad \text{and} \quad \|g_b\|_{B^p} \asymp 1.$$

Since $\mathcal{H}_\mu$ is a bounded operator from B^p into itself, this implies that

$$1 \gtrsim \|\mathcal{H}_\mu(g_b)\|_{B^p}^p. \tag{2.3.26}$$

We have

$$g_b(z) = \sum_{k=0}^{\infty} a_{k,b} z^k, \quad \text{with} \quad a_{k,b} = \left(\log \frac{1}{1-b^2}\right)^{-1/p} \frac{b^k}{k}.$$

Since the $a_{k,b}$'s are positive it follows that the sequence $\{\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b}\}_{n=0}^{\infty}$ of the Taylor coefficients of $\mathcal{H}_\mu(g_b)$ is a decreasing sequence of non-negative real numbers. Using this, Theorem 22, and (2.3.26) we see that

$$\begin{aligned}
1 &\gtrsim \|\mathcal{H}_\mu(g_b)\|_{B^p}^p \gtrsim \sum_{n=1}^{\infty} n^{p-1} \left(\sum_{k=1}^{\infty} \mu_{n+k} a_{k,b}\right)^p \\
&= \left(\log \frac{1}{1-b^2}\right)^{-1} \sum_{n=1}^{\infty} n^{p-1} \left(\sum_{k=1}^{\infty} \frac{b^k}{k} \int_{[0,1)} t^{n+k} d\mu(t)\right)^p \\
&\geq \left(\log \frac{1}{1-b^2}\right)^{-1} \sum_{n=1}^{\infty} n^{p-1} \left(\sum_{k=1}^{\infty} \frac{b^k}{k} \int_{[b,1)} t^{n+k} d\mu(t)\right)^p \\
&\geq \left(\log \frac{1}{1-b^2}\right)^{-1} \sum_{n=1}^{\infty} n^{p-1} \left(\sum_{k=1}^{\infty} \frac{b^{n+2k}}{k}\right)^p \mu\left([b,1)\right)^p \\
&= \left(\log \frac{1}{1-b^2}\right)^{-1} \sum_{n=1}^{\infty} n^{p-1} b^{np} \left(\sum_{k=1}^{\infty} \frac{b^{2k}}{k}\right)^p \mu\left([b,1)\right)^p \\
&= \left(\log \frac{1}{1-b^2}\right)^{p-1} \frac{1}{(1-b^p)^p} \mu\left([b,1)\right)^p \\
&\asymp \left(\log \frac{1}{1-b^2}\right)^{p-1} \frac{1}{(1-b)^p} \mu\left([b,1)\right)^p.
\end{aligned}$$

Then it follows that $\mu\left([b,1)\right) \lesssim \frac{1-b}{\left(\log \frac{1}{1-b}\right)^{1-\frac{1}{p}}}$. This finishes the proof. □

Proof of Theorem 20. By the closed graph theorem it suffices to show that $\mathcal{H}_\mu(B^p) \subset B^p$.

Take $f \in B^p$. Since μ is a γ-logarithmic 1-Carleson measure, using Lemma 4 we see that

$$\mathcal{H}_\mu(f)(z) \;=\; I_\mu(f)(z) \;=\; \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t)\, d\mu(t) \right) z^n, \quad z \in \mathbb{D}.$$

Also, using Corollary 1, we see that

$$\mathcal{H}_\mu(f) \;\in\; B^p \;\Leftrightarrow\; \sum_{n=1}^{\infty} 2^{-n(p-1)} \|\Delta_n \left(\mathcal{H}_\mu(f)'\right)\|_{H^p}^p \;<\; \infty. \tag{2.3.27}$$

Now, we have

$$\Delta_n \left(\mathcal{H}_\mu(f)'\right)(z) \;=\; \sum_{k=2^n}^{2^{n+1}-1} (k+1) \left(\int_{[0,1)} t^{k+1} f(t)\, d\mu(t) \right) z^k.$$

Using Lemma 5 we obtain that

$$\|\Delta_n \left(\mathcal{H}_\mu(f)'\right)\|_{H^p} \;\lesssim\; \left(\int_{[0,1)} t^{2^{n-2}+1} |f(t)|\, d\mu(t) \right) \|\Delta_n F\|_{H^p}$$

with $F(z) = \sum_{k=0}^{\infty} (k+1) z^k$ $(z \in \mathbb{D})$. Now, we have that $M_p(r,F) \;=\; \mathrm{O}\left(\frac{1}{(1-r)^{2-\frac{1}{p}}} \right)$ and then it follows that $\|\Delta_n F\|_{H^p} \;=\; \mathrm{O}\left(2^{n(2-\frac{1}{p})}\right)$ (see, e. g., [71]). Using this and the estimate $|f(t)| \lesssim \left(\log \frac{2}{1-t}\right)^{1/p'}$, we obtain

$$\|\Delta_n \left(\mathcal{H}_\mu(f)'\right)\|_{H^p} \;\lesssim\; 2^{n(2-\frac{1}{p})} \left(\int_{[0,1)} t^{2^{n-2}+1} \left(\log \frac{2}{1-t} \right)^{1/p'} d\mu(t) \right),$$

which using the fact that μ is a γ-logarithmic 1-Carleson measure and Lemma 2 implies

$$\|\Delta_n \left(\mathcal{H}_\mu(f)'\right)\|_{H^p} \;\lesssim\; 2^{n(2-\frac{1}{p})} 2^{-n} n^{\frac{1}{p'}-\gamma} \;=\; 2^{n/p'} n^{\frac{1}{p'}-\gamma}.$$

This, together with the fact that $\gamma > 1$, implies that

$$\begin{aligned} \sum_{n=1}^{\infty} 2^{-n(p-1)} \|\Delta_n \left(\mathcal{H}_\mu(f)'\right)\|_{H^p}^p \;&\lesssim\; \sum_{n=1}^{\infty} 2^{-n(p-1)} 2^{np/p'} n^{p(1-\gamma)-1} \\ &= \sum_{n=1}^{\infty} n^{p(1-\gamma)-1} \;<\; \infty. \end{aligned}$$

Bearing in mind (2.3.27), this shows that $\mathcal{H}_\mu(f) \in B^p$ and finishes the proof. □

2.4 A generalized Hilbert matrix acting on mean Lipschitz spaces

The mean Lipschitz space $\Lambda^2_{1/2}$ showed up in Theorem 21 of Section 2.3. This section will be devoted to study the operator $\mathcal{H}_\mu$ acting on general mean Lipschitz spaces. Let us present the definition of these spaces.

If $f \in \mathcal{H}ol(\mathbb{D})$ has a non-tangential limit $f(e^{i\theta})$ at almost every $e^{i\theta} \in \partial\mathbb{D}$ and $\delta > 0$, we define

$$\omega_p(\delta, f) = \sup_{0<|t|\leq\delta} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(e^{i(\theta+t)}) - f(e^{i\theta}) \right|^p d\theta \right)^{1/p}, \quad \text{if } 1 \leq p < \infty,$$

$$\omega_\infty(\delta, f) = \sup_{0<|t|\leq\delta} \left(\operatorname*{ess.\,sup}_{\theta\in[-\pi,\pi]} |f(e^{i(\theta+t)}) - f(e^{i\theta})| \right).$$

Then $\omega_p(\cdot, f)$ is the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f.

Given $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, the mean Lipschitz space Λ^p_α consists of those functions $f \in \mathcal{H}ol(\mathbb{D})$ having a non-tangential limit almost everywhere for which $\omega_p(\delta, f) = O(\delta^\alpha)$, as $\delta \to 0$. If $p = \infty$ we write Λ_α instead of Λ^∞_α. This is the usual Lipschitz space of order α.

A classical result of Hardy and Littlewood [58] (see also [40, Chapter 5]) asserts that for $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, we have that $\Lambda^p_\alpha \subset H^p$ and

$$\Lambda^p_\alpha = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_p(r, f') = \mathrm{O}\left(\frac{1}{(1-r)^{1-\alpha}} \right) \right\}.$$

It is known that if $1 < p < \infty$ and $\alpha > \frac{1}{p}$ then each $f \in \Lambda^p_\alpha$ is bounded and has a continuous extension to the closed unit disc [25, p.88]. This is not true for $\alpha = \frac{1}{p}$, because the function $f(z) = \log(1-z)$ belongs to $\Lambda^p_{1/p}$ for all $p \in (1, \infty)$. By a theorem of Hardy and Littlewood [40, Theorem 5.9] and of [25, Theorem 2.5] we have

$$\Lambda^p_{1/p} \subset \Lambda^q_{1/q} \subset BMOA \quad 1 \leq p < q < \infty.$$

The inclusion $\Lambda^p_{1/p} \subset BMOA$, $1 \leq p < \infty$ was proved to be sharp in a very strong sense in [20, 50, 51] using the following generalization of the spaces Λ^p_α which occurs frequently in the literature.

Let $\omega : [0, \pi] \to [0, \infty)$ be a continuous and increasing function with $\omega(0) = 0$ and $\omega(t) > 0$ if $t > 0$. Then, for $1 \leq p \leq \infty$, the mean Lipschitz space $\Lambda(p, \omega)$ consists of those functions $f \in H^p$ such that

$$\omega_p(\delta, f) = O(\omega(\delta)), \quad \text{as } \delta \to 0.$$

With this notation we have $\Lambda_\alpha^p = \Lambda(p, \delta^\alpha)$.

The question of finding conditions on ω so that it is possible to obtain results on the spaces $\Lambda(p, \omega)$ analogous to those proved by Hardy and Littlewood for the spaces Λ_α^p has been studied by several authors (see [22, 23, 25]). We shall say that ω satisfies the Dini condition or that ω is a Dini-weight if there exists a positive constant C such that

$$\int_0^\delta \frac{\omega(t)}{t}\, dt \leq C\omega(\delta), \quad 0 < \delta < 1.$$

We shall say that ω satisfies the condition b_1 or that $\omega \in b_1$ if there exists a positive constant C such that

$$\int_\delta^\pi \frac{\omega(t)}{t^2}\, dt \leq C\frac{\omega(\delta)}{\delta}, \quad 0 < \delta < 1.$$

In order to simplify our notation, let $\mathcal{AW}$ denote the family of all functions $\omega : [0, \pi] \to [0, \infty)$ which satisfy the following conditions:

(i) ω is continuous and increasing in $[0, \pi]$.

(ii) $\omega(0) = 0$ and $\omega(t) > 0$ if $t > 0$.

(iii) ω is a Dini-weight.

(iv) ω satisfies the condition b_1.

The elements of $\mathcal{AW}$ will be called admissible weights. Characterizations and examples of admissible weights can be found in [22, 23].

Blasco and de Souza extended the above mentioned result of Hardy and Littlewood showing in [22, Theorem 2.1] that if $\omega \in \mathcal{AW}$ then,

$$\Lambda(p, \omega) = \left\{ f \text{ analytic in } \mathbb{D} : M_p(r, f') = \mathrm{O}\left(\frac{\omega(1-r)}{1-r}\right), \text{ as } r \to 1 \right\}.$$

In [20, 51, 50] it is proved that if $1 \leq p < \infty$ and ω is an admissible weight such that

$$\frac{\omega(\delta)}{\delta^{1/p}} \to \infty, \text{ as } \delta \to 0,$$

then there exists a function $f \in \Lambda(p, \omega)$ which is a not a normal function (see [6] for the definition). Since any Bloch function is normal, if follows that for such admissible weights ω one has that $\Lambda(p, \omega) \not\subset \mathcal{B}$.

Theorem 21 of the above section gives a result for a Banach space X satisfying that $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. This result can be improved changing $\Lambda^2_{1/2}$ by $\Lambda^p_{1/p}$ for any $p > 1$.

Theorem 23 ([72])**.** *Suppose that $1 < p < \infty$. Let μ be a positive Borel measure on $[0, 1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^p_{1/p} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.*

(i) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into the Bloch space $\mathcal{B}$.

(ii) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^p_{1/p}$.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) = \mathrm{O}\left(\frac{1}{n}\right)$.

As an immediate consequence of Theorem 23 we obtain the following result.

Corollary 2. *Let μ be a positive Borel measure on $[0, 1)$ and $1 < p < \infty$. Then the operator $\mathcal{H}_\mu$ is well defined in $\Lambda^p_{1/p}$ and, furthermore, it is a bounded operator from $\Lambda^p_{1/p}$ into itself if and only if μ is a 1-logarithmic 1-Carleson measure.*

Let us turn our attention now to the spaces $\Lambda(p, \omega)$ with $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$, as $\delta \searrow 0$ which, as noted before, are not included in the Bloch space. We have the following result which shows that the situation is different from the one covered in Theorem 23.

Theorem 24 ([72])**.** *Let $1 < p < \infty$, $\omega \in \mathcal{AW}$ with $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$ when $\delta \searrow 0$. The following conditions are equivalent:*

(i) The operator $\mathcal{H}_\mu$ is well defined in $\Lambda(p, \omega)$ and, furthermore, it is a bounded operator from $\Lambda(p, \omega)$ into itself.

(ii) The measure μ is a Carleson measure.

2.4.1 Preliminary results

In this section we shall present some results that we will use in the proofs of the above theorems.

A key ingredient in the proof of Theorem 21 is the fact that for any space X with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$ the functions $f \in X$ of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ whose sequence of Taylor coefficients $\{a_n\}$ is a decreasing sequence of non-negative numbers are the same. Indeed, for such a function f and such a space X we have that $f \in X \;\Leftrightarrow\; a_n = \mathrm{O}\left(\frac{1}{n}\right)$. This result remains true if we substitute $\Lambda^2_{1/2}$ by $\Lambda^p_{1/p}$ for any $p > 1$. That is, the following result holds:

Lemma 6. *Suppose that $1 < p < \infty$ and let $f \in \mathcal{H}ol(\mathbb{D})$ be of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\{a_n\}_{n=0}^{\infty}$ being a decreasing sequence of nonnegative numbers. If X is a subspace of $\mathcal{H}ol(\mathbb{D})$ with $\Lambda^p_{1/p} \subset X \subset \mathcal{B}$, then*

$$f \in X \quad \Leftrightarrow \quad a_n = \mathrm{O}\left(\frac{1}{n}\right).$$

Lemma 6 is a consequence of the following one which gives a characterization for functions with decreasing coefficients in all $\Lambda(p,\omega)$ spaces with $1 < p < \infty$ and ω being an admissible weight.

Lemma 7. *Let $1 < p < \infty$, $\omega \in \mathcal{AW}$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\{a_n\}_{n=0}^{\infty}$ being a decreasing sequence of nonnegative numbers. Then*

$$f \in \Lambda(p,\omega) \quad \Leftrightarrow \quad a_n \;=\; \mathrm{O}\left(\frac{\omega(1/n)}{n^{1-1/p}}\right). \tag{2.4.1}$$

The proof of Lemma 7 is based in the following result of Girela and González [53, Theorem 2]. We recall that for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\mathbb{D}$, the polynomials $\Delta_j f$ are defined as follows:

$$\Delta_j f(z) = \sum_{k=2^j}^{2^{j+1}-1} a_k z^k, \quad \text{for } j \geq 1,$$

$$\Delta_0 f(z) = a_0 + a_1 z.$$

Theorem N. *Let $1 < p < \infty$ and let ω be an admissible weight. If $f \in \mathcal{H}ol(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then*

$$f \in \Lambda(p,\omega) \;\Leftrightarrow\; \|\Delta_N f\|_{H^p} = O\left(\omega\left(\frac{1}{2^N}\right)\right).$$

Proof of Lemma 7. By Lemma A of [78], since $a_n \searrow 0$, we have

$$\|\Delta_N f\|_{H^p} \asymp a_{2^N} 2^{N(1-1/p)}, \quad N \geq 1.$$

So by Theorem N we have that

$$f \in \Lambda(p,\omega) \Leftrightarrow a_{2^N} \lesssim \frac{\omega\left(1/2^N\right)}{2^{N(1-1/p)}}, \quad N \geq 1.$$

This easily implies (2.4.1). □

We also need the following result to prove Theorem 24.

Lemma 8. *Suppose that* $1 < p < \infty$. *Let* ν *be a positive Borel measure on* $[0,1)$, *and let* $\omega \in \mathcal{AW}$ *satisfying that* $x^{-1/p}\omega(x) \nearrow \infty$, *as* $x \searrow 0$. *Then following conditions are equivalent:*

(i) $\nu_n \lesssim \frac{\omega(1/n)}{n^{1-1/p}}$, $n \geq 2$.

(ii) $\nu([b,1]) \lesssim (1-b)^{1-1/p}\omega(1-b)$, $b \in [0,1)$.

Proof. Suppose (i). Then we have that

$$\begin{aligned}
1 \gtrsim & \frac{n^{1-1/p}\,\nu_n}{\omega(1/n)} = \frac{n^{1-1/p}}{\omega(1/n)} \int_{[0,1)} t^n \, d\nu(t) \geq \frac{n^{1-1/p}}{\omega(1/n)} \int_{[1-1/n,1)} t^n \, d\nu(t) \\
\geq & \frac{n^{1-1/p}}{\omega(1/n)} \nu([1-1/n,1)) \left(1-\frac{1}{n}\right)^n \\
\geq & \frac{n^{1-1/p}}{\omega(1/n)} \nu([1-1/n,1)) \inf_{m\geq 2} \left(1-\frac{1}{m}\right)^m \\
\gtrsim & \frac{n^{1-1/p}}{\omega(1/n)} \nu([1-1/n,1)).
\end{aligned}$$

So $\nu([1-1/n,1)) \lesssim \frac{\omega(1/n)}{n^{1-1/p}}$ for $n \geq 2$.

Let now $b \in [1/2,1)$. There exists $n \geq 2$ such that $1-\frac{1}{n} \leq b < 1-\frac{1}{n+1}$ so using the above we have that

$$\nu([b,1)) \leq \nu([1-1/n,1)) \lesssim \frac{\omega(1/n)}{n^{1-1/p}}.$$

This, and the facts that $\omega(1/n)n^{1/p} \leq \omega(1/(n+1))(n+1)^{1/p}$ and that the weight ω increases give (ii).

Suppose now (ii). Then

$$\begin{aligned}\nu_n &= \int_{[0,1)} t^n \, d\nu(t) = n \int_0^1 \nu([t,1)) t^{n-1} \, dt \\ &\lesssim n \int_0^1 (1-t)^{1-1/p} \omega(1-t) t^{n-1} \, dt \\ &= n \int_0^{1-\frac{1}{n}} + \int_{1-\frac{1}{n}}^1 \left((1-t)^{1-1/p} \omega(1-t) t^{n-1} \, dt \right).\end{aligned}$$

The first integral can be estimated bearing in mind that $(1-t)^{-1/p}\omega(1-t) \nearrow \infty$ when $t \nearrow 1$ as follows

$$\begin{aligned}& n \int_0^{1-\frac{1}{n}} (1-t)^{1-1/p} \omega(1-t) t^{n-1} \, dt \\ \leq & n^{1+1/p} \omega(1/n) \int_0^{1-\frac{1}{n}} (1-t) t^{n-1} \, dt \\ = & n^{1+1/p} \omega(1/n) \left(1 - \frac{1}{n}\right)^n \left(\frac{1}{n} - \frac{n-1}{n(n+1)}\right) \\ \lesssim & \frac{\omega(1/n)}{n^{1-1/p}}.\end{aligned}$$

To estimate of the second integral we use that $(1-t)^{1-1/p}\omega(1-t) \searrow 0$ when $t \nearrow 1$ to obtain

$$\begin{aligned}& n \int_{1-\frac{1}{n}}^1 (1-t)^{1-1/p} \omega(1-t) t^{n-1} \, dt \\ \leq & n^{1/p} \omega(1/n) \int_{1-\frac{1}{n}}^1 t^{n-1} \, dt \\ = & \frac{\omega(1/n)}{n^{1-1/p}} \left(1 - \left(1 - \frac{1}{n}\right)^n\right) \\ \lesssim & \frac{\omega(1/n)}{n^{1-1/p}}.\end{aligned}$$

Then (i) follows. □

2.4.2 Proofs

To prove Theorem 23 we only need to use Lemma 6 and follow the proof of Theorem 21. Let us prove Theorem 24.

Proof of Theorem 24. (i) $\Rightarrow$ (ii) Suppose that $\mathcal{H}_\mu : \Lambda(p,\omega) \to \Lambda(p,\omega)$ is bounded. By Lemma 7 we have that the function f defined by $f(z) = \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1-1/p}} z^n$ belongs to the space $\Lambda(p,\omega)$ so, by the hypothesis, $\mathcal{H}_\mu(f)$ belongs also to $\Lambda(p,\omega)$. Now

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \right) z^n.$$

Notice that $\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \searrow 0$, as $n \nearrow \infty$, so using again Lemma 7 it holds that

$$\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} = \int_{[0,1)} t^n \sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} t^k \, d\mu(t) \lesssim \frac{\omega(1/n)}{n^{1-1/p}},$$

that is, the moments of the measure ν defined by

$$d\nu(t) = \sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} t^k \, d\mu(t)$$

satisfy that

$$\nu_n \lesssim \frac{\omega(1/n)}{n^{1-1/p}},$$

so by Lemma 8 we have that $\nu([b,1)) \lesssim (1-b)^{1-1/p}\omega(1-b)$, $b \in [0,1)$.

According to the definition of the measure

$$\begin{aligned}
(1-b)^{1-1/p}\omega(1-b) \gtrsim \nu([b,1)) &= \int_{[b,1)} d\nu(t) \\
&= \int_{[b,1)} \sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} t^k \, d\mu(t) \\
&\geq \mu\left([b,1)\right) \sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} b^k
\end{aligned}$$

and the sum can be estimated as follows

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} b^k &\asymp \int_1^{\infty} \frac{\omega(1/x)}{x^{1-1/p}} b^x \, dx \\
&\geq \int_1^{\frac{1}{1-b}} \frac{\omega(1/x)}{x^{1-1/p}} b^x \, dx \\
&\geq (1-b)^{1-1/p}\omega(1-b) b^{\frac{1}{1-b}} \left(\frac{1}{1-b} - 1 \right) \\
&\gtrsim \frac{\omega(1-b)}{(1-b)^{1/p}}.
\end{aligned}$$

Finally, putting all together we have that

$$\mu([b,1)) \lesssim 1-b$$

so μ is a Carleson measure.

(ii) $\Rightarrow$ (i) To prove this implication we need to use again the integral operator I_μ considered before.

Suppose that μ is a Carleson measure supported on $[0,1)$ and let $f \in \Lambda(p,\omega)$. We claim that

$$\int_{[0,1)} \frac{|f(t)|}{|1-tz|}\, d\mu(t) < \infty. \tag{2.4.2}$$

Indeed, using Lemma 3 of [53] we have that

$$f \in \Lambda(p,\omega) \Rightarrow |f(z)| \lesssim \frac{\omega(1-|z|)}{(1-|z|)^{1/p}}, \quad z \in \mathbb{D}. \tag{2.4.3}$$

Then we obtain

$$\begin{aligned}\int_{[0,1)} \frac{|f(t)|}{|1-tz|}\, d\mu(t) &\leq \frac{1}{1-|z|}\int_{[0,1)} |f(t)|\, d\mu(t)\\ &\lesssim \frac{1}{1-|z|}\int_{[0,1)} \frac{\omega(1-t)}{(1-t)^{1/p}}\, d\mu(t).\end{aligned}$$

If we choose $r \in [0,1)$ we can split the integral in the intervals $[0,r)$ and $[r,1)$. In the first one, as ω is an increasing weight we have

$$\begin{aligned}\int_{[0,r)} \frac{\omega(1-t)}{(1-t)^{1/p}}\, d\mu(t) &\leq \omega(1)\int_{[0,r)} \frac{d\mu(t)}{(1-t)^{1/p}}\\ &\leq \omega(1)\int_{[0,1)} \frac{d\mu(t)}{(1-t)^{1/p}}\\ &\lesssim 1,\end{aligned}$$

because μ is a Carleson measure. Using this and the condition $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$, as $\delta \searrow 0$ we can estimate the other integral as follows

$$\begin{aligned}\int_{[r,1)} \frac{\omega(1-t)}{(1-t)^{1/p}}\, d\mu(t) &\leq \omega(1-r)\int_{[r,1)} \frac{1}{(1-t)^{1/p}} d\mu(t)\\ &\lesssim \omega(1-r)(1-r)^{1-1/p}\\ &\lesssim 1.\end{aligned}$$

So we have that for $f \in \Lambda(p,\omega)$ and $z \in \mathbb{D}$, (2.4.2) holds. This implies that $I_\mu(f)$ is well defined, and, using Fubini's theorem and standard arguments it follows easily that $\mathcal{H}_\mu(f)$ is also well defined and that, furthermore,

$$\mathcal{H}_\mu(f)(z) = I_\mu(f)(z), \quad z \in \mathbb{D}.$$

Now we have,

$$I_\mu(f)'(z) = \int_{[0,1)} \frac{tf(t)}{(1-tz)^2}\, d\mu(t), \quad z \in \mathbb{D},$$

so the mean of order p of $I_\mu(f)'$ has the form

$$M_p\left(r, I_\mu(f)'\right) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_{[0,1)} \frac{tf(t)}{(1-tre^{i\theta})^2}\, d\mu(t)\right|^p d\theta\right)^{1/p}.$$

Using again (2.4.3), the Minkowski inequality and a classical estimation of integrals we obtain that

$$\begin{aligned} M_p\left(r, I_\mu(f)'\right) &\lesssim \int_{[0,1)} |f(t)| \left(\int_{-\pi}^{\pi} \frac{d\theta}{|1-tre^{i\theta}|^{2p}}\right)^{1/p} d\mu(t) \\ &\lesssim \int_{[0,1)} \frac{|f(t)|}{(1-tr)^{2-1/p}}\, d\mu(t) \\ &\lesssim \int_{[0,1)} \frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}}\, d\mu(t). \end{aligned}$$

At this point we split the integrals on the sets $[0,r)$ and $[r,1)$.

In the first integral we use that $x^{-1/p}\omega(x) \nearrow \infty$, as $x \searrow 0$, and the fact that if μ is a Carleson measure (so that $\mu_n = \int_{[0,1)} t^n\, d\mu(t) \lesssim \frac{1}{n}$) to obtain

$$\begin{aligned} \int_{[0,r)} \frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}}\, d\mu(t) &\leq \frac{\omega(1-r)}{(1-r)^{1/p}} \int_{[0,r)} \frac{d\mu(t)}{(1-tr)^{2-1/p}} \\ &\leq \frac{\omega(1-r)}{(1-r)^{1/p}} \int_{[0,1)} \frac{d\mu(t)}{(1-tr)^{2-1/p}} \\ &\lesssim \frac{\omega(1-r)}{(1-r)^{1/p}} \sum_{n=1}^{\infty} n^{1-1/p} r^n \int_{[0,1)} t^n\, d\mu(t) \\ &\lesssim \frac{\omega(1-r)}{(1-r)^{1/p}} \sum_{n=1}^{\infty} \frac{r^n}{n^{1/p}} \\ &\lesssim \frac{\omega(1-r)}{(1-r)}. \end{aligned}$$

In the second integral we use that ω is an increasing weight and the fact that the measure μ being a Carleson measure is equivalent to saying that the measure ν defined by $d\nu(t) = \frac{d\mu(t)}{(1-t)^{1/p}}$ is a $1-\frac{1}{p}$-Carleson measure so that the moments ν_n of ν satisfy $\nu_n \lesssim \frac{1}{n^{1-\frac{1}{p}}}$. Then we obtain

$$\begin{aligned}
\int_{[r,1)} \frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}}\, d\mu(t) &\leq \omega(1-r) \int_{[r,1)} \frac{d\nu(t)}{(1-tr)^{2-1/p}} \\
&\leq \omega(1-r) \int_{[0,1)} \frac{d\nu(t)}{(1-tr)^{2-1/p}} \\
&\lesssim \omega(1-r) \sum_{n=1}^{\infty} n^{1-1/p} r^n \int_{[0,1)} t^n \, d\nu(t) \\
&\lesssim \omega(1-r) \sum_{n=1}^{\infty} r^n \\
&= \frac{\omega(1-r)}{(1-r)}.
\end{aligned}$$

Therefore $I_\mu(f) \in \Lambda(p,\omega)$ and then the operator I_μ (and hence the operator $\mathcal{H}_\mu$) is bounded from $\Lambda(p,\omega)$ into itself. $\square$

Chapter 3

Morrey spaces

This chapter is devoted to Morrey spaces, which were introduced by Charles B. Morrey Jr. [73] in 1938 in connection with partial differential equations, and were subsequently studied as function classes in harmonic analysis on Euclidean spaces, extending the notion of functions of bounded mean oscillation. The analytic Morrey spaces were introduced more recently and they have been studied by several authors, see for example [68], [102], [106], and [107].

We recall the definition and some of their properties. Observe that

$$\int_I \left|f(e^{i\theta}) - f_I\right|^2 d\theta \to 0 \qquad \text{as } |I| \to 0,$$

for every $f \in H^2$, and the rate of this convergence to 0 depends clearly on the degree of oscillation of f around its average f_I on I. Given $\lambda \in (0,1]$ we can isolate functions f for which this rate of convergence is comparable to $|I|^\lambda$.

Thus for $f \in H^2$ and $0 < \lambda \leq 1$ we say that f belongs to the Morrey space $\mathcal{L}^{2,\lambda}$ if

$$\|f\|_{\mathcal{L}^{2,\lambda}} \stackrel{\text{def}}{=} |f(0)| + \|f\|_{\mathcal{L}^{2,\lambda}(\mathbb{T})} < \infty,$$

where

$$\|f\|_{\mathcal{L}^{2,\lambda}(\mathbb{T})} = \sup_{\substack{I \subset \mathbb{T} \\ I \text{ interval}}} \left(\frac{1}{|I|^\lambda} \int_I \left|f(e^{i\theta}) - f_I\right|^2 d\theta \right)^{1/2}. \tag{3.0.1}$$

Clearly, $\mathcal{L}^{2,1} = BMOA$. The Morrey spaces increase when the parameter λ decreases, so we have the following relation:

$$BMOA \subset \mathcal{L}^{2,\lambda_2} \subset \mathcal{L}^{2,\lambda_1} \subset H^2, \quad 0 < \lambda_1 \leq \lambda_2 \leq 1. \tag{3.0.2}$$

It turns out that an equivalent norm is

$$\|f\|_{\mathcal{L}^{2,\lambda}} = |f(0)| + \sup_{a\in\mathbb{D}}(1-|a|^2)^{\frac{1}{2}(1-\lambda)}\|f\circ\varphi_a - f(a)\|_{H^2}, \tag{3.0.3}$$

and the following Carleson measure characterization is also valid

$$f \in \mathcal{L}^{2,\lambda} \Leftrightarrow \sup_{I\subset\mathbb{T}} \frac{1}{|I|^\lambda}\int_{S(I)} |f'(z)|^2(1-|z|^2)\,dA(z) < \infty. \tag{3.0.4}$$

See Lemma 2.3 of [66] for both characterizations.

In the same way as it is defined the space $VMOA$ we can consider the space $\mathcal{L}_0^{2,\lambda}$ as the space of functions in $\mathcal{L}^{2,\lambda}$ such that

$$\lim_{|I|\to 0} \frac{1}{|I|^\lambda}\int_I \left|f(e^{i\theta}) - f_I\right|^2 d\theta = 0.$$

Characterizations similar to (3.0.3) and (3.0.4) can be obtained for these spaces.

One important thing when we research in complex analysis is to have a good variety of examples of functions which belong to the spaces we are working with. Because of that, the next section is devoted to explore the structure of Morrey spaces, characterizing for some typical classes of analytic functions $\mathcal{C}$ those functions in $\mathcal{C}$ which lie in the Morrey spaces, and paying attention to the differences and similarities with Hardy spaces and $BMOA$.

The second section will be devoted to the actions of semigroups of composition operators on these Morrey spaces. Most of our results concerning this topic are included in [47].

3.1 Structure of Morrey spaces

One of the most important types of analytic functions are the lacunary series. We say that a power series centered at 0 is a *lacunary power series* or a *power series with Hadamard gaps* if it is of the form

$$\sum_{k=0}^{\infty} a_k z^{n_k}$$

where $\{n_k\}_{k=0}^{\infty}$ is a sequence of non-negative integers for which there exists $l > 1$ such that

$$n_{k+1} \geq l n_k, \text{ for all } k \geq 0.$$

It is well known (see [52, Theorem 9.3]) that the lacunary power series in $BMOA$ coincide with those in H^2.

Theorem O. *Let $l > 1$ and let $f \in \mathcal{H}ol(\mathbb{D})$ which is given by a power series with Hadamard gaps of the form $\sum_{k=0}^{\infty} a_k z^{n_k}$ with $n_{k+1} \geq l n_k$ for all $k \geq 0$. Then, the following are equivalent.*

(i) $f \in BMOA$.

(ii) $f \in H^2$.

(iii) $\sum\limits_{k=0}^{\infty} |a_k|^2 < \infty$.

(iv) $f \in H^p$ *for some* $p \in (0, \infty)$.

Since Morrey spaces are between $BMOA$ and H^2 we have the same characterization for them, so we can not distinguish lacunary series in Morrey spaces from those in $BMOA$ or H^2.

It is well known that functions in $BMOA$ have logarithmic growth,

$$f \in \mathrm{BMOA} \Rightarrow |f(z)| \lesssim \log \frac{2}{1-|z|}, \quad z \in \mathbb{D}. \tag{3.1.1}$$

This does not remain true for functions in Morrey spaces. Indeed, we have by Lemma 2 of [67] that the chain of contentions in (3.0.2) can be improved as follows:

$$H^{\frac{2}{1-\lambda}} \subset \mathcal{L}^{2,\lambda} \subset H^2, \quad 0 < \lambda < 1. \tag{3.1.2}$$

This easily implies that for $\varepsilon > 0$ the function $f(z) = (1-z)^{-\frac{1-\lambda}{2}+\varepsilon}$ belongs to the Morrey space $\mathcal{L}^{2,\lambda}$.

The substitute of (3.1.1) for Morrey spaces is the following result, which can be found in [66].

Theorem P. *Let $0 < \lambda < 1$. If $f \in \mathcal{L}^{2,\lambda}$ then*

$$|f(z)| \lesssim \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|)^{\frac{1-\lambda}{2}}}, \quad z \in \mathbb{D}. \tag{3.1.3}$$

Thanks to the fact that Morrey spaces are between two Hardy spaces we observe that certain random power series in Hardy spaces and Morrey spaces are the same. We shall consider random power series of the form

$$\sum_{n=0}^{\infty} \varepsilon_n a_n z^n,$$

where $\{\varepsilon_n\}_{n=0}^{\infty}$ is a choice of signs, that is, $\{\varepsilon_n\}_{n=0}^{\infty} \subset \{-1, 1\}$. By (3.1.2), the following result ([40, Theorem A.5]) can be extended to all Morrey spaces $\mathcal{L}^{2,\lambda}$ with $0 < \lambda < 1$.

Theorem Q. *Let $f(z) = \sum_{n=0}^{\infty} \varepsilon_n a_n z^n$ be an analytic function in the disc where $\{\varepsilon_n\}_{n=0}^{\infty}$ is a choice of signs. Then*

(i) If $\sum\limits_{n=0}^{\infty} |a_n|^2 < \infty$, then for almost every choice of signs $\{\varepsilon_n\}_{n=0}^{\infty}$,

$$f \in H^p \quad \text{for all } p < \infty.$$

(ii) If $\sum\limits_{n=0}^{\infty} |a_n|^2 = \infty$ then for almost every choice of signs $\{\varepsilon_n\}_{n=0}^{\infty}$, f has a radial limit almost nowhere and hence $f \notin H^p$ for any $p < \infty$.

Here we notice a difference between Morrey spaces and $BMOA$. By results in [6] and [42] we have the following.

Theorem R. *Let $f(z) = \sum_{n=0}^{\infty} \varepsilon_n a_n z^n$ be an analytic function in the disc where $\{\varepsilon_n\}_{n=0}^{\infty}$ is a choice of signs. Then*

(i) If $\sum\limits_{n=0}^{\infty} |a_n|^2 \log n < \infty$, then for almost every choice of signs $\{\varepsilon_n\}_{n=0}^{\infty}$, $f \in BMOA$.

(ii) Given a sequence of non-negative numbers $\{c_n\}_{n=0}^{\infty}$ which decreases to 0, there exists a sequence of positive numbers $\{a_n\}_{n=0}^{\infty}$ such that $\sum\limits_{n=0}^{\infty} c_n a_n^2 \log n < \infty$ but $f \notin \mathcal{B}$ for almost every choice of signs $\{\varepsilon_n\}_{n=0}^{\infty}$ (Note that $BMOA \subset \mathcal{B}$).

So there exist random power functions which belong to every Morrey spaces $\mathcal{L}^{2,\lambda}$ with $0 < \lambda < 1$ for almost every choice of signs but do not belong to $BMOA$ for almost any choice of signs.

Coming back to the chain (3.1.2), we have that the first inclusion, $H^{\frac{2}{1-\lambda}} \subset \mathcal{L}^{2,\lambda}$, $0 < \lambda < 1$, is proper because, for example, the function $f(z) = (1-z)^{-\frac{1-\lambda}{2}}$, which gives the maximum growth, does not belong to the Hardy space $H^{\frac{2}{1-\lambda}}$ (Theorem 5.9 of [40]) but it is in the Morrey space $\mathcal{L}^{2,\lambda}$. We can prove this in some different ways, maybe the most interesting is by a characterization of functions with non-negative Taylor coefficients in Morrey spaces.

Theorem 25. *Let* $0 < \lambda \leq 1$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *a function in* $\mathcal{H}ol(\mathbb{D})$. *Then,* $f \in \mathcal{L}^{2,\lambda}$ *if and only if*

$$\sup_{w\in\mathbb{D}} \sum_{n=0}^{\infty} \frac{(1-|w|^2)^{2-\lambda}}{(n+1)^2} \left| \sum_{k=0}^{n} (k+1) a_{k+1} \overline{w}^{n-k} \right|^2 < \infty.$$

We remark that for $\lambda = 1$, this reduces to the known result for $BMOA$ which can be found in [52].

If we consider the case where $a_n \geq 0$ for every $n \geq 0$ then, the above result reduces to the following:

The function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to $\mathcal{L}^{2,\lambda}$ if and only if

$$\sup_{0\leq r<1} \sum_{n=0}^{\infty} \frac{(1-r^2)^{2-\lambda}}{(n+1)^2} \left(\sum_{k=0}^{n} (k+1) a_{k+1} r^{n-k} \right)^2 < \infty. \tag{3.1.4}$$

In this case we can give a simpler equivalent characterization.

Theorem 26. *Let* $0 < \lambda \leq 1$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *a function in* $\mathcal{H}ol(\mathbb{D})$ *with* $a_n \geq 0$ *for every* $n \geq 0$. *Then* $f \in \mathcal{L}^{2,\lambda}$ *if and only if*

$$\sup_{n\geq 1} \frac{1}{n^{1-\lambda}} \sum_{k=0}^{\infty} \left(\sum_{j=kn}^{(k+1)n-1} a_j \right)^2 < \infty. \tag{3.1.5}$$

Let us see now an easy characterization of functions in Morrey spaces with have non-negative and non-increasing Taylor coefficients.

Theorem 27. *Let* $0 < \lambda < 1$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *a function in* $\mathcal{H}ol(\mathbb{D})$ *with* $a_n \geq 0$ *for every* $n \geq 0$ *and* $\{a_n\}$ *non-increasing. Then*

$$f \in \mathcal{L}^{2,\lambda} \Leftrightarrow a_n \lesssim n^{-\frac{1+\lambda}{2}}.$$

Using this characterization we obtain a new proof of the fact that, for $0 < \lambda < 1$, the function $f(z) = (1-z)^{-\frac{1-\lambda}{2}}$ belongs to the Morrey space $\mathcal{L}^{2,\lambda}$. Indeed, by Theorem 2.31 of [112] we know that $f(z) = \sum_{n=0}^{\infty} A_n z^n$, where $A_n = \frac{\prod_{k=0}^{n-1}(\alpha+k)}{n!}$ with $\alpha = \frac{1+\lambda}{2} < 1$, so by the theorem above we obtain directly that $f \in \mathcal{L}^{2,\lambda}$.

Theorem 27 gives us also the following result

Corollary 3. *Let $0 < \lambda < 1$. We define $\mathcal{P}$ as the class of analytic functions in the disc with non-negative and non-increasing Taylor coefficients,*

$$\mathcal{P} = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}ol(\mathbb{D}) : a_n \geq 0 \text{ and } \{a_n\} \text{ non-increasing} \right\}.$$

Then

$$\mathcal{L}^{2,\lambda} \cap \mathcal{P} \subset \bigcap_{p < \frac{2}{1-\lambda}} H^p.$$

We can also find differences between $BMOA$ and Morrey spaces if we regard the univalent functions. We recall that a function is said univalent if it is analytic and injective. We denote $\mathcal{U}$ as the class of univalent functions in the unit disc. There exists a geometric characterization for univalent functions in the spaces $BMOA$ and the Bloch space which can be found in [35, 87].

Theorem S. *Let $f \in \mathcal{U}$. Then the following conditions are equivalent.*

- $f \in \mathcal{B}$.
- $f \in BMOA$.
- $f(\mathbb{D})$ *does not contain arbitrarily large discs.*

The function $f(z) = (1-z)^{-\frac{1-\lambda}{2}}$ mentioned above, is univalent and belongs to the space $\mathcal{L}^{2,\lambda}$. It is clear that $f(\mathbb{D})$ is a sector of the complex plane in that arbitrarily large discs are contained, so the class of univalent functions in a Morrey space $\mathcal{L}^{2,\lambda}$ with $0 < \lambda < 1$ is pretty larger than that of the univalent functions in $BMOA$ or the Bloch space.

The importance of univalent functions in Morrey spaces lies in the following result, which gives us a similar contention as in Corollary 3.

Theorem 28. *Let $0 < \lambda < 1$. Then*

$$\mathcal{L}^{2,\lambda} \cap \mathcal{U} \subset \bigcap_{p < \frac{2}{1-\lambda}} H^p.$$

We do not know if the results of Corollary 3 and Theorem 28 can be extended to the whole Morrey space. We leave this question as a conjecture.

Question 2. Let $0 < \lambda < 1$. It is true that

$$\mathcal{L}^{2,\lambda} \subset \bigcap_{p < \frac{2}{1-\lambda}} H^p \quad ?$$

3.1.1 Proofs

Proof of Theorem 25. We know that

$$\begin{aligned}\|f-f(0)\|^2_{\mathcal{L}^{2,\lambda}} &\asymp \sup_{w\in\mathbb{D}}(1-|w|^2)^{1-\lambda}\int_{\mathbb{D}}|f'(z)|^2(1-|\sigma_w(z)|^2)\,dA(z)\\ &= \sup_{w\in\mathbb{D}}(1-|w|^2)^{2-\lambda}\int_{\mathbb{D}}|f'(z)|^2\frac{1-|z|^2}{|1-\overline{w}z|^2}\,dA(z)\end{aligned}$$

In the proof of Theorem 9.9 of [52], it is obtained by Parseval's identity that for $w\in\mathbb{D}$

$$\int_{\mathbb{D}}|f'(z)|^2\frac{1-|z|^2}{|1-\overline{w}z|^2}\,dA(z)=\pi\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}\left|\sum_{k=0}^{n}(k+1)a_{k+1}\overline{w}^{n-k}\right|^2$$

so considering the weight and taking suprema we have that

$$\begin{aligned}\|f-f(0)\|^2_{\mathcal{L}^{2,\lambda}} &\asymp \sup_{w\in\mathbb{D}}(1-|w|^2)^{2-\lambda}\int_{\mathbb{D}}|f'(z)|^2\frac{1-|z|^2}{|1-\overline{w}z|^2}\,dA(z)\\ &= \pi\sup_{w\in\mathbb{D}}(1-|w|^2)^{2-\lambda}\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}\left|\sum_{k=0}^{n}(k+1)a_{k+1}\overline{w}^{n-k}\right|^2\\ &\asymp \sup_{w\in\mathbb{D}}(1-|w|^2)^{2-\lambda}\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\left|\sum_{k=0}^{n}(k+1)a_{k+1}\overline{w}^{n-k}\right|^2.\end{aligned}$$

□

Proof of Theorem 26. We have just to prove that (3.1.4) and (3.1.5) are equivalent for a sequence $\{a_n\}_{n=0}^{\infty}$ with $a_n\geq 0$ for all n.

So take such a sequence $\{a_n\}$ of non negative numbers and suppose first that (3.1.4) holds. Let A be the supremum in (3.1.4). Bearing in mind that there exist two absolute constant $c_1,c_2>0$ (independent of n) such that

$$c_1\leq\left(1-\frac{1}{n+1}\right)^j\leq c_2,$$

if $0\leq j\leq 2n$, by Theorem 9.11 of [52] we have that for every $n\geq 1$

$$\sum_{k=0}^{\infty}\left(\sum_{j=kn}^{(k+1)n-1}a_{j+1}\right)^2\leq C\frac{1}{n+1}\sum_{s=0}^{\infty}\frac{1}{(s+1)^2}\left(\sum_{j=0}^{s}(j+1)a_{j+1}\left(1-\frac{1}{n+1}\right)^{s-j}\right)^2$$

so

$$\frac{1}{(n+1)^{1-\lambda}}\sum_{k=0}^{\infty}\left(\sum_{j=kn}^{(k+1)n-1} a_{j+1}\right)^2 \leq$$

$$C\frac{1}{(n+1)^{2-\lambda}}\sum_{s=0}^{\infty}\frac{1}{(s+1)^2}\left(\sum_{j=0}^{s}(j+1)a_{j+1}\left(1-\frac{1}{n+1}\right)^{s-j}\right)^2 \leq CA.$$

This gives (3.1.5).

Conversely, suppose that $a_n \geq 0$, for all n, and that (3.1.5) is satisfied. It is easy to see that (3.1.5) implies that there exist positive constants c_1, c_2, c_3 such that

$$\sum_{j=n}^{2n} a_j \leq c_1 n^{\frac{1-\lambda}{2}}, \quad \text{for all } n, \tag{3.1.6}$$

$$\sum_{m=1}^{n} m a_m \leq c_2 n^{\frac{3-\lambda}{2}}, \quad \text{for all } n, \tag{3.1.7}$$

and

$$\sum_{k=0}^{\infty}\left(\sum_{j=\alpha+kN}^{\alpha+(k+1)N} a_j\right)^2 \leq c_3 N^{1-\lambda}, \quad \text{for all } \alpha, N \in \mathbb{N}. \tag{3.1.8}$$

Fix $r \in (0,1)$. Let N be the positive integer satisfying

$$\frac{1}{N} < 1-r \leq \frac{1}{N-1}$$

(then $N \geq 2$). Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(1-r^2)^{2-\lambda}}{(n+1)^2}&\left(\sum_{k=0}^{n}(k+1)a_{k+1}r^{n-k}\right)^2\\ &\leq \frac{C}{N^{2-\lambda}}\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\left(\sum_{k=0}^{n}(k+1)a_{k+1}\left(1-\frac{1}{N}\right)^{n-k}\right)^2\\ &\leq \frac{C}{N^{2-\lambda}}(A_N+B_N),\end{aligned}$$

where

$$A_N = \sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\left(\sum_{0\leq k\leq \frac{n}{2}}(k+1)a_{k+1}\left(1-\frac{1}{N}\right)^{n-k}\right)^2 \tag{3.1.9}$$

and

$$B_N = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \left(\sum_{\frac{n}{2}<k\leq n} (k+1)a_{k+1} \left(1-\frac{1}{N}\right)^{n-k} \right)^2 .$$

It is clear that (3.1.4) will follow from

$$A_N + B_N = O(N^{2-\lambda}), \quad \text{as } N \to \infty. \tag{3.1.10}$$

Using (3.1.9) and (3.1.7) we obtain

$$\begin{aligned} A_N &\leq C \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \left(\sum_{0\leq k\leq \frac{n}{2}} (k+1)a_{k+1} \right)^2 \left(1-\frac{1}{N}\right)^{n/2} \\ &\leq C \sum_{n=0}^{\infty} (n+1)^{1-\lambda} \left(1-\frac{1}{N}\right)^{n} \leq CN^{2-\lambda}. \end{aligned} \tag{3.1.11}$$

On the other hand

$$B_N \leq C \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \left(\sum_{k=n}^{2n} k a_k \left(1-\frac{1}{N}\right)^{2n-k} \right)^2 \leq C I_N \tag{3.1.12}$$

where

$$I_N = \sum_{n=0}^{\infty} \left(\sum_{k=n}^{2n} a_k \left(1-\frac{1}{N}\right)^{2n-k} \right)^2 .$$

Fix a large positive integer M such that

$$\left(\frac{1-\frac{1}{N}}{1-\frac{1}{MN}} \right)^{2MN} \leq \frac{1}{4M^{2-\lambda}}, \quad \text{for all } N. \tag{3.1.13}$$

Using (3.1.6), (3.1.8) and (3.1.13), we obtain

$$
\begin{aligned}
I_N \leq & \sum_{n=0}^{NM}\left(\sum_{k=n}^{2n} a_k\right)^2 \\
& + \sum_{n=NM}^{\infty}\left(\sum_{k=2n-MN}^{2n} a_k + \sum_{k=n}^{2n-MN} a_k\left(1-\frac{1}{N}\right)^{2n-k}\right)^2 \\
\leq & C(NM)^{2-\lambda} + 2\sum_{n=NM}^{\infty}\left(\sum_{k=2n-MN}^{2n} a_k\right)^2 \\
& + 2\sum_{n=NM}^{\infty}\left(\sum_{k=n}^{2n-MN} a_k\left(1-\frac{1}{N}\right)^{2n-k}\right)^2 \\
\leq & C(NM)^{2-\lambda} + 2\sum_{\alpha=0}^{MN-1}\sum_{k=0}^{\infty}\left(\sum_{j=\alpha+MNk}^{\alpha+MN(k+1)} a_j\right)^2 \\
& + 2\left(\frac{1-\frac{1}{N}}{1-\frac{1}{MN}}\right)^{2MN}\sum_{n=0}^{\infty}\left(\sum_{k=n}^{2n} a_k\left(1-\frac{1}{MN}\right)^{2n-k}\right)^2 \\
\leq & C(NM)^{2-\lambda} + 2C(NM)^{2-\lambda} + 2\left(\frac{1-\frac{1}{N}}{1-\frac{1}{MN}}\right)^{2MN} I_{MN} \\
\leq & C(NM)^{2-\lambda} + \frac{1}{2M^{2-\lambda}} I_{MN}, \quad \text{for all N.}
\end{aligned}
$$

Then it follows that

$$\sup_N \frac{I_N}{N^{2-\lambda}} \leq CM^{2-\lambda} + \frac{1}{2}\sup_N \frac{I_{MN}}{(MN)^{2-\lambda}}.$$

Clearly, this implies that

$$\sup_N \frac{I_N}{N^{2-\lambda}} \leq CM^{2-\lambda}$$

if the sequence $\{a_n\}$ contains only finitely many non zero terms. Then a limit argument shows that this is true in the general case. Using this, (3.1.12), (3.1.11), we obtain (3.1.10). This finishes the proof. □

Proof of Theorem 27. We suppose that $a_n \lesssim n^{-\frac{1+\lambda}{2}}$. Using the characterization

obtained in Theorem 26, we have that

$$\frac{1}{n^{1-\lambda}}\sum_{k=0}^{\infty}\left(\sum_{j=kn}^{kn+n-1} a_j\right)^2 \lesssim \frac{1}{n^{1-\lambda}}\sum_{k=0}^{\infty}\left(\sum_{j=kn}^{kn+n-1} j^{-\frac{1+\lambda}{2}}\right)^2 \leq \frac{1}{n^{1-\lambda}}\sum_{k=0}^{\infty}\left(nk^{-\frac{1+\lambda}{2}}n^{-\frac{1+\lambda}{2}}\right)^2$$
$$= \frac{1}{n^{1-\lambda}}\sum_{k=0}^{\infty}\frac{n^{1-\lambda}}{k^{1+\lambda}} \leq C.$$

so $f \in \mathcal{L}^{2,\lambda}$.

We suppose now that $f \in \mathcal{L}^{2,\lambda}$, being a_n non-negative and non-increasing. Then

$$n^{1-\lambda} \gtrsim \sum_{k=0}^{\infty}\left(\sum_{j=kn}^{kn+n-1} a_j\right)^2 \geq \left(\sum_{j=n}^{2n-1} a_j\right)^2 \geq \left(a_{2n-1}\sum_{j=n}^{2n-1}\right)^2 = n^2 a_{2n-1}^2$$

hence

$$a_{2n-1} \lesssim n^{-\frac{1+\lambda}{2}}.$$

□

The proof of Corollary 3 is a direct consequence of Theorem 27 and Theorem A of [78].

Proof of Theorem 28. Let $p < \frac{2}{1-\lambda}$. If $f \in \mathcal{U}$ we know by Theorem A of [84] that

$$f \in H^p \Leftrightarrow \int_0^1 M_\infty^p(r,f)\,dr < \infty.$$

Since $f \in \mathcal{L}^{2,\lambda}$, using (3.1.3) we obtain that

$$\int_0^1 M_\infty^p(r,f)\,dr \lesssim \int_0^1 \frac{dr}{(1-r)^{\frac{p(1-\lambda)}{2}}} = \frac{1}{1-\frac{p(1-\lambda)}{2}} < \infty.$$

□

3.2 Semigroups on Morrey spaces

This section is devoted to the action of semigroups of composition operators on Morrey spaces. Let us start with the definition of a semigroup and some of its main elements.

A (one-parameter) semigroup of analytic functions is a continuous homomorphism $\Phi : t \mapsto \Phi(t) = \varphi_t$ from the additive semigroup of nonnegative real numbers

into the composition semigroup of all analytic functions which map $\mathbb{D}$ into $\mathbb{D}$. In other words, $\Phi = (\varphi_t)$ consists of analytic functions on $\mathbb{D}$ with $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ and for which the following three conditions hold:

(i) φ_0 is the identity in $\mathbb{D}$,

(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,

(iii) $\varphi_t \to \varphi_0$, as $t \to 0$, uniformly on compact subsets of $\mathbb{D}$.

It is well known that condition (iii) above can be replaced by

$$(iii') \text{ For each } z \in \mathbb{D},\ \varphi_t(z) \to z, \text{ as } t \to 0.$$

Some basic examples of semigroups are:

(i) The trivial semigroup, $\varphi_t(z) = z$, $t \geq 0$.

(ii) The dilatations of the disc with respect to the origin, $\varphi_t(z) = e^{-t}z$, $t \geq 0$.

(iii) The rotations of the disc, $\varphi_t(z) = e^{it}z$, $t \geq 0$.

Each such semigroup gives rise to a semigroup (C_t) consisting of composition operators on $\mathcal{H}ol(\mathbb{D})$,

$$C_t(f) \stackrel{\text{def}}{=} f \circ \varphi_t, \quad f \in \mathcal{H}ol(\mathbb{D}).$$

There is a good number of works about semigroups of composition operators focused on the restriction of (C_t) to certain linear subspaces of $\mathcal{H}ol(\mathbb{D})$. Given a Banach space X consisting of functions in $\mathcal{H}ol(\mathbb{D})$ and a semigroup (φ_t), we say that (φ_t) generates a semigroup of operators on X if (C_t) is a well-defined strongly continuous semigroup of bounded operators in X. This exactly means that for every $f \in X$, we have $C_t(f) \in X$ for all $t \geq 0$ and

$$\lim_{t \to 0^+} \|C_t(f) - f\|_X = 0.$$

Thus the crucial step to showing that (φ_t) generates a semigroup of operators in X is to pass from the pointwise convergence $\lim_{t \to 0^+} f \circ \varphi_t(z) = f(z)$ on $\mathbb{D}$ to the convergence in the norm of X.

This connection between composition operators and semigroups opens the possibility of studying spectral properties, operator ideal properties or dynamical properties of the semigroup of operators (C_t) in terms of the theory of functions. The papers [17] and [85] are considered the starting point in this direction.

Classical choices of X treated in the literature are the Hardy spaces H^p, the disc algebra $\mathcal{A}$, the Bergman spaces A^p, the Dirichlet space $\mathcal{D}$ and the chain of spaces Q_p and $Q_{p,0}$ which include the spaces $BMOA$, the Bloch space as well as their *little oh* analogues. See [93, 94, 100] for composition semigroups on these spaces.

Some results of semigroups on these spaces are the following:

(i) Every semigroup of analytic functions generates a semigroup of operators on the Hardy spaces H^p $(1 \leq p < \infty)$ [17], the Bergman spaces A^p $(1 \leq p < \infty)$ [92], the Dirichlet space [93], and on the spaces $VMOA$ and the little Bloch space $\mathcal{B}_0$ [100].

(ii) No non-trivial semigroup generates a semigroup of operators in the space H^∞ of bounded analytic functions [5, 19].

(iii) There are plenty of semigroups (but not all) which generate semigroups of operators in the disc algebra. Indeed, they can be well characterized in several analytical terms [31].

Recently, it has been discovered ([5], [19] and [18]) that $BMOA$ and the Bloch space are in the second case, i. e., the only semigroup (φ_t) such that $C_t(X) \subset X$ for all $t \geq 0$ and $\lim_{t\to 0^+} \|C_t(f) - f\|_X = 0$ where $X = BMOA, \mathcal{B}$ is the trivial one.

Let us introduce some notation and basic facts about semigroups. All this basic information can be found in [38, Chapter VII] and [94].

Given a semigroup (φ_t) and a Banach space X, we will denote by $[\varphi_t, X]$ the maximal closed linear subspace of X such that (φ_t) generates a semigroup of operators on it.

Another important tool in the study of semigroups is the infinitesimal generator. We define it as

$$G(z) \stackrel{\text{def}}{=} \lim_{t\to 0^+} \frac{\varphi_t(z) - z}{t}, \; z \in \mathbb{D}.$$

This convergence holds uniformly on compact subsets of $\mathbb{D}$ so $G \in \mathcal{H}ol(\mathbb{D})$. Moreover G satisfies

$$G\left(\varphi_t(z)\right) = \frac{\partial \varphi_t(z)}{\partial t} = G(z)\frac{\partial \varphi_t(z)}{\partial z},\ z \in \mathbb{D},\ t \geq 0.$$

Further G has a unique representation

$$G(z) = (\bar{b}z - 1)(z - b)P(z),\ z \in \mathbb{D},$$

where $b \in \overline{\mathbb{D}}$ and $P \in \mathcal{H}ol(\mathbb{D})$ with Re $P(z) \geq 0$ for all $z \in \mathbb{D}$. If G is not identically null, that is, if (φ_t) is not trivial, the couple (b, P) is uniquely determined from (φ_t) and the point b is called the Denjoy-Wolff point of the semigroup. This point plays a crucial role in the dynamical behavior of the semigroup [32]. The next fundamental results about the structure of semigroups depending on the Denjoy-Wolff point can be consulted in [17, Section 3] and [94, Section 3]. Before of that we need to define some geometric concepts which can be found in [41, Section 2.7] and [86, Section 2.3].

A *logarithmic spiral* is a curve in the complex plane of the form

$$\omega(t) = \omega_0 e^{-ct}, \quad t \in \mathbb{R},$$

where ω_0 and c are complex constants with $\omega_0 \neq 0$. A domain D containing the origin is said to be *spirallike* if there exists $c \in \mathbb{C}$ such that for each point $\omega_0 \neq 0$ in D the arc of the spiral $\omega(t) = \omega_0 e^{-ct}$ from ω_0 to the origin lies entirely in D.

A domain D containing the origin is said to be *close-to-convex* if $\mathbb{C} \setminus D$ is the union of closed halflines such that the corresponding open half-lines are disjoint.

Let us come back to the structure of semigroups. Under normalization, the Denjoy-Wolff point $b \in \overline{\mathbb{D}}$ may be assumed to be 0 (if $b \in \mathbb{D}$) or 1 (if $b \in \partial\mathbb{D}$). If $b = 0$, then

$$\varphi_t(z) = h^{-1}(e^{-ct}h(z)),$$

where h is a univalent function from $\mathbb{D}$ onto a spirallike domain Ω, $h(0) = 0$, Re $c \geq 0$, and $\omega e^{-ct} \in \Omega$ for each $\omega \in \Omega$, $t \geq 0$. If $b = 1$, then

$$\varphi_t(z) = h^{-1}(h(z) + ct),$$

where h is a univalent function from $\mathbb{D}$ onto a close-to-convex domain Ω, $h(0) = 0$, where Re $c \geq 0$, and $\omega + ct \in \Omega$ for each $\omega \in \Omega, t \geq 0$.

Our work here has been to prove that for $0 < \lambda < 1$, Morrey spaces $\mathcal{L}^{2,\lambda}$ are in the same case that H^∞, $BMOA$ and the Bloch space: No non-trivial semigroup generates a semigroup of operators on them.

First of all we give some results about semigroups on Morrey spaces. The first one is a result about the existence of the maximal subspace referred before for all semigroup (φ_t). The second one is a characterization of this maximal subspace via the infinitesimal generator.

Theorem 29. *Let $0 < \lambda < 1$ be and (φ_t) a semigroup of analytic functions. Then there exists a closed subspace $Y \subset \mathcal{L}^{2,\lambda}$ such that (φ_t) generates a semigroup of operators on Y and such that any other subspace of $\mathcal{L}^{2,\lambda}$ with this property is contained in Y.*

As we have said above, we note that space Y as $[\varphi_t, \mathcal{L}^{2,\lambda}]$.

Theorem 30. *Let $0 < \lambda < 1$ and let (φ_t) be a semigroup of analytic functions with infinitesimal generator G. Then*

$$[\varphi_t, \mathcal{L}^{2,\lambda}] = \overline{\{f \in \mathcal{L}^{2,\lambda} : Gf' \in \mathcal{L}^{2,\lambda}\}}.$$

The following result ensures that little Morrey spaces are in the same case that $VMOA$ and the little Bloch space.

Theorem 31. *For $0 < \lambda < 1$, every semigroup (φ_t) generates a semigroup of operators on $\mathcal{L}_0^{2,\lambda}$.*

This in particular means that in our notation,

$$\mathcal{L}_0^{2,\lambda} \subset [\varphi_t, \mathcal{L}^{2,\lambda}] \subset \mathcal{L}^{2,\lambda}, \tag{3.2.1}$$

for every $0 < \lambda < 1$ and every semigroup (φ_t).

This chain of inclusions leads us to wonder about those semigroups with an extreme character, that is, those giving equality

$$\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}] \quad \text{or} \quad [\varphi_t, \mathcal{L}^{2,\lambda}] = \mathcal{L}^{2,\lambda}.$$

We can prove that for dilatations and rotations, the left hand side equality holds.

Theorem 32. *Suppose $0 < \lambda < 1$ and $f \in \mathcal{L}^{2,\lambda}$; then the following are equivalent:*

(i) $f \in \mathcal{L}_0^{2,\lambda}$.

(ii) $\lim_{t\to 0^+} \|f(e^{it}z) - f\|_{\mathcal{L}^{2,\lambda}} = 0$.

(iii) $\lim_{t\to 0^+} \|f(e^{-t}z) - f\|_{\mathcal{L}^{2,\lambda}} = 0$.

In our notation this theorem can be written as

$$\mathcal{L}_0^{2,\lambda} = [e^{it}z, \mathcal{L}^{2,\lambda}] = [e^{-t}z, \mathcal{L}^{2,\lambda}], \quad \text{for } 0 < \lambda < 1.$$

However, in general the first inclusion in (3.2.1) can be proper. An example of this type is the semigroup

$$\varphi_t(z) = e^{-t}z + 1 - e^{-t}, \quad t \geq 0, z \in \mathbb{D}.$$

For this semigroup and for $0 < \lambda < 1$, the function $f(z) = (1-z)^{-\frac{1-\lambda}{2}}$ which belongs to $\mathcal{L}^{2,\lambda}$ but not to $\mathcal{L}_0^{2,\lambda}$, satisfies

$$\|f \circ \varphi_t - f\|_{\mathcal{L}^{2,\lambda}} = \|e^{t\frac{1-\lambda}{2}}(1-z)^{-\frac{1-\lambda}{2}} - (1-z)^{-\frac{1-\lambda}{2}}\|_{\mathcal{L}^{2,\lambda}} = C\left(e^{t\frac{1-\lambda}{2}} - 1\right) \longrightarrow 0,$$

thus $f \in [\varphi_t, \mathcal{L}^{2,\lambda}]$.

We have obtained some necessary and sufficient conditions for equality in the left hand side of (3.2.1).

Theorem 33. *Let (φ_t) be a semigroup with infinitesimal generator G. Let $0 < \lambda < 1$. Assume that for some $0 < \alpha < 1/2$,*

$$\frac{(1-|z|)^\alpha}{G(z)} = O(1) \quad \text{as } |z| \to 1.$$

Then $\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$.

Clearly, the semigroups $\varphi_t(z) = e^{it}z$ and $\varphi_t(z) = e^{-t}z$ of Theorem 32 satisfy this condition because, in both cases, the infinitesimal generator is $G(z) = cz$, where $c \in \mathbb{C} \setminus \{0\}$.

Theorem 33 can be proved as a consequence of a stronger theorem.

Theorem 34. *Let (φ_t) be a semigroup with infinitesimal generator G. Let $0 < \lambda < 1$. Assume that*

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_{S(I)} \frac{1-|z|}{|G(z)|^2}\, dA(z) = 0$$

then $\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$.

If G satisfies the hypothesis of Theorem 33 then $\frac{(1-|z|)^{2\alpha}}{|G(z)|^2} \lesssim 1$ so

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_{S(I)} \frac{1-|z|}{|G(z)|^2}\, dA(z) \lesssim \lim_{|I|\to 0} \frac{1}{|I|} \int_{S(I)} (1-|z|)^{1-2\alpha}\, dA(z)$$
$$\lesssim \lim_{|I|\to 0} |I|^{2-2\alpha} = 0.$$

We have this necessary condition for semigroups with inner Denjoy-Wolff point.

Theorem 35. *Let (φ_t) be a semigroup with infinitesimal generator G and Denjoy-Wolff point $b \in \mathbb{D}$. If $\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$, then*

$$\lim_{|z|\to 1} \frac{(1-|z|)^{\frac{3-\lambda}{2}}}{G(z)} = 0.$$

Finally, we close this chapter with a result about the right hand side inclusion of (3.2.1). Let us start first with some background about the same problem in the Bloch space and $BMOA$. In Theorem 3 of [18] it is proved that if (φ_t) is a non-trivial semigroup then $[\varphi_t, \mathcal{B}] \subsetneq \mathcal{B}$. In fact the result is true for the more general class of Bloch spaces $\mathcal{B}^\alpha$, $\alpha > 0$, defined by

$$\mathcal{B}^\alpha = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty \right\}.$$

In this proof it is used strongly that, for all $\alpha > 0$, $\mathcal{B}^\alpha$ is a Grothendieck space with the Dunford-Pettis property. This geometric property of α-Bloch spaces is really hard to check (we do not know whether Morrey spaces satisfy it or not) and it is false for $BMOA$. For that, this method does not work for this space and this question has remained open for $BMOA$ for some years. Recently, in [5] the authors solved this problem proving the following.

Theorem T. *Let X be a Banach space. Suppose $H^\infty \subset X \subset \mathcal{B}$. Then there are no non-trivial semigroups such that $[\varphi_t, X] = X$.*

In particular, there are no non-trivial semigroups such that $[\varphi_t, BMOA] = BMOA$.

We have been able to adapt their steps in order to prove the following.

Theorem 36. *Let X be a Banach space. For $0 < \lambda < 1$, suppose $\mathcal{L}^{2,\lambda} \subset X \subset \mathcal{B}^{\frac{3-\lambda}{2}}$ and let (φ_t) be a non trivial semigroup of analytic functions. Then $[\varphi_t, X] \subsetneq X$. In particular there are no non-trivial semigroups such that $[\varphi_t, \mathcal{L}^{2,\lambda}] = \mathcal{L}^{2,\lambda}$.*

3.2.1 Proofs

This section is devoted to prove all the results about semigroups on Morrey spaces.

Proof of Theorem 29. Let $0 < \lambda < 1$ be. By Corollary 1 of [106] we have that $C_t : \mathcal{L}^{2,\lambda} \to \mathcal{L}^{2,\lambda}$ are bounded for all $t \geq 0$ and

$$\|f \circ \varphi_t\|_{\mathcal{L}^{2,\lambda}} \leq \left(\frac{1+|\varphi_t(0)|}{1-|\varphi_t(0)|}\right)^{\frac{1-\lambda}{2}} \|f\|_{\mathcal{L}^{2,\lambda}}, \quad t \geq 0.$$

Since $\sup_{t\in[0,1]} |\varphi_t(0)| = M < 1$ we have that

$$\sup_{t\in[0,1]} \|C_t\|_{\mathcal{L}^{2,\lambda}} \leq C \sup_{t\in[0,1]} \left(\frac{1+|\varphi_t(0)|}{1-|\varphi_t(0)|}\right)^{\frac{1-\lambda}{2}} \leq \frac{C}{(1-M)^{\frac{1-\lambda}{2}}} < \infty.$$

Bearing this in mind, we obtain directly the result by Proposition 1 of [18]. □

Proof of Theorem 30. As we claimed in the proof of Theorem 29, for every $0 < \lambda < 1$

$$\sup_{t\in[0,1]} \|C_t\|_{\mathcal{L}^{2,\lambda}} = M < \infty. \tag{3.2.2}$$

Since Morrey spaces trivially contain the constant functions we have directly the result by Theorem 1 of [18]. □

Proof of Theorem 31. Let $0 < \lambda < 1$. If (φ_t) is a semigroup of analytic functions, then every composition operator $C_t(f) = f \circ \varphi_t$ is bounded on $\mathcal{L}_0^{2,\lambda}$ for $0 < \lambda < 1$. This is because each φ_t belongs to the Dirichlet space $\mathcal{D}$ (recall that φ_t is univalent) and therefore also in $\mathcal{L}_0^{2,\lambda}$. Thus the composition semigroup (C_t) consists of bounded operators on $\mathcal{L}_0^{2,\lambda}$.

The only thing we have to prove is that $\lim_{t\to 0^+} \|C_t(f) - f\|_{\mathcal{L}^{2,\lambda}} = 0$ for every $f \in \mathcal{L}_0^{2,\lambda}$. Bearing in mind (3.2.2) we have that for any polynomial P

$$\|C_t(f) - f\|_{\mathcal{L}^{2,\lambda}} \leq (M+1)\|f - P\|_{\mathcal{L}^{2,\lambda}} + \|C_t(P) - P\|_{\mathcal{L}^{2,\lambda}}$$

holds, so, since polynomials are dense in $(\mathcal{L}_0^{2,\lambda}, \|\cdot\|_{\mathcal{L}^{2,\lambda}})$, it is enough to prove $\lim_{t\to 0^+} \|C_t(Q) - Q\|_{\mathcal{L}^{2,\lambda}} = 0$ for a polynomial Q. This is straightforward because $\lim_{t\to 0^+} \|Q \circ \varphi_t - Q\|_{\mathcal{D}} = 0$ and $\|\cdot\|_{\mathcal{L}^{2,\lambda}} \lesssim \|\cdot\|_{\mathcal{D}}$. □

Proof of Theorem 34. It suffices to show that

$$\{f \in \mathcal{L}^{2,\lambda} : Gf' \in \mathcal{L}^{2,\lambda}\} \subset \mathcal{L}_0^{2,\lambda}.$$

Of course, if $\frac{1-|z|}{|G(z)|^2}\,dA(z)$ is a vanishing Carleson measure then is a Carleson measure and also a λ-vanishing Carleson measure.
For an interval $I \subset \mathbb{T}$ of center $e^{i\theta}$ and its Carleson box $S(I)$ we consider the point $z_I = (1-|I|)e^{i\theta}$.

We assume that:

- $f \in \mathcal{L}^{2,\lambda}$. Then $\sup\limits_I \frac{1}{|I|^\lambda}\int_{S(I)} |f'(z)|^2(1-|z|)\,dA(z) \lesssim 1$.

- $f'G \in \mathcal{L}^{2,\lambda}$.

Let us prove that $\lim\limits_{|I|\to 0} \frac{1}{|I|^\lambda}\int_{S(I)} |f'(z)|^2(1-|z|)\,dA(z) = 0$.

$$\begin{aligned}
&\frac{1}{|I|^\lambda}\int_{S(I)} |f'(z)|^2(1-|z|)\,dA(z)\\
&=\frac{1}{|I|^\lambda}\int_{S(I)} |f'(z)G(z)|^2\frac{(1-|z|)}{|G(z)|^2}\,dA(z)\\
F = f'G \in \mathcal{L}^{2,\lambda}&\\
&=\frac{1}{|I|^\lambda}\int_{S(I)} |F(z)|^2\frac{(1-|z|)}{|G(z)|^2}\,dA(z)\\
&\lesssim\frac{1}{|I|^\lambda}\int_{S(I)} |F(z)-F(z_I)|^2\frac{(1-|z|)}{|G(z)|^2}\,dA(z)\\
&\quad+\frac{1}{|I|^\lambda}\int_{S(I)} |F(z_I)|^2\frac{(1-|z|)}{|G(z)|^2}\,dA(z)\\
&=\frac{1}{|I|^\lambda}\int_{S(I)} |F(z)-F(z_I)|^2\frac{(1-|z|)}{|G(z)|^2}\,dA(z)\\
&\quad+|F(z_I)|^2\frac{1}{|I|^\lambda}\int_{S(I)} \frac{(1-|z|)}{|G(z)|^2}\,dA(z)
\end{aligned}$$

now using the growth condition of Morrey spaces we have

$$\begin{aligned}
&\lesssim\frac{1}{|I|^\lambda}\int_{S(I)} |F(z)-F(z_I)|^2\frac{(1-|z|)}{|G(z)|^2}\,dA(z)\\
&\quad+\frac{\|F\|^2_{\mathcal{L}^{2,\lambda}}}{(1-|z_I|)^{1-\lambda}}\frac{1}{|I|^\lambda}\int_{S(I)} \frac{(1-|z|)}{|G(z)|^2}\,dA(z)
\end{aligned}$$

and since $1-|z_I|=|I|$

$$\begin{aligned}
&=\frac{1}{|I|^\lambda}\int_{S(I)}|F(z)-F(z_I)|^2\frac{(1-|z|)}{|G(z)|^2}\,dA(z)\\
&\quad+\|F\|^2_{\mathcal{L}^{2,\lambda}}\frac{1}{|I|}\int_{S(I)}\frac{(1-|z|)}{|G(z)|^2}\,dA(z)\\
&=\mathbb{A}_I+\mathbb{B}_I.
\end{aligned}$$

Since $\lim\limits_{|I|\to 0}\frac{1}{|I|}\int_{S(I)}\frac{1-|z|}{|G(z)|^2}\,dA(z)=0$ we have $\lim\limits_{|I|\to 0}\mathbb{B}_I=0$

$$\begin{aligned}
\mathbb{A}_I&=\frac{1}{|I|^\lambda}\int_{S(I)}|F(z)-F(z_I)|^2\frac{(1-|z|)}{|G(z)|^2}\,dA(z)\\
&\lesssim\frac{|I|^2}{|I|^\lambda}\int_{S(I)}\left|\frac{F(z)-F(z_I)}{1-\overline{z_I}z}\right|^2\frac{(1-|z|)}{|G(z)|^2}\,dA(z).
\end{aligned}$$

Let μ be the measure defined by $d\mu(z)=\frac{(1-|z|)}{|G(z)|^2}\,dA(z)$ in $S(I)$ and the null measure in $\mathbb{D}\setminus S(I)$. Then

$$\mathbb{A}_I\lesssim|I|^{2-\lambda}\int_{\mathbb{D}}\left|\frac{F(z)-F(z_I)}{1-\overline{z_I}z}\right|^2\,d\mu(z)$$

since μ is a vanishing Carleson measure it is also a Carleson measure, so we deduce that

$$\begin{aligned}
\mathbb{A}_I&\lesssim\left(\sup_J\frac{\mu(S(J))}{|J|}\right)|I|^{2-\lambda}\int_{\mathbb{T}}\left|\frac{F(\xi)-F(z_I)}{1-\overline{z_I}\xi}\right|^2\,|d\xi|\\
&\lesssim\left(\sup_J\frac{\mu(S(J))}{|J|}\right)(1-|z_I|)^{1-\lambda}\int_{\mathbb{T}}|F(\xi)-F(z_I)|^2\frac{1-|z_I|}{|1-\overline{z_I}\xi|^2}\,|d\xi|\\
&\leq\left(\sup_J\frac{\mu(S(J))}{|J|}\right)\|F\|^2_{\mathcal{L}^{2,\lambda}}.
\end{aligned}$$

Actually

$$\frac{\mu(S(J))}{|J|}=\frac{\mu\left(S(J)\cap S(I)\right)}{|J|}$$

so we need only consider arcs J with $J\cap I\neq\emptyset$.

- If $|J|>|I|$

$$\frac{\mu(S(J))}{|J|}=\frac{\mu\left(S(J)\cap S(I)\right)}{|J|}\leq\frac{\mu(S(I))}{|J|}\leq\frac{\mu(S(I))}{|I|}\to 0,\quad |I|\to 0.$$

- If $|J| \leq |I|$, then $J \subset 3I$ where $3I$ is the arc with same center as I an length $3|I|$.

So in general

$$\sup_J \frac{\mu(S(J))}{|J|} \leq \sup_{J \subset 3I} \frac{1}{|J|} \int_{S(J)} \frac{1-|z|}{|G(z)|^2}\, dA(z)$$

so if $|I| \to 0$ then $|J| \to 0$ too, so

$$\lim_{|I| \to 0} \sup_{J \subset 3I} \frac{1}{|J|} \int_{S(J)} \frac{1-|z|}{|G(z)|^2}\, dA(z) = 0$$

because

$$\lim_{|J| \to 0} \frac{1}{|J|} \int_{S(J)} \frac{1-|z|}{|G(z)|^2}\, dA(z) = 0.$$

Thus in any case

$$\frac{\mu(S(J))}{|J|} \to 0,$$

and then it follows that

$$\mathbb{A}_I \to 0.$$

□

Proof of Theorem 35. Without loss of generality, we may assume that $b = 0$. The infinitesimal generator then is

$$G(z) = -zP(z),$$

where P is analytic with Re $P \geq 0$. If P is constant, the result is clear. Otherwise consider the function

$$m(z) = \int_0^z \frac{u}{G(u)}\, du = -\int_0^z \frac{1}{P(u)}\, du.$$

As in Theorem 3.3 of [19] we deduce that $m \in BMOA$ and hence in $\mathcal{L}^{2,\lambda}$. As in the calculations of this theorem we observe that

$$(m \circ \varphi_t)'(z) - m'(z) = \int_0^t \varphi_s'(z) ds.$$

Hence

$$|(m \circ \varphi_t)'(z) - m'(z)|^2 = \left| \int_0^t \varphi_s'(z) ds \right|^2 \leq \int_0^t |\varphi_s'(z)|^2 ds.$$

By the Theorem 3.3 of [19] we obtain

$$\|m \circ \varphi_t - m\|_{\mathcal{L}^{2,\lambda}} \lesssim \|m \circ \varphi_t - m\|_{BMOA} \leq |m(\varphi_t(0)) - m(0)| + Ct,$$

so $\lim_{t\to 0} \|m \circ \varphi_t - m\|_{\mathcal{L}^{2,\lambda}} = 0$. Thus $m \in [\varphi_t, \mathcal{L}^{2,\lambda}]$ and by the hypothesis $m \in \mathcal{L}_0^{2,\lambda}$. The following standard argument for functions in $\mathcal{L}_0^{2,\lambda}$ completes the proof.

For $a \in \mathbb{D}$ we consider σ_a, then

$$\begin{aligned}\left(1-|a|^2\right)^{3-\lambda} |m'(a)|^2 &= \left(1-|a|^2\right)^{1-\lambda} |(m\circ\sigma_a)'(0)|^2 \\ &\lesssim \left(1-|a|^2\right)^{1-\lambda} \int_{\mathbb{D}} |(m\circ\sigma_a)'(z)|^2 (1-|z|^2)\, dA(z)\end{aligned}$$

(by the change of variables $w = \sigma_a(z)$)

$$\begin{aligned}&= \left(1-|a|^2\right)^{1-\lambda} \int_{\mathbb{D}} |m'(w)|^2 (1-|\sigma_a(w)|^2)\, dA(w) \\ &= \left(1-|a|^2\right)^{1-\lambda} \int_{\mathbb{D}} |m'(w)|^2 \frac{(1-|a|^2)(1-|w|^2)}{|1-\overline{a}w|^2}\, dA(w)\end{aligned}$$

and this last integral tends to 0 as $|a| \to 1$ because $m \in \mathcal{L}_0^{2,\lambda}$. It follows that

$$\lim_{|a|\to 1} \frac{(1-|a|^2)^{\frac{3-\lambda}{2}}}{G(a)} = \lim_{|a|\to 1} \frac{(1-|a|^2)^{\frac{3-\lambda}{2}} m'(a)}{a} = 0.$$

□

Getting into the proof of Theorem 36 we need the following result.

Theorem 37. *Given any nontrivial semigroup (φ_t), and $0 < \lambda < 1$ there exists $f \in \mathcal{L}^{2,\lambda}$ such that*

$$\liminf_{t\to 0} \|f \circ \varphi_t - f\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} \geq 1.$$

Prime ends are a key ingredient in the proof of this theorem, so we will now review some basic facts about prime ends introduced by Carathéodory in order to describe the boundary behavior of a univalent function h from $\mathbb{D}$ onto a simply connected domain $\Omega \subset \mathbb{C} \cup \{\infty\}$; see [88, Section 2.4]. A *crosscut* C of Ω is an open Jordan arc in Ω such that $\overline{C} \setminus C$ consists of one or two points on $\partial\Omega$. Here $\overline{C}$ denotes the closure of C in the Riemann sphere. If C is a crosscut of Ω, then $\Omega \setminus C$ has exactly two components. The diameter of a set $E \subset \mathbb{C} \cup \{\infty\}$ in the spherical metric is denoted $\operatorname{diam}^{\#} E$.

A *null-chain* $(C_n)_{n\geq 0}$ of Ω is defined as a sequence of crosscuts of Ω such that

(i) $\overline{C_n} \cap \overline{C_{n+1}} = \emptyset$ for all $n \geq 0$.

(ii) C_n separates C_0 and C_{n+1} for all $n \geq 0$.

(iii) $\operatorname{diam}^{\#} C_n \to 0$ as $n \to \infty$.

Let V_n be the component of $\Omega \setminus C_n$ not containing C_0, and define V_n' similarly for (C_n'). The null-chains (C_n) and (C_n') are called *equivalent* if, for every sufficiently large m, there exists n such that $V_n \subset V_m'$ and $V_n' \subset V_m$. This is an equivalence relation on the set of all null-chains of Ω. The equivalence classes are called the *prime ends* of Ω. A point $a \in \mathbb{C} \cup \{\infty\}$ is called a *principal point* of the prime end P if there exists a null-chain (C_n) representing P such that $C_n \to \{a\}$ in the spherical metric as $n \to \infty$. The set $I(P) = \bigcap_n \overline{V_n}$ is non-empty, compact and connected in $\mathbb{C} \cup \{\infty\}$. We call $I(P)$ the *impression* of P. If $I(P)$ is a single point we call the prime end *degenerate*.

We call a prime end P *accessible* if there exists a Jordan arc that lies, except for one endpoint on $\partial\Omega$, in Ω and intersects all but finitely many crosscuts of every null-chain (C_n) that represents P.

We will also need the following result from univalent function theory which states that univalent functions have no Koebe arcs. For our purposes, it may be stated as follows:

Theorem U. *[86, Lemma 9.3 and Corollary 9.1] Suppose that $h : \mathbb{D} \to \mathbb{C}$ is univalent, $\{\eta_n\}$ is a sequence of Jordan arcs in $\mathbb{D}$, and $h(\eta_n)$ converges to a point $\omega_0 \in \mathbb{C} \cup \{\infty\}$, i.e.,*

$$h(z) \to \omega_0, \quad z \in \eta_n, \quad n \to \infty.$$

Then the Euclidean diameter of η_n satisfies $\operatorname{diam} \eta_n \to 0$, as $n \to \infty$.

Proof of Theorem 37. Let (φ_t) be a nontrivial semigroup, and let b be the corresponding Denjoy-Wolff point. After normalization, we may assume that b is either 0 or 1. First we deal with the case that $b = 0$, so that each φ_t is given by

$$\varphi_t(z) = h^{-1}(e^{-ct} h(z)),$$

where h is a univalent function from $\mathbb{D}$ onto a spirallike domain Ω, $h(0) = 0$, $\operatorname{Re} c \geq 0$, and $\omega e^{-ct} \in \Omega$ for each $\omega \in \Omega$, $t \geq 0$.

When Re $c = 0$, the (φ_t) are rotations of the disc. The function $f(z) = (1-z)^{-(1-\lambda)/2}$ satisfies $f \in \mathcal{L}^{2,\lambda}$ and

$$\lim_{r\to 1^-} |f'(r)|(1-r)^{\frac{3-\lambda}{2}} = \frac{1-\lambda}{2} > 0.$$

However, for $\theta \neq 0$

$$\lim_{r\to 1^-} |f'(re^{i\theta})|(1-r)^{\frac{3-\lambda}{2}} = 0.$$

If $\varphi_t(z) = ze^{iat}$ for real $a \neq 0$, then, for all t between 0 and $2\pi/|a|$,

$$\begin{aligned}\|f\circ\varphi_t - f\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} &\geq \sup_{0<r<1} |f'(\varphi_t(r))\varphi_t'(r) - f'(r)|(1-r)^{\frac{3-\lambda}{2}} \\ &\geq \frac{1-\lambda}{2}.\end{aligned}$$

Replacing f with $\frac{2f}{1-\lambda}$ gives the result.

Next consider the case where Re $c > 0$, so that (φ_t) does not consist of automorphisms. Since Ω is spirallike about 0, we can choose $\omega_0 \in \partial\Omega$ such that

$$|\omega_0| = \inf\{|\omega| : \omega \in \partial\Omega\}.$$

Then $[0, \omega_0) \subset \Omega$. For all sufficiently large values of n, let C_n be the connected component of $\{\omega \in \Omega : |\omega - \omega_0| = 1/n\}$ that intersects $[0, \omega_0)$. Then (C_n) is a null-chain that represents an accessible prime end P with principal point ω_0. As in Theorem 3.1 of [5], $\lim_{r\to 1^-} h(r\gamma_0)$ exists (and is equal to ω_0), where $\gamma_0 \in \partial\mathbb{D}$ corresponds to P. Thus,

$$\lim_{r\to 1^-} \varphi_t(r\gamma_0) = h^{-1}(e^{-ct}\omega_0) \in \mathbb{D}, \quad t > 0.$$

Since φ_t is univalent and bounded, φ_t is in the Dirichlet spaces, and $\varphi_t \in \mathcal{B}_0^{\frac{3-\lambda}{2}}$. Hence

$$\lim_{r\to 1^-} |\varphi_t'(r\gamma_0)|(1-r)^{\frac{3-\lambda}{2}} = 0.$$

Letting $f(z) = (1-\overline{\gamma_0}z)^{-(1-\lambda)/2}$, we have

$$\lim_{r\to 1^-} |f'(r\gamma_0)|(1-r)^{\frac{3-\lambda}{2}} = \frac{1-\lambda}{2} > 0.$$

However, f' is continuous on $\mathbb{D}$, so for fixed $t > 0$

$$\lim_{r\to 1^-} |f'(\varphi_t(r\gamma_0))| = |f'(h^{-1}(e^{-ct}\omega_0))| < \infty.$$

Thus, for all $t > 0$,

$$\|f \circ \varphi_t - f\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} \geq \limsup_{r \to 1^-} |f'(\varphi_t(r\gamma_0))\varphi_t'(r\gamma_0) - f'(r\gamma_0)|(1-r)^{\frac{3-\lambda}{2}} \geq \frac{1-\lambda}{2}, \quad (3.2.3)$$

and replacing f with $\frac{2f}{1-\lambda}$ gives the result.

If the Denjoy-Wolff point b is 1 then $\varphi_t(z) = h^{-1}(h(z)+ct)$ where h is a univalent function from $\mathbb{D}$ onto a close-to-convex domain Ω, $h(0) = 0$, $\operatorname{Re} c \geq 0$, and $\omega + ct \in \Omega$ for each $\omega \in \Omega, t \geq 0$. If the φ_t are automorphisms, then the map $\omega \mapsto \omega + ct$ is an automorphism of Ω. It follows that Ω is a half-plane or strip, and $\partial\Omega$ in $\mathbb{C}$ consists of impressions of degenerate prime ends which are not fixed under $\omega \mapsto \omega + ct$, $t > 0$. Let $\omega_0 \in \mathbb{C}$ be one such impression, and let γ_0 be the corresponding point in $\partial\mathbb{D}$. Then $\varphi_t(\gamma_0) \in \partial\mathbb{D}$ but $\varphi_t(\gamma_0) \neq \gamma_0$ for all $t > 0$. Let $f(z) = (1 - \overline{\gamma_0}z)^{-(1-\lambda)/2}$. Then

$$\lim_{r \to 1^-} |f'(r\gamma_0)|(1-r)^{\frac{3-\lambda}{2}} = \frac{1-\lambda}{2} > 0,$$

but

$$\lim_{r \to 1^-} |f'(r\gamma)|(1-r)^{\frac{3-\lambda}{2}} = 0$$

for all $\gamma \in \partial\mathbb{D}$, $\gamma \neq \gamma_0$. The function f satisfies that f' extends continuously to $\overline{\mathbb{D}} \setminus \{\gamma_0\}$. Now fix some $t > 0$. Since γ_0 is not a fixed point of φ_t, composition with φ_t moves the radius $[0, \gamma_0)$ away to where f is well-behaved. For $\gamma_t = \varphi_t(\gamma_0)$, we have that f' extends to be continuous at γ_t, so

$$\lim_{r \to 1^-} f'(\varphi_t(r\gamma_0)) = f'(\gamma_t).$$

Since φ_t is an automorphism, φ_t' is bounded on $\mathbb{D}$. For fixed $t > 0$,

$$\|f \circ \varphi_t - f\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} \geq \limsup_{r \to 1^-} |f'(\varphi_t(r\gamma_0))\varphi_t'(r\gamma_0) - f'(r\gamma_0)|(1-r)^{\frac{3-\lambda}{2}} \geq \frac{1-\lambda}{2}.$$

As before, replacing f with $\frac{2f}{1-\lambda}$ gives the result.

In the non-automorphism case, for $t > 0$ the map given for $\omega \in \Omega$ by $\omega \mapsto \omega + ct$ is not onto. Let $t > 0$ and $\omega \in \Omega \setminus (\Omega + ct)$. Then there is $t_0 \in (0, t]$ such that $\omega_0 = \omega - ct_0 \in \partial\Omega$, but $(\omega_0, \omega] \subset \Omega$. As in the case $b = 0$, ω_0 is the principal point of an accessible prime end, and the same argument terminating with (3.2.3) completes the proof. □

Proof of Theorem 36. Each test function f in Theorem 37 is in $\mathcal{L}^{2,\lambda}$, and hence in X from the hypothesis that $\mathcal{L}^{2,\lambda} \subset X$. Since $X \subset \mathcal{B}^{\frac{3-\lambda}{2}}$, the Closed Graph Theorem

shows that $\|\cdot\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} \lesssim \|\cdot\|_X$ and bounding the $\frac{3-\lambda}{2}$-Bloch norm away from 0 bounds the X norm as well. Thus it follows from Theorem 37 that $f \notin [\varphi_t, X]$, and so $[\varphi_t, X] \subsetneq X$. $\square$

Chapter 4

Dirichlet-Morrey spaces

The final chapter is devoted to explore a class of spaces of analytic functions which shares properties with Dirichlet spaces and Morrey spaces.

We mentioned the Dirichlet spaces $\mathcal{D}_p$ in Section 2.3. If $0 \leq p < \infty$ they can be also defined as the spaces of analytic functions $f \in \mathcal{H}ol(\mathbb{D})$ for which

$$\|f\|^2_{\mathcal{D}_p} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^p \, dA(z) < \infty.$$

The quantity $\|\cdot\|_{\mathcal{D}_p}$ is a norm. Clearly $\mathcal{D}_1 = H^2$ with equivalence of norms, and $\mathcal{D}_0$ is the classical Dirichlet space denoted by $\mathcal{D}$. For $p > 1$, $\mathcal{D}_p$ coincides with the weighted Bergman space A^2_{p-2}. If $0 < p < q$ then

$$\mathcal{D} \subset \mathcal{D}_p \subset \mathcal{D}_q,$$

and there is a constant $C = C(p,q)$ such that $\|f\|_{\mathcal{D}_q} \leq C\|f\|_{\mathcal{D}_p}$ for each $f \in \mathcal{D}_p$.

The conformally invariant version of these spaces are the spaces Q_p,

$$\|f\|_{Q_p} = |f(0)| + \sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{\mathcal{D}_p},$$

We recall that $Q_0 = \mathcal{D}$, $Q_1 = BMOA$, while for all $p > 1$, Q_p coincides with the Bloch space $\mathcal{B}$.

If $0 \leq p \leq 1$ and $f \in \mathcal{D}_p$ the following estimate is valid

$$\|f \circ \varphi_a - f(a)\|_{\mathcal{D}_p} \leq \frac{C\|f\|_{\mathcal{D}_p}}{(1-|a|^2)^{\frac{p}{2}}}, \quad a \in \mathbb{D},$$

with the constant C depending only on p. In the case of the Hardy space $H^2 = \mathcal{D}_1$ condition (3.0.3) says that $f \in \mathcal{L}^{2,\lambda}$ for $0 < \lambda < 1$ if the stronger growth bound

$$\|f \circ \varphi_a - f(a)\|_{H^2} \leq \frac{C\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|a|^2)^{\frac{1}{2}(1-\lambda)}}, \quad a \in \mathbb{D},$$

holds. Motivated by this we define the Dirichlet-Morrey spaces as follows.

Let λ, $p \in [0,1]$. We say that $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the Dirichlet-Morrey space $\mathcal{D}_p^\lambda$ if

$$\|f\|_{\mathcal{D}_p^\lambda} = |f(0)| + \sup_{a\in\mathbb{D}}(1-|a|^2)^{\frac{p}{2}(1-\lambda)}\|f \circ \varphi_a - f(a)\|_{\mathcal{D}_p} < \infty. \tag{4.0.1}$$

It is clear $\mathcal{D}_p^\lambda$ is a linear space and the above quantity is a norm, under which $\mathcal{D}_p^\lambda$ is a Banach space. We see that $\mathcal{D}_1^\lambda = \mathcal{L}^{2,\lambda}$ and that for each p, $\mathcal{D}_p^1 = Q_p$ and $\mathcal{D}_p^0 = \mathcal{D}_p$. Furthermore we have

$$Q_p \subseteq \mathcal{D}_p^\lambda \subseteq \mathcal{D}_p, \qquad 0 < \lambda < 1.$$

In the next section we will state some basic properties of Dirichlet-Morrey spaces and discuss briefly their characterization in terms of boundary values. In Section 4.2 we will concentrate on the boundedness of integration operators and pointwise multipliers on these spaces. Most of our results concerning this topic are included in [46].

4.1 Structure and properties of Dirichlet-Morrey spaces

The following proposition gives a Carleson measure characterization of $\mathcal{D}_p^\lambda$, which is analogous to (3.0.4) for Morrey spaces.

Proposition 2. *Let $0 < p, \lambda < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$. Then the following are equivalent,*

(i) $f \in \mathcal{D}_p^\lambda$.

(ii) $\|f\|_{p,\lambda,*} = \sup\limits_{\substack{I\subset\mathbb{T} \\ I \text{ interval}}} \left(\frac{1}{|I|^{p\lambda}}\int_{S(I)} |f'(z)|^2(1-|z|^2)^p\, dA(z)\right)^{1/2} < \infty,$

and the norm $\|f\|_{\mathcal{D}_p^\lambda}$ is comparable to $|f(0)| + \|f\|_{p,\lambda,*}$.

In the next proposition we give a result about the radial growth of functions in Dirichlet-Morrey spaces and show that this condition is sharp.

Proposition 3. *Let $0 < p, \lambda < 1$ then,*

(i) There is a constant $C = C(p, \lambda)$ such that any $f \in \mathcal{D}_p^\lambda$ satisfies

$$|f(z)| \leq \frac{C\|f\|_{\mathcal{D}_p^\lambda}}{(1-|z|)^{\frac{p}{2}(1-\lambda)}}, \quad z \in \mathbb{D}. \tag{4.1.1}$$

(ii) The function $f_{p,\lambda}(z) = (1-z)^{-\frac{p}{2}(1-\lambda)}$ belongs to $\mathcal{D}_p^\lambda$.

Observe that both parts of the above Proposition are also valid when $p = 1$ for $0 < \lambda < 1$.

We set in the next result a necessary and sufficient condition to a Dirichlet-Morrey space is contained in another one.

Proposition 4. *Let $\lambda_1, p_1, \lambda_2, p_2 \in (0, 1)$. Then*

$$\mathcal{D}_{p_1}^{\lambda_1} \subseteq \mathcal{D}_{p_2}^{\lambda_2} \Longleftrightarrow p_1 \leq p_2 \text{ and } p_1(1-\lambda_1) \leq p_2(1-\lambda_2).$$

Xiao obtained in [103, Lemma 6.1.1] and [103, Theorem 6.1.1] the following characterizations of Dirichlet spaces and Q_p spaces in terms of boundary values.

Lemma A. *(i) If $f \in H^2$ and $0 < p < 1$, then $f \in \mathcal{D}_p$ if and only if*

$$\|f\|_{\mathcal{D}_p^*}^2 = \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}\,|du||dv| < \infty,$$

and furthermore, $\|f\|_{\mathcal{D}_p} \asymp |f(0)| + \|f\|_{\mathcal{D}_p^*}$.

(ii) If $f \in H^2$ and $0 < p < 1$, then $f \in Q_p$ if and only if

$$\sup_{\substack{I \subset \mathbb{T} \\ I \text{ interval}}} \int_I\int_I \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}\,|du||dv| < \infty.$$

We have used the simplified notation $u = e^{i\theta} \in \mathbb{T}$ and $|du| = d\theta$.

The proofs of these results can be adapted to obtain the following characterization of Dirichlet-Morrey spaces.

Theorem 38. *Suppose $f \in H^2$ and let $0 < p, \lambda < 1$. Then $f \in \mathcal{D}_p^\lambda$ if and only if*

$$\|f\|_{\mathcal{D}_p^{\lambda,*}}^2 = \sup_{\substack{I \subset \mathbb{T} \\ I \text{ interval}}} \frac{1}{|I|^{p\lambda}} \int_I\int_I \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}\,|du|\,|dv| < \infty. \tag{4.1.2}$$

4.1.1 Preliminary results

In this section we shall collect a number of results which will be needed in the proof of Theorem 38.

As in [103, Theorem 6.1.1] by the change of variables, $z = \varphi_a(u)$, $u \in \mathbb{D}$, we easily establish that

$$\|f \circ \varphi_a\|^2_{\mathcal{D}^*_p} = (2\pi)^p \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f(u) - f(v)|^2}{|u - v|^{2-p}} \left(P_a(u)P_a(v)\right)^{p/2} |du|\,|dv|, \tag{4.1.3}$$

where

$$2\pi P_a(u) = \frac{1 - |a|^2}{|1 - \overline{a}u|^2}$$

is the Poisson kernel.

We recall also the boundary characterization of functions in Morrey spaces, given in (3.0.1), for $f \in H^2$ and $0 < \lambda \leq 1$, $f \in \mathcal{L}^{2,\lambda}$ if and only if

$$\|f\|^2_{\mathcal{L}^{2,\lambda}(\mathbb{T})} = \sup_{\substack{I \subset \mathbb{T} \\ I \text{ interval}}} \frac{1}{|I|^\lambda} \int_I |f(u) - f_I|^2 \, |du| < \infty,$$

where f_I is the average of the function f over the arc $I \subset \mathbb{T}$, that is

$$f_I = \frac{1}{|I|} \int_I f(v)|dv|.$$

For any arc $I \subset \mathbb{T}$ we use the Cauchy-Schwarz inequality to obtain the following

$$\begin{aligned} |f(u) - f_I|^2 &\leq \left(\frac{1}{|I|} \int_I |f(u) - f(v)| \, |dv| \right)^2 \\ &\leq \frac{1}{|I|} \int_I |f(u) - f(v)|^2 \, |dv|. \end{aligned}$$

Doing some calculations we prove also that

$$\begin{aligned} \frac{1}{|I|} \int_I \int_I |f(u) - f(v)|^2 \, |du||dv| &= \frac{1}{|I|} \int_I \int_I |f(u) - f_I + f_I - f(v)|^2 \, |du||dv| \\ &\leq 4 \frac{1}{|I|} \int_I \int_I |f(u) - f_I|^2 \, |du||dv| \\ &= 4 \int_I |f(u) - f_I|^2 \, |du|. \end{aligned}$$

So we have that

$$\|f\|^2_{\mathcal{L}^{2,\lambda}(\mathbb{T})} \asymp \sup_{\substack{I\subset\mathbb{T}\\ I \text{ interval}}} \frac{1}{|I|^{1+\lambda}} \int_I\int_I |f(u)-f(v)|^2\,|du|\,|dv| < \infty. \tag{4.1.4}$$

In the next lemma we compare the quantities defined in (4.1.2) and (4.1.4). Notice that it holds for some values of p bigger than 1.

Lemma 9. *If $0<\lambda<1$ and $0<p<\frac{1}{1-\lambda}$ then for $f\in L^2(\mathbb{T})$,*

$$\|f\|_{\mathcal{L}^{2,1-p(1-\lambda)}(\mathbb{T})} \lesssim \|f\|_{\mathcal{D}^{\lambda}_{p,*}}.$$

Proof. First, we prove the case $p\le 2$. Let I be an arc of $\mathbb{T}$. Using that

$$|u-v|^{2-p} \le |I|^{2-p} \quad \text{for } u,v\in I$$

we get the following

$$\begin{aligned}
&\frac{1}{|I|^{2-p+p\lambda}}\int_I\int_I |f(u)-f(v)|^2\,|du|\,|dv|\\
=&\frac{1}{|I|^{2-p+p\lambda}}\int_I\int_I \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}|u-v|^{2-p}\,|du|\,|dv|\\
\le&\frac{1}{|I|^{p\lambda}}\int_I\int_I \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}\,|du|\,|dv|,
\end{aligned}$$

and then $\|f\|_{\mathcal{L}^{2,1-p(1-\lambda)}(\mathbb{T})} \le \|f\|_{\mathcal{D}^{\lambda,*}_p}$.

Assume now that $p>2$. Observe that, for any interval $I\subseteq\mathbb{T}$,

$$\begin{aligned}
&\frac{1}{|I|^{2-p+p\lambda}}\int_I\int_I |f(u)-f(v)|^2\,|du||dv|\\
\lesssim&\frac{1}{|I|^{2-p+p\lambda}}\sum_{k=1}^{\infty}\iint_{2^{-k}|I|<|u-v|<2^{1-k}|I|} |f(u)-f(v)|^2\,|du|\,|dv|\\
\lesssim&\frac{1}{|I|^{2-p+p\lambda}}\sum_{k=1}^{\infty}\left(\frac{|I|}{2^k}\right)^{2-p}\iint_{|u-v|<2^{1-k}|I|}\frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}\,|du|\,|dv|\\
\lesssim&\frac{1}{|I|^{p\lambda}}\sum_{k=1}^{\infty}2^{k(p-2)}2^k\left(\frac{|I|}{2^{k-1}}\right)^{p\lambda}\|f\|^2_{\mathcal{D}^{\lambda,*}_p}\\
\lesssim&\sum_{k=1}^{\infty}\frac{1}{2^{k(1-p(1-\lambda))}}\|f\|^2_{\mathcal{D}^{\lambda,*}_p}\\
\lesssim&\|f\|^2_{\mathcal{D}^{\lambda,*}_p}.
\end{aligned}$$

□

Finally, we will need the following technical result.

Lemma 10. *Let I and J be two intervals of $\mathbb{T}$ such that $I \subseteq J$ and $\gamma \in (0,1)$. If $f \in L^2(\mathbb{T})$ then*

$$|f_J - f_I|^2 \leq \frac{|J|^{\gamma+1}}{|I|^2} \|f\|^2_{\mathcal{L}^{2,\gamma}(\mathbb{T})}.$$

Proof. Using the Cauchy-Schwarz inequality and the characterization in (3.0.1) we obtain

$$\begin{aligned}
|f_J - f_I| &\leq \frac{1}{|I|} \int_I |f(u) - f_J| \, |du| \\
&\leq \frac{1}{|I|} \left(\int_J |f(u) - f_J|^2 \, |du| \right)^{1/2} |J|^{1/2} \\
&= \frac{|J|^{(\gamma+1)/2}}{|I|} \left(\frac{1}{|J|^\gamma} \int_J |f(u) - f_J|^2 \, |du| \right)^{1/2} \\
&\leq \frac{|J|^{(\gamma+1)/2}}{|I|} \|f\|_{\mathcal{L}^{2,\gamma}(\mathbb{T})}.
\end{aligned}$$

□

4.1.2 Proofs

Proof of Proposition 2. Assume $f \in \mathcal{D}_p^\lambda$. For an interval $I \subset \mathbb{T}$ let ζ be the midpoint of I and let $a = a_I = (1 - |I|)\zeta$. Note that

$$|1 - \overline{a}z| \asymp |I| = 1 - |a| \asymp 1 - |a|^2, \qquad z \in S(I),$$

thus

$$\begin{aligned}
&\frac{1}{|I|^{p\lambda}}\int_{S(I)}|f'(z)|^2(1-|z|^2)^p\,dA(z)\\
&\asymp \frac{1}{|I|^{p\lambda}}\int_{S(I)}|f'(z)|^2(1-|z|^2)^p\frac{(1-|a|^2)^{2p}}{|1-\overline{a}z|^{2p}}\,dA(z)\\
&\asymp |I|^{p(1-\lambda)}\int_{S(I)}|f'(z)|^2(1-|\varphi_a(z)|^2)^p\,dA(z)\\
&\lesssim (1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}|f'(z)|^2(1-|\varphi_a(z)|^2)^p\,dA(z)\\
&= (1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}|(f\circ\varphi_a)'(z)|^2(1-|z|^2)^p\,dA(z)\\
&= (1-|a|^2)^{p(1-\lambda)}\|f\circ\varphi_a-f(a)\|^2_{\mathcal{D}_p}\\
&\leq \|f\|^2_{\mathcal{D}_p^\lambda}.
\end{aligned}$$

This is valid for each interval $I\subset\mathbb{T}$ and taking supremum shows that (i) implies (ii).

Conversely suppose (ii) holds. That is, for the nonnegative measure $d\mu(z)=|f'(z)|^2(1-|z|^2)^p\,dA(z)$ there is a constant C such that

$$\mu(S(I))=\int_{S(I)}d\mu(z)\leq C|I|^{p\lambda}$$

for all $I\subset\mathbb{T}$, i.e. μ is a $p\lambda$-Carleson measure. Then for $a\in\mathbb{D}$,

$$\begin{aligned}
\|f\circ\varphi_a-f(a)\|^2_{\mathcal{D}_p}&=\int_{\mathbb{D}}|f'(z)|^2(1-|\varphi_a(z)|^2)^p\,dA(z)\\
&=\int_{\mathbb{D}}|f'(z)|^2\frac{(1-|z|^2)^p(1-|a|^2)^p}{|1-\overline{a}z|^{2p}}\,dA(z)\\
&=\int_{\mathbb{D}}\frac{(1-|a|^2)^p}{|1-\overline{a}z|^{2p}}\,d\mu(z).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\|f\circ\varphi_a-f(a)\|^2_{\mathcal{D}_p}&=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^{2p-p\lambda}}{|1-\overline{a}z|^{2p}}\,d\mu(z)\\
&=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^{q}}{|1-\overline{a}z|^{q+p\lambda}}\,d\mu(z)\\
&<\infty,
\end{aligned}$$

by using the characterization of Carleson measures in [104, Lemma 3.1.1] with $q = 2p - p\lambda > 0$, completing the proof. □

Proof of Proposition 3. (i) Suppose $f \in \mathcal{D}_p^\lambda$. We apply the inequality

$$|g(0)|^2 \leq (p+1) \int_{\mathbb{D}} |g(z)|^2 (1-|z|^2)^p \, dA(z),$$

see [111, Lemma 4.12], valid for all analytic g on $\mathbb{D}$, to the function $g = (f \circ \varphi_w - f(w))'$ to obtain

$$\begin{aligned} |f'(w)|^2 (1-|w|^2)^2 &\leq (p+1) \int_{\mathbb{D}} |(f \circ \varphi_w)'(z)|^2 (1-|z|^2)^p \, dA(z) \\ &= \frac{p+1}{(1-|w|^2)^{p(1-\lambda)}} \left((1-|w|^2)^{p(1-\lambda)} \|f \circ \varphi_w - f(w)\|_{\mathcal{D}_p}^2 \right) \\ &\leq \frac{p+1}{(1-|w|^2)^{p(1-\lambda)}} \|f\|_{\mathcal{D}_p^\lambda}^2, \end{aligned}$$

for each $w \in \mathbb{D}$. Thus

$$|f'(w)| \leq \frac{(p+1)^{1/2}}{(1-|w|^2)^{1+\frac{p}{2}(1-\lambda)}} \|f\|_{\mathcal{D}_p^\lambda}, \quad w \in \mathbb{D}.$$

Using this and the integration $f(z) - f(0) = \int_0^z f'(\zeta) \, d\zeta$ we obtain the desired growth inequality.

(ii) We will verify that $|f'_{p,\lambda}(z)|^2 (1-|z|^2)^p \, dA(z)$ is a $p\lambda$-Carleson measure and then Proposition 2 gives the conclusion. In doing so, it is more convenient to work with the equivalent family of Carleson lune-shaped sets $S(b,h) = \{z \in \mathbb{D} : |b - z| < h\}$, where $b \in \mathbb{T}$ and $0 < h < 1$, than with the Carleson boxes $S(I), I \subset \mathbb{T}$. Thus it suffices to show that

$$\sup_{\substack{b \in \mathbb{T} \\ 0<h<1}} \frac{1}{h^{p\lambda}} \int_{S(b,h)} |f'_{p,\lambda}(z)|^2 (1-|z|^2)^p \, dA(z) < \infty. \tag{4.1.5}$$

We have

$$\begin{aligned}\int_{S(b,h)} |f'_{p,\lambda}(z)|^2(1-|z|^2)^p\,dA(z) &= C_1\int_{S(b,h)} \frac{(1-|z|^2)^p}{|1-z|^{2+p(1-\lambda)}}\,dA(z)\\ &\lesssim \int_{S(b,h)} \frac{1}{|1-z|^{2-p\lambda}}\,dA(z)\\ &\lesssim \int_{S(1,h)} \frac{1}{|1-z|^{2-p\lambda}}\,dA(z)\\ &\lesssim \int_{|w|<h} \frac{1}{|w|^{2-p\lambda}}\,dA(w)\\ &= \int_0^h \frac{1}{r^{1-p\lambda}}dr\\ &= h^{p\lambda}.\end{aligned}$$

Thus (4.1.5) holds and the proof is finished. □

Proof of Proposition 4. Assume $p_1 \le p_2$ and $p_1(1-\lambda_1) \le p_2(1-\lambda_2)$ and let $f \in \mathcal{D}_{p_1}^{\lambda_1}$ and $I \subset \mathbb{T}$. Then

$$\begin{aligned}&\frac{1}{|I|^{p_2\lambda_2}}\int_{S(I)} |f'(z)|^2(1-|z|^2)^{p_2}\,dA(z)\\ &\quad= \frac{1}{|I|^{p_2\lambda_2}}\int_{S(I)} |f'(z)|^2(1-|z|^2)^{p_1}(1-|z|^2)^{p_2-p_1}\,dA(z)\\ &\quad\le \frac{|I|^{p_2-p_1}}{|I|^{p_2\lambda_2}}\int_{S(I)} |f'(z)|^2(1-|z|^2)^{p_1}\,dA(z)\\ &\quad= |I|^{p_2(1-\lambda_2)-p_1(1-\lambda_1)}\left(\frac{1}{|I|^{p_1\lambda_1}}\int_{S(I)} |f'(z)|^2(1-|z|^2)^{p_1}\,dA(z)\right)\end{aligned}$$

and by Proposition 2, it follows $\mathcal{D}_{p_1}^{\lambda_1} \subseteq \mathcal{D}_{p_2}^{\lambda_2}$.

Assume now that $\mathcal{D}_{p_1}^{\lambda_1} \subseteq \mathcal{D}_{p_2}^{\lambda_2}$. Then it is necessary that $p_1 \le p_2$. The easiest way to see this is to use the class HG of functions in $\mathcal{H}ol(\mathbb{D})$ whose Taylor series with center at 0 has Hadamard gaps. According to [103, Theorem 1.2.1] for $0 < p < 1$ we have $HG \cap Q_p = HG \cap \mathcal{D}_p$, and for $0 < p < q < 1$ we have $HG \cap \mathcal{D}_p \subsetneq HG \cap \mathcal{D}_q$. If we assume that $p_2 < p_1$ then $\mathcal{D}_{p_2} \subseteq \mathcal{D}_{p_1}$ and using the assumption $\mathcal{D}_{p_1}^{\lambda_1} \subseteq \mathcal{D}_{p_2}^{\lambda_2}$ we will have further $Q_{p_1} \subseteq \mathcal{D}_{p_1}^{\lambda_1} \subseteq \mathcal{D}_{p_2}^{\lambda_2} \subseteq \mathcal{D}_{p_2} \subseteq \mathcal{D}_{p_1}$. This would imply that $HG \cap \mathcal{D}_{p_1} = HG \cap \mathcal{D}_{p_2}$ which contradicts part of the above mentioned theorem. In addition from Proposition 3 it follows easily that $p_1(1-\lambda_1) \le p_2(1-\lambda_2)$. □

Proof of Theorem 38. Consider a function $f \in \mathcal{D}_p^{\lambda}$. By Lemma A and (4.1.3)

$$\begin{aligned}\|f\circ\varphi_a - f(a)\|^2_{\mathcal{D}_p} &\gtrsim \|f\circ\varphi_a - f(a)\|^2_{\mathcal{D}^*_p}\\ &\gtrsim \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}\,(P_a(u)P_a(v))^{p/2}\,|du|\,|dv|\\ &\geq \int_I\int_I \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}\,(P_a(u)P_a(v))^{p/2}\,|du|\,|dv|\,,\end{aligned}$$

for any subarc $I\subseteq\mathbb{T}$. If $I\neq\mathbb{T}$, then we choose a point $a\in\mathbb{D}\setminus\{0\}$ such that $\frac{a}{|a|}$ is the center of I and $2\pi(1-|a|)$ the arclength. If $I=\mathbb{T}$ we take $a=0$. With such a point a as well as the inequality $\cos t\geq 1-2^{-1}t^2$ for $t\in(-\infty,\infty)$, we get that for $u\in I$

$$P_a(u)\geq\frac{1}{1-|a|}=\frac{2\pi}{|I|}\,,\quad u\in I.$$

Thus

$$\|f\circ\varphi_a - f(a)\|^2_{\mathcal{D}_p}\gtrsim\frac{1}{|I|^p}\int_I\int_I\frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}\,|du|\,|dv|,$$

which implies that

$$(1-|a|^2)^{p(1-\lambda)}\|f\circ\varphi_a - f(a)\|^2_{\mathcal{D}_p}\gtrsim\frac{1}{|I|^{p\lambda}}\int_I\int_I\frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}\,|du|\,|dv|,$$

so finally we get

$$\|f\|^2_{\mathcal{D}^\lambda_p}\gtrsim\|f\|^2_{\mathcal{D}^{\lambda,*}_p}\,.$$

For the other inequality we work as follows. To each point $a\in\mathbb{D}\setminus\{0\}$ we associate the subarc I_a with center $\frac{a}{|a|}$ and arclength $2\pi(1-|a|)$. For $a=0$ we set $I_a=\mathbb{T}$. Also we set

$$I^n=2^nI_a,\quad n=0,1,\ldots,N-1,$$

where N is the smallest integer such that $2^N|I_a|\geq 2\pi$. Then, we put $I^N=\mathbb{T}$.

Using the elementary inequality $\cos t\leq 1-2\pi^{-2}t^2$ for $t\in[-\pi,\pi]$, we know that for every point $u\in\mathbb{T}$,

$$P_a(u)\lesssim\frac{1}{1-|a|}. \tag{4.1.6}$$

Furthermore, for $u\in\mathbb{T}\setminus I^n$ we have

$$P_a(u)\lesssim\frac{1}{2^{2n}|a||I_a|}.$$

In the sequel, we may assume $|a| \geq 1/2$, otherwise, the result is obviously true. Therefore, if $u \in I^{n+1} \setminus I^n$, then

$$P_a(u) \lesssim \frac{1}{2^{2n}|I_a|}. \tag{4.1.7}$$

With the above notations, we break $(1-|a|^2)^{p(1-\lambda)}\|f \circ \varphi_a - f(a)\|^2_{\mathcal{D}^*_p}$ into two parts,

$$(1-|a|^2)^{p(1-\lambda)}\|f \circ \varphi_a - f(a)\|^2_{\mathcal{D}^*_p} = X_1 + X_2,$$

where

$$X_1 = (2\pi)^p(1-|a|^2)^{p(1-\lambda)} \int_{\mathbb{T}}\int_{I_a} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} \, (P_a(u)P_a(v))^{p/2} \, |du|\,|dv|$$

and

$$X_2 = (2\pi)^p(1-|a|^2)^{p(1-\lambda)} \sum_{n=0}^{N-1} \int_{\mathbb{T}}\int_{I^{n+1}\setminus I^n} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} \, (P_a(u)P_a(v))^{p/2} \, |du|\,|dv|.$$

Consider first X_1. By (4.1.6) and (4.1.7) we have that

$$\begin{aligned}
\frac{X_1}{(2\pi)^p} &= (1-|a|^2)^{p(1-\lambda)} \int_{I_a}\int_{I_a} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} \, (P_a(u)P_a(v))^{p/2} \, |du|\,|dv| \\
&\quad + (1-|a|^2)^{p(1-\lambda)} \sum_{n=0}^{N-1} \int_{I^{n+1}\setminus I^n}\int_{I_a} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} \, (P_a(u)P_a(v))^{p/2} \, |du|\,|dv| \\
&\lesssim \frac{1}{|I_a|^{p\lambda}} \int_{I_a}\int_{I_a} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} \, |du|\,|dv| \\
&\quad + |I_a|^{p(1-\lambda)} \sum_{n=0}^{N-1} \frac{1}{(2^n|I_a|)^p} \int_{I^{n+1}\setminus I^n}\int_{I_a} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} |du|\,|dv| \\
&= \frac{1}{|I_a|^{p\lambda}} \int_{I_a}\int_{I_a} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} \, |du|\,|dv| \\
&\quad + \frac{1}{|I_a|^{p\lambda}} \int_{2I_a\setminus I_a}\int_{I_a} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} |du|\,|dv| \\
&\quad + |I_a|^{p(1-\lambda)} \sum_{n=1}^{N-1} \frac{1}{(2^n|I_a|)^p} \int_{I^{n+1}\setminus I^n}\int_{I_a} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} |du|\,|dv|.
\end{aligned}$$

Therefore

$$X_1 \lesssim \|f\|^2_{\mathcal{D}^{\lambda,*}_p} + \frac{1}{|I_a|^{2-p(1-\lambda)}} \sum_{n=1}^{N-1} \frac{1}{2^{2n}} \int_{I^{n+1}\setminus I^n}\int_{I_a} |f(u)-f(v)|^2 |du|\,|dv|.$$

Using the following identity

$$\frac{1}{|I|}\int_I |f(z)-b|^2|dz| = \frac{1}{|I|}\int_I |f(z)-f_I|^2|dz| + |f_I - b|^2, \quad b\in\mathbb{C}, \tag{4.1.8}$$

and the characterization of Morrey spaces given in (3.0.1) we have the following

$$\begin{aligned}
&\frac{1}{|I_a|^{2-p(1-\lambda)}}\sum_{n=1}^{N-1}\frac{1}{2^{2n}}\int_{I^{n+1}\setminus I^n}\int_{I_a}|f(u)-f(v)|^2|du|\,|dv| \\
=&\frac{1}{|I_a|^{1-p(1-\lambda)}}\sum_{n=1}^{N-1}\frac{1}{2^{2n}}\int_{I^{n+1}\setminus I^n}\frac{1}{|I_a|}\int_{I_a}|f(u)-f(v)|^2|du|\,|dv| \\
=&\frac{1}{|I_a|^{1-p(1-\lambda)}}\sum_{n=1}^{N-1}\frac{1}{2^{2n}}\int_{I^{n+1}\setminus I^n}\left(\frac{1}{|I_a|}\int_{I_a}|f(u)-f_{I_a}|^2|du| + |f(v)-f_{I_a}|^2\right)|dv|.
\end{aligned}$$

The first term can be bounded as follows using Lemma 9

$$\frac{1}{|I_a|^{2-p(1-\lambda)}}\int_{I_a}|f(u)-f_{I_a}|^2|du|\sum_{n=1}^{N-1}\frac{1}{2^{2n}}\int_{I^{n+1}\setminus I^n}|dv| \lesssim \|f\|^2_{\mathcal{L}^{2,1-p(1-\lambda)}(\mathbb{T})}\sum_{n=1}^{N-1}\frac{1}{2^n}\lesssim \|f\|^2_{\mathcal{D}_p^{\lambda,*}}.$$

Finally, considering (4.1.8), Lemma 9, Lemma 10 and the triangle inequality we get that the second term can be bounded in the following way

$$\begin{aligned}
&|I_a|^{p(1-\lambda)}\sum_{n=1}^{N-1}\frac{1}{2^{2n}}\frac{1}{|I_a|}\int_{I^{n+1}\setminus I^n}|f(v)-f_{I_a}|^2|dv| \\
\lesssim&|I_a|^{p(1-\lambda)}\sum_{n=1}^{N-1}\frac{1}{2^n}\frac{1}{|I^{n+1}|}\int_{I^{n+1}}|f(v)-f_{I_a}|^2|dv| \\
=&|I_a|^{p(1-\lambda)}\sum_{n=1}^{N-1}\frac{1}{2^n}\left(\frac{1}{|I^{n+1}|}\int_{I^{n+1}}|f(v)-f_{I^{n+1}}|^2|dv| + |f_{I^{n+1}}-f_{I_a}|^2\right) \\
\lesssim&\sum_{n=1}^{N-1}\frac{1}{2^{(1+p(1-\lambda))n}}\frac{1}{|I^{n+1}|^{1-p(1-\lambda)}}\int_{I^{n+1}}|f(v)-f_{I^{n+1}}|^2|dv| \\
&+|I_a|^{p(1-\lambda)}\sum_{n=1}^{N-1}\frac{n+1}{2^n}\sum_{k=1}^{n+1}|f_{I^k}-f_{I^{k-1}}|^2 \\
\lesssim&\|f\|^2_{\mathcal{L}^{2,1-p(1-\lambda)}(\mathbb{T})}\left(\sum_{n=1}^{N-1}\frac{1}{2^{(1+p(1-\lambda))n}}+\sum_{n=1}^{N-1}\frac{n+1}{2^n}\sum_{k=1}^{n+1}\frac{1}{2^{p(1-\lambda)k}}\right) \\
\lesssim&\|f\|^2_{\mathcal{D}_p^{\lambda,*}}.
\end{aligned}$$

Consider now X_2. Using again the same techniques we obtain

$$\begin{aligned}
\frac{X_2}{(2\pi)^p} &= (1-|a|^2)^{p(1-\lambda)} \sum_{n=0}^{N-1} \int_{I_a} \int_{I^{n+1}\setminus I^n} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} \left(P_a(u)P_a(v)\right)^{p/2} |du|\,|dv| \\
&\quad + (1-|a|^2)^{p(1-\lambda)} \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} \int_{I^{m+1}\setminus I^m} \int_{I^{n+1}\setminus I^n} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} \left(P_a(u)P_a(v)\right)^{p/2} |du|\,|dv| \\
&\lesssim \|f\|^2_{\mathcal{D}_p^{\lambda,*}} \\
&\quad + (1-|a|^2)^{p(1-\lambda)} \sum_{m=0}^{N-1} \int_{I^{m+1}\setminus I^m} \int_{I^1\setminus I_a} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} \left(P_a(u)P_a(v)\right)^{p/2} |du|\,|dv| \\
&\quad + (1-|a|^2)^{p(1-\lambda)} \sum_{m=0}^{N-1}\sum_{n=1}^{N-1} \int_{I^{m+1}\setminus I^m} \int_{I^{n+1}\setminus I^n} \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}} \left(P_a(u)P_a(v)\right)^{p/2} |du|\,|dv|,
\end{aligned}$$

where the second term is bounded by $\|f\|^2_{\mathcal{D}_p^{\lambda,*}}$ and the third one is estimated as

$$\|f\|^2_{\mathcal{D}_p^{\lambda,*}} + \sum_{n=0}^{\infty} \frac{n}{2^{p(1-\lambda)n}} \|f\|^2_{\mathcal{D}_p^{\lambda,*}} \lesssim \|f\|^2_{\mathcal{D}_p^{\lambda,*}}.$$

Combining the estimates of X_1 and X_2 we get the desired result

$$\|f\|_{\mathcal{D}_p^{\lambda}} \lesssim \|f\|_{\mathcal{D}_p^{\lambda,*}}.$$

□

4.2 Pointwise multipliers

Let X be a Banach space of analytic functions on $\mathbb{D}$. A function $g \in \mathcal{H}ol(\mathbb{D})$ is said to be a multiplier of X if the multiplication operator

$$M_g(f)(z) = g(z)f(z), \quad f \in X,$$

is a bounded operator on X. For this it is usually enough to check that $M_g(X) \subset X$ and apply the closed graph theorem. The space of all multipliers of X is denoted by $M(X)$. Multiplication operators are closely related to integration operators J_g and I_g. These are induced by symbols $g \in \mathcal{H}ol(\mathbb{D})$ as follows

$$J_g(f)(z) = \int_0^z f(w)g'(w)\,dw, \quad z \in \mathbb{D},$$

and

$$I_g(f)(z) = \int_0^z f'(w)g(w)\,dw, \quad z \in \mathbb{D},$$

and act on functions $f \in \mathcal{H}ol(\mathbb{D})$. The operators I_g, J_g have been studied in a number of papers, see for example [1, 3, 44, 52, 66]. Their relation with M_g comes from the integration by parts formula

$$J_g(f)(z) = M_g(f)(z) - f(0)g(0) - I_g(f)(z). \tag{4.2.1}$$

This essentially says that if g is a symbol for which two of the operators I_g, J_g, M_g are bounded on a space X so is the third. It also says that it is possible for two of the operators to be unbounded but the third is bounded due to cancelation.

The space of multipliers is known for several of the classical spaces such as Hardy and Bergman spaces. In particular for $H^2 = \mathcal{D}_1$ the space of multipliers is $M(H^2) = H^\infty$, the algebra of bounded analytic functions. For other Dirichlet spaces $\mathcal{D}_p$, $p \in (0,1)$, the situation is more complicated. The description of $M(\mathcal{D}_p)$ is in terms of $\mathcal{D}_p$-Carleson measures. Recall that a positive Borel measure μ on the disc is a $\mathcal{D}_p$-Carleson measure if there is a constant $C = C(\mu)$ such that

$$\int_{\mathbb{D}} |f(z)|^2\,d\mu(z) \le C\|f\|^2_{\mathcal{D}_p}, \qquad f \in \mathcal{D}_p.$$

These measures were described initially by Stegenga [96] with the help of Bessel capacities, and similar characterizations were given by other authors. In another approach, Arcozzi, Rochberg and Sawyer [9] described these measures by a different condition, a simplified form of which is given in [48]. Accordingly, a finite measure μ is a $\mathcal{D}_p$-Carleson measure if and only if

$$\sup_{w\in\mathbb{D}} \frac{1}{\mu(S(w))} \int_{S(w)} \frac{(\mu(S(z)\cap S(w)))^2}{(1-|z|^2)^{2+p}}\,dA(z) < \infty,$$

where for $w \in \mathbb{D}$ the set $S(w)$ on which integration takes place is the Carleson box $S(w) = \{z \in \mathbb{D} : 1-|z| \le 1-|w|,\ |arg(\bar{z}w)| \le \pi(1-|w|)\}$.

It is convenient at this point to use the space $\mathcal{W}_p$ of functions $g \in \mathcal{H}ol(\mathbb{D})$ such that the measure

$$d\mu_g(z) = |g'(z)|^2(1-|z|^2)^p\,dA(z)$$

is a $\mathcal{D}_p$-Carleson measure. This space has been studied in [90] and [101]. The multipliers of $\mathcal{D}_p$ were described in [96] as follows.

Theorem V. *Suppose* $0 < p < 1$ *and* $g \in \mathcal{H}ol(\mathbb{D})$. *Then* $g \in M(\mathcal{D}_p)$ *if and only if* $g \in H^\infty$ *and* $d\mu_g(z) = |g'(z)|^2(1-|z|^2)^p\, dA(z)$ *is a* $\mathcal{D}_p$*-Carleson measure. In other words,*

$$M(\mathcal{D}_p) = H^\infty \cap \mathcal{W}_p.$$

On the other hand the multipliers of Q_p are completely described in [75, 105] as follows.

Theorem W. *Suppose* $0 < p < 1$ *and* $g \in \mathcal{H}ol(\mathbb{D})$. *Then* $g \in M(Q_p)$ *if and only if* $g \in H^\infty$ *and*

$$\sup_{I \subseteq \mathbb{T}} \frac{\left(\log \frac{1}{|I|}\right)^2}{|I|^p} \int_{S(I)} |g'(z)|^2(1-|z|^2)^p\, dA(z) < \infty. \tag{4.2.2}$$

Thus if we denote by $Q_{p,\log}$ the space of functions that satisfy (4.2.2) then the above theorem says

$$M(Q_p) = H^\infty \cap Q_{p,\log}.$$

It is not difficult to check that $Q_{p,\log} \subset \mathcal{W}_p$. On the other hand it was shown in [9] that $\mathcal{W}_p \subset Q_p$ and there is a simplified proof of this in [69, Lemma 4]. Thus we have

$$Q_{p,\log} \subset \mathcal{W}_p \subset Q_p, \quad 0 < p < 1.$$

In what follows we study the action of the operators I_g, J_g on the spaces $\mathcal{D}_p^\lambda$, and obtain information on pointwise multipliers. Our first result is the following.

Theorem 39. *Let* $0 < p, \lambda < 1$ *and* $g \in \mathcal{H}ol(\mathbb{D})$. *Then* $I_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ *is bounded if and only if* $g \in H^\infty$.

Concerning the action of J_g on $\mathcal{D}_p^\lambda$ we have the following necessary condition.

Theorem 40. *Let* $0 < p, \lambda < 1$ *and* $g \in \mathcal{H}ol(\mathbb{D})$. *If* $J_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ *is bounded then* $g \in Q_p$.

We now find sufficient conditions on g for J_g to be bounded on $\mathcal{D}_p^\lambda$.

Theorem 41. *Suppose* $0 < p < 1$.

(i) If $0 < q < p$ *and* $g \in Q_q$ *then* $J_g : \mathcal{D}_p^{q/p} \to \mathcal{D}_p^{q/p}$ *is bounded.*

(ii) If $0 < \lambda < 1$ and $g \in \mathcal{W}_p$ then $J_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ is bounded.

The above theorems in combination with (4.2.1) give the following corollary for multipliers of $\mathcal{D}_p^\lambda$.

Corollary 4. *Suppose $0 < p, \lambda < 1$ and $g \in \mathcal{H}ol(\mathbb{D})$. Then*

(i) If $g \in \mathcal{W}_p \cap H^\infty$ then $M_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ is bounded.

(ii) If $g \in Q_{p\lambda} \cap H^\infty$ then $M_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ is bounded.

(iii) If $M_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ is bounded then $g \in Q_p \cap H^\infty$.

We conclude this section with the following remark.

Let $0 < p < 1$. We know that $\mathcal{W}_p \subset Q_p$, and this inclusion is strict [104, Theorem 6.3.4]. At the same time for $0 < q < p$ we have $Q_q \subset Q_p$ with strict inclusion. For each $q < p$ we give an example of a function f such that $f \in \mathcal{W}_p$ but f does not belong to Q_q. Thus $\mathcal{W}_p \nsubseteq Q_q$ for any $q < p$.

Indeed, with q, p as above consider the function

$$f(z) = \sum_{k=1}^{\infty} a_k z^{2^k}, \quad z \in \mathbb{D}$$

where $a_k = 1/2^{k(1-q)/2}$. By a theorem of Yamashita [108, Theorem 1(i)] for such Hadamard gap series, and since

$$\limsup_{k\to\infty} |a_k| 2^{k\left(1-\frac{1+q}{2}\right)} = 1 < \infty,$$

it follows that f satisfies the growth condition

$$\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|)^{\frac{1+q}{2}} < \infty.$$

Applying Proposition 4.2 of [12] (after adjusting the parameters involved to our notation) we find that this function is a multiplier of $\mathcal{D}_p$ because $q < p$. Thus the bounded function f belongs to $\mathcal{W}_p$.

On the other hand

$$\sum_{k=0}^{\infty} 2^{k(1-q)} \left(\sum_{2^k \le n_j < 2^{k+1}} |a_j|^2 \right) = \sum_{k=0}^{\infty} 1 = \infty,$$

and therefore by [103, Theorem 1.2.1] for such Hadamard gap series, $f \notin Q_q$.

The complete description of the multiplier space $M(\mathcal{D}_p^\lambda)$ and of the symbols g for which J_g is bounded on $\mathcal{D}_p^\lambda$ addreses the open question in [76] and it seems to be a hard problem.

4.2.1 Proofs

We will need the following technical lemma from [76, p. 488]. We state only the part of it that we need.

Lemma B. *Let $u \in \mathbb{D}$, $|v| \leq 1$ and $s > -1$, r, $t > 0$. Then*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\bar{u}z|^r|1-\bar{v}z|^t}\, dA(z) \leq \frac{C}{(1-|u|^2)^{r+t-s-2}}\,, \quad 0 < r+t-s-2 < r$$

where C is an absolute, positive constant.

Using this estimate we obtain a family of test functions in $\mathcal{D}_p^\lambda$.

Lemma 11. *Let $0 < p, \lambda < 1$ and $c \in \mathbb{D}$. Then the functions*

$$f_c(z) = \frac{1}{(1-\bar{c}z)^{p(1-\lambda)/2}}, \quad z \in \mathbb{D},$$

belong to $\mathcal{D}_p^\lambda$ and $K = \sup_{c\in\mathbb{D}} \|f_c\|_{\mathcal{D}_p^\lambda} < \infty$.

Proof. Fix $c \in \mathbb{D}$. Then for $a \in \mathbb{D}$,

$$\begin{aligned}
&(1-|a|^2)^{p(1-\lambda)} \int_{\mathbb{D}} |f_c'(z)|^2 (1-|\varphi_a(z)|^2)^p\, dA(z) \\
&= (1-|a|^2)^{p(2-\lambda)} \int_{\mathbb{D}} \frac{(1-|z|^2)^p}{|1-\bar{c}z|^{2+p(1-\lambda)}|1-\bar{a}z|^{2p}}\, dA(z)\,.
\end{aligned}$$

Now for $r = 2p$, $t = 2 + p(1-\lambda)$, $s = p$, Lemma B gives the desired result. □

Proof of Theorem 39. Let $g \in H^\infty$ then

$$\begin{aligned}
\|I_g(f)\|_{\mathcal{D}_p^\lambda}^2 &\asymp \sup_{I\subset\mathbb{T}} \frac{1}{|I|^{p\lambda}} \int_{S(I)} |I_g(f)'(z)|^2 (1-|z|^2)^p\, dA(z) \\
&= \sup_{I\subset\mathbb{T}} \frac{1}{|I|^{p\lambda}} \int_{S(I)} |f'(z)|^2 |g(z)|^2 (1-|z|^2)^p\, dA(z) \\
&\lesssim \|g\|_\infty^2 \|f\|_{\mathcal{D}_p^\lambda}^2
\end{aligned}$$

for every $f \in \mathcal{D}_p^\lambda$. So $\|I_g\| \leq C\|g\|_\infty$ where C is a constant.

On the other hand, assume that I_g is bounded on $\mathcal{D}_p^\lambda$. We will use the test functions $\{f_c\}$ of Lemma 11 for $\{|c| > \frac{1}{2}\}$. Then from the Lemma there is a constant

C such that $1 \leq \|f_c\|_{\mathcal{D}_p^\lambda} \leq C$ for all c, so that $\|I_g\|^2 \geq \frac{1}{C^2}\|I_g(f_c)\|^2_{\mathcal{D}_p^\lambda}$ and,

$$\begin{aligned}
\|I_g(f_c)\|^2_{\mathcal{D}_p^\lambda} &= \sup_{a\in\mathbb{D}} (1-|a|^2)^{p(1-\lambda)} \int_{\mathbb{D}} |I_g(f_c)'(z)|^2 (1-|\varphi_a(z)|^2)^p \, dA(z) \\
&\gtrsim (1-|c|^2)^{p(1-\lambda)} \int_{\mathbb{D}} |I_g(f_c)'(z)|^2 (1-|\varphi_c(z)|^2)^p \, dA(z) \\
&= (1-|c|^2)^{p(1-\lambda)} \int_{\mathbb{D}} |f_c'(z)|^2 |g(z)|^2 (1-|\varphi_c(z)|^2)^p \, dA(z) \\
&\asymp |c|(1-|c|^2)^{p(1-\lambda)} \int_{\mathbb{D}} \frac{|g(z)|^2 (1-|\varphi_c(z)|^2)^p}{|1-\bar{c}z|^{2+p(1-\lambda)}} \, dA(z)\,,
\end{aligned}$$

now by restricting the above integral on a disc with center the point c and radius $\frac{1-|c|}{2}$ and by applying the mean value property of subharmonic functions we get that

$$\|I_g\|^2 \gtrsim |g(c)|^2$$

for any $\{|c| > \frac{1}{2}\}$. It follows that g is a bounded analytic function on $\mathbb{D}$. □

Proof of Theorem 40. We use the test functions $f_c(z) = (1-\bar{c}z)^{-p(1-\lambda)/2}$ of Lemma 11. From the hypothesis there is a constant C such that

$$\|J_g(f_c)\|_{\mathcal{D}_p^\lambda} \leq C\|f_c\|_{\mathcal{D}_p^\lambda} \leq C \sup_{c\in\mathbb{D}} \|f_c\|_{\mathcal{D}_p^\lambda} = CK < \infty,$$

for all $c \in \mathbb{D}$. This means that

$$\sup_{I\subset\mathbb{T}} \frac{1}{|I|^{p\lambda}} \int_{S(I)} |f_c(z)|^2 |g'(z)|^2 (1-|z|^2)^p \, dA(z) \leq K' < \infty$$

for all $c \in \mathbb{D}$. For each interval I choose $c = c_I = (1-|I|)e^{i\theta}$ where $e^{i\theta}$ is the center of I, then $|1-\bar{c}z| \asymp |I|$ for $z \in S(I)$ and we have

$$\begin{aligned}
K' \geq& \frac{1}{|I|^{p\lambda}} \int_{S(I)} \frac{1}{|1-\bar{c}z|^{p(1-\lambda)}} |g'(z)|^2 (1-|z|^2)^p \, dA(z) \\
\asymp& \frac{1}{|I|^p} \int_{S(I)} |g'(z)|^2 (1-|z|^2)^p \, dA(z)
\end{aligned}$$

with K' independent of I. Taking the supremum of the last integral over all $I \subset \mathbb{T}$ we see that $g \in Q_p$. □

Proof of Theorem 41. (i). Set $\lambda = q/p < 1$ and suppose $I \subset \mathbb{T}$ is an interval. Using

the growth condition (4.1.1) for $f \in \mathcal{D}_p^\lambda$ we have

$$\begin{aligned}
&\frac{1}{|I|^{p\lambda}} \int_{S(I)} |J_g(f)'(z)|^2 (1-|z|^2)^p \, dA(z) \\
=&\frac{1}{|I|^q} \int_{S(I)} |f(z)|^2 |g'(z)|^2 (1-|z|^2)^p \, dA(z) \\
\lesssim&\frac{1}{|I|^q} \int_{S(I)} \frac{1}{(1-|z|^2)^{p(1-\lambda)}} |g'(z)|^2 (1-|z|^2)^p \, dA(z) \, \|f\|^2_{\mathcal{D}_p^\lambda} \\
=&\frac{1}{|I|^q} \int_{S(I)} |g'(z)|^2 (1-|z|^2)^q \, dA(z) \, \|f\|^2_{\mathcal{D}_p^\lambda} \\
\lesssim&\|g\|^2_{Q_q} \|f\|^2_{\mathcal{D}_p^\lambda},
\end{aligned}$$

and the assertion follows by taking supremum on the left.

(ii). Let $f \in \mathcal{D}_p^\lambda$. For an interval $I \subset \mathbb{T}$ let $w = w_I = (1-|I|)e^{i\theta}$ where $e^{i\theta}$ is the center of I. Then

$$\begin{aligned}
&\frac{1}{|I|^{p\lambda}} \int_{S(I)} |J_g(f)'(z)|^2 (1-|z|^2)^p \, dA(z) \\
=&\frac{1}{|I|^{p\lambda}} \int_{S(I)} |f(z)|^2 |g'(z)|^2 (1-|z|^2)^p \, dA(z) \\
\leq&\frac{2}{|I|^{p\lambda}} \int_{S(I)} |f(w)|^2 |g'(z)|^2 (1-|z|^2)^p \, dA(z) \\
&+ \frac{2}{|I|^{p\lambda}} \int_{S(I)} |f(z)-f(w)|^2 |g'(z)|^2 (1-|z|^2)^p \, dA(z) \\
=&A_I + B_I.
\end{aligned}$$

For the first integral, using (4.1.1) and recalling that $\mathcal{W}_p \subset Q_p$ we have

$$A_I \lesssim \|f\|^2_{\mathcal{D}_p^\lambda} \frac{1}{|I|^p} \int_{S(I)} |g'(z)|^2 (1-|z|^2)^p \, dA(z) \lesssim \|f\|^2_{\mathcal{D}_p^\lambda} \|g\|^2_{Q_p}.$$

For the second integral we write

$$
\begin{aligned}
B_I &= \frac{2}{|I|^{p\lambda}} \int_{S(I)} \left| \frac{f(z)-f(w)}{(1-\overline{w}z)^p} \right|^2 |1-\overline{w}z|^{2p} |g'(z)|^2 (1-|z|^2)^p \, dA(z) \\
&\lesssim |I|^{p(2-\lambda)} \int_{S(I)} \left| \frac{f(z)-f(w)}{(1-\overline{w}z)^p} \right|^2 |g'(z)|^2 (1-|z|^2)^p \, dA(z) \\
&= (1-|w|)^{p(2-\lambda)} \int_{S(I)} \left| \frac{f(z)-f(w)}{(1-\overline{w}z)^p} \right|^2 |g'(z)|^2 (1-|z|^2)^p \, dA(z) \\
&\lesssim (1-|w|)^{p(2-\lambda)} |f(0)-f(w)|^2 \\
&\quad + (1-|w|)^{p(2-\lambda)} \int_{\mathbb{D}} \left| \frac{d}{dz} \left(\frac{f(z)-f(w)}{(1-\overline{w}z)^p} \right) \right|^2 (1-|z|^2)^p \, dA(z) \\
&= (1-|w|)^{p(2-\lambda)} |f(0)-f(w)|^2 + C_w,
\end{aligned}
$$

where we have used the hypothesis that $d\mu_g(z) = |g'(z)|^2(1-|z|^2)^p \, dA(z)$ is a $\mathcal{D}_p$-Carleson measure. The first term in the last sum is

$$
(1-|w|)^{p(2-\lambda)} |f(0)-f(w)|^2 \leq (1-|w|)^{p(1-\lambda)} |f(0)-f(w)|^2 \lesssim \|f\|^2_{\mathcal{D}^\lambda_p}
$$

by using (4.1.1) once more. For the second term we have

$$
\begin{aligned}
C_w &= (1-|w|)^{p(2-\lambda)} \int_{\mathbb{D}} \left| \frac{d}{dz} \left(\frac{f(z)-f(w)}{(1-\overline{w}z)^p} \right) \right|^2 (1-|z|^2)^p \, dA(z) \\
&\leq 2(1-|w|)^{p(2-\lambda)} \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|z|^2)^p}{|1-\overline{w}z|^{2p}} \, dA(z) \\
&\quad + 2p^2 |w|^2 (1-|w|)^{p(2-\lambda)} \int_{\mathbb{D}} \left| \frac{f(z)-f(w)}{(1-\overline{w}z)^{1+p}} \right|^2 (1-|z|^2)^p \, dA(z) \\
&= 2(1-|w|)^{p(1-\lambda)} \int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_w(z)|^2)^p \, dA(z) \\
&\quad + 2p^2 |w|^2 (1-|w|)^{p(1-\lambda)} \int_{\mathbb{D}} \left| \frac{f(z)-f(w)}{1-\overline{w}z} \right|^2 (1-|\varphi_w(z)|^2)^p \, dA(z) \\
&\lesssim \|f\|^2_{\mathcal{D}^\lambda_p} + (1-|w|)^{p(1-\lambda)} \int_{\mathbb{D}} \left| \frac{f(z)-f(w)}{1-\overline{w}z} \right|^2 (1-|\varphi_w(z)|^2)^p \, dA(z) \\
&= \|f\|^2_{\mathcal{D}^\lambda_p} + D_w.
\end{aligned}
$$

Observe that

$$\begin{aligned}
D_w &= (1-|w|)^{p(1-\lambda)} \int_{\mathbb{D}} \left| \frac{f(z)-f(w)}{1-\overline{w}z} \right|^2 (1-|\varphi_w(z)|^2)^p \, dA(z) \\
&= (1-|w|)^{p(1-\lambda)} \int_{\mathbb{D}} \left| \frac{f\circ\varphi_w(z)-f\circ\varphi_w(0)}{1-\overline{w}\varphi_w(z)} \right|^2 |\varphi_w'(z)|^2 (1-|z|^2)^p \, dA(z) \\
&= (1-|w|)^{p(1-\lambda)} \int_{\mathbb{D}} \left| \frac{f\circ\varphi_w(z)-f\circ\varphi_w(0)}{1-\overline{w}z} \right|^2 (1-|z|^2)^p \, dA(z).
\end{aligned}$$

To find an upper estimate for D_w, we follow the argument of [75, pages 551-552] (see also [105, page 2080]). The argument consists in applying a reproducing formula from [90], the Cauchy-Schwarz inequality, Fubini's theorem and the estimate [111, Lemma 3.10(b)]. We refrain from writing all the details since the argument applies *mutatis mutandis*. The final steps of the calculation are as follows

$$\begin{aligned}
D_w &\lesssim (1-|w|)^{p(1-\lambda)} \int_{\mathbb{D}} |(f\circ\varphi_w)'(z)|^2 \frac{(1-|z|^2)^{2+p}}{|1-\overline{w}z|^2} \, dA(z) \\
&\lesssim (1-|w|)^{p(1-\lambda)} \int_{\mathbb{D}} |(f\circ\varphi_w)'(z)|^2 (1-|z|^2)^p \, dA(z) \\
&\lesssim (1-|w|)^{p(1-\lambda)} \int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_w(z)|^2)^p \, dA(z) \\
&\lesssim \|f\|^2_{\mathcal{D}^\lambda_p}.
\end{aligned}$$

Collecting all the above estimates gives $\|J_g(f)\|_{\mathcal{D}^\lambda_p} \leq C\|f\|_{\mathcal{D}^\lambda_p}$ which is the desired conclusion. □

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