ABSTRACT TOPOLOGICAL COMPLEXITY AND APPLICATIONS

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Abstract. We study the behaviour of the abstract topological complexity in different proper model structures on topological spaces. These results are extended to the abstract LS-category working with J-categories. As an application, we provide a combinatorial way for computing these invariants.

Introduction

The notion of Lusternik-Schnirelmann category, denoted by $\mathbf{cat}(X)$, which is one less the least number of open sets contractible in X covering the whole space, was introduced as a lower bound of the number of critical points of any smooth function. While the aim was analytic in nature it had many different consequences in geometry and algebra as well.

The sectional category of a fibration $p : E \to B$, denoted by $\mathbf{secat}(p)$, which is the minimum number n such that B can be covered by $n+1$ open subsets on each of which p admits a local section, is an extension of the previous invariant. Indeed, it is a lower bound of the LS-category of the base space and, when the total space E is contractible, equals it. Despite the original applications of the sectional category were about classification of bundles and the embedding problem (see [11]), it has been extended in several directions, both towards abstract (rational homotopy theory, cohomological and homotopical theorems) and applied fields (robotics, motions constrained in mechanical systems, complexity of algorithms of space motion to name a few).

One of the most interesting applications is the topological complexity of a given path-connected space X, denoted by $\mathbf{TC}(X)$. Introduced by M. Farber in [7], it can be defined to be one more the sectional category of the diagonal morphism $\Delta : X \to X \times X$ of a given topological space X. When X is thought to be the configuration space associated to the motion of a given mechanical system, this invariant measures, roughly speaking, the minimum amount of instructions needed for controlling the given system.

First, J.P. Doeraene in [6] and then F. Díaz Díaz, J. M García Calcines, P. R. García Díaz, A. Murillo Mas and J. Remedios Gómez in [4] extended the LS-category and the sectional category respectively to a categorical setting, like a Quillen model category. Indeed, J.P. Doeraene gives two generalization of the classical LS-category called of Ganea and of Whitehead, which agree in a J-category (see [6]) through the crucial cube

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axiom, and in [4] the notion of abstract sectional category and abstract topological complexity is introduced.
In principle, there is no reason why these numerical invariants have to coincide through functors relating different proper model categories or J-categories, even if these satisfy sufficiently good properties. The aim of this paper is to show that this is the case in the main and most useful model structures on topological spaces under reasonable hypotheses. Indeed, through an accurate study of the mixed structure introduced by M. Cole in [2] and of the join computed in this structure, we succeed in proving the following (see Theorem 2.2 and Theorem 3.1 for precise statement and next section for basic definitions and notations)

Theorem 0.1. *Let X be a topological space with the homotopy type of a locally compact CW-complex. Then the following equality holds:*

$$TC^Q(X) = TC^S(X),$$

where $TC^Q(X)$ is the abstract topological complexity computed in the Quillen model structure and $TC^S(X)$ is the one computed in the Strøm one.

In the remainder of the paper, we will prove that the mixed model structure on well-pointed topological spaces with the homotopy type of a CW-complex is a J-category and so it satisfies the cube axiom. This result leads us to prove an analogous Theorem of the previous one about the abstract LS-category.

Theorem 0.2. *For a well-pointed space with the homotopy type of a CW-complex the following equality holds:*

$$cat^Q(X) = cat^S(X),$$

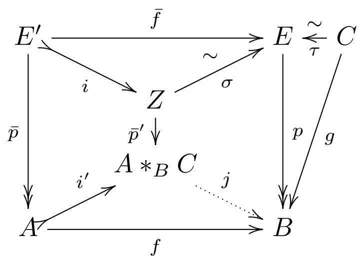
where $cat^Q(X)$ is the abstract LS-category computed in the Quillen structure and $cat^S(X)$ in the Strøm one.

Finally, as an application, we will give a first combinatorial way for computing these homotopy invariants, idea that will be extended in a forthcoming work.

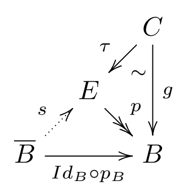
1. Preliminaries

In this section we state the axioms of a category **C** that we need for our purposes. In particular, we present the basic constructions and some results about the abstract sectional category and the abstract topological complexity obtained in [4].
From now on a category **C** will denote a proper model category (see [1]). Recall that such a category, if pointed, is a J-category (see [4] and [6]) except that it may not satisfy the cube axiom. Recall that in **Top**, the category of topological spaces and continuous maps, there are two standard structures that make it a proper model category. The first one, denoted by $\mathbf{Top}^Q$, in which the fibrations are the Serre fibrations, the weak equivalences are the weak homotopy equivalences and the cofibrations are those morphisms that have the left lifting property with respect to all trivial fibrations, was defined by D.G. Quillen

in [10]. The other, introduced by A. Strøm in [13] and denoted by $\mathbf{Top}^S$, has as fibrations the Hurewicz fibrations, as cofibrations the closed topological cofibrations and as weak equivalences the homotopy equivalences.
On the other hand **sSet**, the category of simplicial sets and simplicial morphisms, is also a proper model category with the structure given by D.G. Quillen in [10].
In a proper model category the *join* $A *_B C$ of two morphisms $f : A \to B$ and $g : C \to B$ is the object obtained by the following pushout:



where $g = p \circ \tau$ is any F-factorization, E' is the pullback of f and p and $\bar{f} = \sigma \circ i$ is any C-factorization. In particular the dotted map induced by the pushout construction from $A *_B C$ to B is called the *join morphism* of f and g.
Moreover, we will say that a morphim g admits a *weak section*, if the following diagram can be completed in order to commute:



where $g = p \circ \tau$ is any F-factorization and $\overline{B}$ is any cofibrant replacement of B.
Finally, we recall that a covariant functor $\mu : \mathbf{C} \to \mathbf{D}$ is said to be a *modelization functor* if it preserves weak equivalences, homotopy pullbacks and homotopy pushouts. In particular, when $\mathbf{C}$ and $\mathbf{D}$ are pointed, we will say that μ is *pointed* if $\mu(0) = 0$.
A useful technique for proving that a pair of adjoint functors are modelization functors is the following Propostion.

Proposition 1.1. [6, Proposition 6.5] *Assume that we have a pair of adjoint covariant functors $\alpha : \boldsymbol{C} \leftrightarrows \boldsymbol{D} : \beta$ such that*
(1) α and β preserve weak equivalences,
(2) α preserves cofibrations and β preserves fibrations,
(3) the adjunction maps $X \to \beta(\alpha(X))$ and $\alpha(\beta(Y)) \to Y$ are weak equivalences for any object X of $\boldsymbol{C}$ and Y of $\boldsymbol{D}$.
Then α and β are modelization functors. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are pointed, then α and β are also pointed.

Definition 1.2. [4, Definition 8] Let $p : E \to B$ be any morphism in $\mathbf{C}$. For each n we consider the morphism $h_n : *^n_B E \to B$ defined inductively as follows:
(i) $h_0 = p : E \to B$ (and so $*^0_B E = E$)
(ii) Suppose that $h_{n-1} : *^{n-1}_B \to B$ has already been constructed. Then h_n is the join morphism of p and h_{n-1}.
The *abstract sectional category of* p, denoted by *secat(p)*, is the least integer $n \leq \infty$ such that h_n admits a weak section.

Remarks 1.3. This notion is the Ganea sectional category as defined in [4]. Moreover, the abstract sectional category is proved to be invariant up to weakly equivalent morphisms in [4, Proposition 10].

Definition 1.4. Let B be a fibrant object in $\mathbf{C}$. Then, the *abstract topological complexity* of B is defined as the abstract sectional category of the diagonal morphism

$$TC(X) = \mathit{secat}(\Delta : X \to X \times X).$$

Remark 1.5. If we consider the category **Top** with the Strøm model structure we can obtain the classical definition of topological complexity (i.e. $\mathbf{TC}(X) = \mathbf{secat}(PX \to X \times X) + 1$) when we consider a space X such that $X \times X$ is paracompact, since the diagonal map Δ is weakly equivalent to the path fibration $p : PX \to X \times X$ and [[9], Proposition 8.1] holds.

The following Theorem shows under which hypotheses the *abstract topological complexity* is invariant if we move from a proper model category to another.

Theorem 1.6. [4, Corollary 28] *Consider $\mu : \boldsymbol{C} \to \boldsymbol{D}$ and $\nu : \boldsymbol{D} \to \boldsymbol{C}$ modelization functors that preserve the terminal object and let B be an object in $\boldsymbol{C}$ such that $\nu(\mu(B))$ is weakly equivalent to B. Then*

$$TC^{\boldsymbol{D}}(\mu(B)) = TC^{\boldsymbol{C}}(B).$$

Example 1.7. Observe that in general we can't obtain the classical value of the topological complexity if we compute the abstract topological complexity in the Quillen model structure on topological spaces. Indeed, if we take a path-connected compact topological space that has all homotopy groups trivial, but that is not homotopy equivalent to a point (e.g. the Warsaw circle), then Δ turns out to be weak equivalent to the morphism $\Delta' : * \to * \times *$ and, therefore, $TC^Q(X) = 0$. On the other hand, we know that $\mathbf{TC}(X) = 1$ if and only if X is a contractible topological space and so $\mathbf{TC}(X) - 1 = TC^S(X) \neq TC^Q(X)$.

2. The join in the Strøm and mixed model structures on topological spaces

We recall that if a category has two proper model structures $(\mathcal{W}_1, \mathcal{F}_1, \mathcal{C}_1)$ and $(\mathcal{W}_2, \mathcal{F}_2, \mathcal{C}_2)$ that satisfy the inclusions $\mathcal{W}_1 \subset \mathcal{W}_2$ and $\mathcal{F}_1 \subset \mathcal{F}_2$, then there exists a mixed one

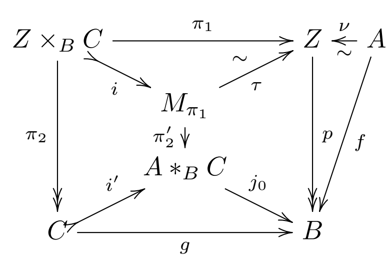
$(\mathcal{W}_m, \mathcal{F}_m, \mathcal{C}_m)$ for which $\mathcal{W}_m = \mathcal{W}_2$ and $\mathcal{F}_m = \mathcal{F}_1$, as proved in [2, Theorem 2.1], [2, Proposition 4.1] and [2, Proposition 4.2]. In particular, if we focus our attention on the category **Top**, we obtain a mixed model structure where the fibrations are Hurewicz fibrations, the weak equivalences are weak homotopy equivalences and the cofibrations are those morphisms that have the left lifting property with respect to all trivial fibrations. In particular, it turns out that the cofibrant objects coincide with the topological spaces with the homotopy type of a CW-complex (see [2, Example 3.8]). We will denote it by $\mathbf{Top}^M$. In particular, the following Proposition holds.

Proposition 2.1. [2, Corollary 3.12] *A closed topological cofibration between cofibrant objects of the mixed structure is a mixed cofibration.*

Hence, we can prove the following crucial Theorem.

Theorem 2.2. *Let us consider the category **Top** and let $f : A \to B$ and $g : C \to B$ be two morphisms between topological spaces with the homotopy type of a CW-complex (i.e. cofibrant objects in the mixed structure). Then the join object $A *_B C$ obtained in the Strøm structure equals up to weak equivalence the one obtained in the mixed one.*

Proof. We know that the first step for constructing the join in the Strøm structure is taking an F-factorization $f = p \circ \nu$ of f, where ν is a homotopy equivalence and p is a Hurewicz fibration. It is clear that this is also an F-factorization in the mixed structure, since the fibrations are the same in the two structures and every homotopy equivalence is also a weak homotopy equivalence. In particular, since ν is a homotopy equivalence, the factorizing object Z turns out to be a cofibrant object in the mixed structure.
The second step is taking the pullback of p and g. Applying [12, Theorem 7.5.9], we obtain that also $Z \times_B C$ is a topological space with the homotopy type of a CW-complex. Then, we have to take a C-factorization of the projection $\pi_1 : Z \times_B C \to Z$ in the Strøm structure. The crucial point of the proof is that this is also a C-factorization in the mixed structure. Indeed, if we take, for example, the C-factorization of π_1 through the mapping cylinder M_{π_1}, $\pi_1 = \tau \circ i$, then it turns out the also M_{π_1} is a cofibrant object in the mixed structure, since τ is a homotopy equivalence. Hence, by Proposition 2.1, i turns out to be also a mixed cofibration and so it follows that this is also a C-factorization in the mixed structure. It follows that the following construction



holds in both the model structures and so that the join object is the same up to weak equivalence. ∎

Remark 2.3. Observe that the join object $A *_B C$ obtained in the previous proof is a cofibrant object in the mixed structure, since [8, Proposition 8.1] holds.

In particular, we obtain the following useful Proposition.

Proposition 2.4. *Let A, B and C be topological spaces with the homotopy type of a CW-complex. Then the join morphism $j : A *_B C \to B$ admits a weak section in the Strøm structure (i.e. a homotopy section) if and only if j admits a weak section in the mixed one (i.e. a weak homotopy section).*

Proof. Let us consider the join morphism $j : A *_B C \to B$ constructed in the proof of Theorem 2.2. It is clear that if j admits a weak section in the Strøm model structure, then j admits a weak section also in the mixed one, since every F-factorization in the Strøm model structure is also an F-factorization in the mixed one.
Conversely, suppose that j admits a weak section in the mixed structure. Since by Remark 2.3 $A *_B C$ is cofibrant in the mixed structure and it is always possible taking an F-factorization through a cofibrant object, it follows that this F-factorization is also a F-factorization in the Strøm model structure, by the Whitehead Theorem. Hence, the thesis holds. ☐

3. The main results

Finally, in this section we can prove that the abstract topological complexity of a topological space with the homotopy type of a locally compact CW-complex gives the same value in both the Quillen and Strøm structures.

Theorem 3.1. *Let X be a topological space with the homotopy type of a locally compact CW-complex. Then the following equality holds:*

$$TC^Q(X) = TC^S(X).$$

Moreover, if X is also path-connected and such that $X \times X$ is paracompact, then the following extended chain of equalities holds:

$$TC^Q(X) = TC^S(X) = \boldsymbol{TC}(X) - 1.$$

Proof. Let us consider the first statement. The first step is to prove that, under these hypotheses, the abstract topological complexity computed in the Strøm structure and in the mixed one gives the same value.
In order to prove this, since $TC(X) = secat(\Delta : X \to X \times X)$, we have just to point out that both X and $X \times X$ have the homotopy type of a CW-complex. Indeed, Theorem 2.2 and Remark 2.3 hold and so each join object in the construction of the abstract sectional category of Δ can be chosen to be cofibrant up to weak equivalence. It follows that Proposition 2.4 can be reformulated in each degree n. This shows that the n-th join $j_n : *^n_{X\times X} X$ admits a weak section in the Strøm structure if and only if it admits

a weak section in the mixed one. If follows that $TC^M(X) = TC^S(X)$, where $TC^M(X)$ is the abstract topological complexity computed in the mixed structure.
The second step is to observe that the following adjoint of functors $Id : \mathbf{Top}^M \rightleftarrows \mathbf{Top}^S : Id$ satisfies the hypotheses of Proposition 1.1. In particular, it satisfies also the hypotheses of Theorem 1.6. It follows that the following chain of equalities holds:

$$TC^Q(X) = TC^M(X) = TC^S(X).$$

The second statement trivially follows from the previous one, since Remark 1.5 holds □

Finally, we have the following Corollary that suggests a way of computing the topological complexity with combinatorial techniques.

Corollary 3.2. *Let X be a path-connected topological space with the homotopy type of a locally compact CW-complex and such that $X \times X$ is paracompact. Then the following equality holds:*

$$\boldsymbol{TC}(X) - 1 = TC^{\boldsymbol{sSet}}(\mathcal{S}(X)),$$

where $\mathcal{S} : \mathbf{Top}^Q \to \boldsymbol{sSet}$ is the singular functor.

Proof. It suffices to observe that the singular functor and the geometric realization functor satisfy the hypotheses of Theorem 1.6. Then by Theorem 3.1, the thesis follows. □

We can now focus our attention on the pointed case (i.e. on the category $\mathbf{Top}_*$) and observe that also in this situation we have three proper model structures. Indeed, we have the Quillen structure, the Strøm structure and the mixed one defined respectively having as three classes of morphisms the same in their respective on the model structures on $\mathbf{Top}$ when we forget the basepoint. In particular, we will denote them as $\mathbf{Top}_*^Q$, $\mathbf{Top}_*^S$ and $\mathbf{Top}_*^M$ respectively.
Hence, we can apply the previous techniques to the *abstract LS-category*, introduced by J.P. Doeraene in [5] and [6]. Indeed, for first, we will prove that the mixed structure on well-pointed topological spaces with the homotopy type of a CW-complex, that is when we restrict it to its cofibrant objects, turns out to be a *J-category*. This result is of deep interest, since it seems that in the literature until now the only example of *J-category* on pointed topological spaces is the Strøm structure restricted to the well-pointed topological spaces (see [6, Proposition A.3]). We will denote it with $\mathbf{Top}^W$.
After having proved this, we will use this result to prove that, working with spaces with the homotopy type of a CW-complex, also the abstract LS-category is invariant in the three model structures on well-pointed topological spaces.

Theorem 3.3. *The category ${\mathbf{Top}_*}^M$ is a J-category if we restrict it to the class of well-pointed spaces with the homotopy type of a CW-complex.*

Note 3.4. During the proof of this Theorem, we will use the same notation for the axioms of a J-category as in [4].

Proof. Observe that under these hypotheses we have that all objects are both cofibrant and fibrant and that the three classes of morphisms are the same of $\mathbf{Top}^W$. Indeed, it follows from the existence of a Quillen adjunction between the mixed and the Strøm structure, when we consider the identity functors, that the weak equivalences are the same and the cofibrations agree by Proposition 2.1.
We can now show that all the axioms are satisfied. Axiom $(J1)$ trivially holds. Also axiom $(J2)$ is satisfied. Indeed, we have already observed in Theorem 2.2 and Remark 2.3 that the required pushouts and the pullbacks exist. In particular, since the three classes of morphisms are contained in the three classes of $\mathbf{Top}^W$, we have that the hypotheses about the induced morphisms are trivially satisfied too.
We can now focus our attention on axiom $(J3)$. The F-factorization can be chosen in $\mathbf{Top}^W$ since the weak equivalences and the fibrations are still weak equivalences and fibrations in our restricted mixed structure. In particular, the factorizing object is still an object in this category since it is a well-pointed topological space with the homotopy type of a CW-complex. Moreover, we have to show that also a C-factorization can be done. In order to do it we can work with the mapping cylinder, as already showed in Theorem 2.2.
Finally, axiom $(J4)$ holds, since all objects are cofibrant and fibrant.
We have to prove now that the cube axiom is satisfied. Let us take a cube such that the bottom face is a homotopy pushout and the vertical faces are homotopy pullbacks in our restricted mixed model category. Hence, when we move into the model category $\mathbf{Top}^W$, these properties of the cube are preserved. In particular, since it is a J-category, the upper face turns out to be a homotopy pushout. Thus, since the the previous axioms hold, it is a homotopy pushout also in our restricted mixed structure. It follows that the cube axiom is satisfied.
Finally, since the 0 element trivially exists, our category is a *J-category.* □

Corollary 3.5. *The category* $\mathbf{Top}^M$*, when restricted to the class of spaces with the homotopy type of a CW-complex, satisfies the cube axiom.*

In the remainder of this section we would like to extend our previous proof for the abstract topological complexity to the abstract LS-category. We begin recalling the Ganea version of LS-category introduced by J.P. Doeraene in [6].

Definition 3.6. [6, Definition 3.8 (ii)] Let $\mathbf{C}$ be a pointed proper model category and let B be an object in $\mathbf{C}$. The *n-th Ganea object* $G_n(B)$ and the *n-th Ganea map* $p_n : G_n(B) \to B$, $n \geq 0$, are constructed inductively as follows.
(i) For $n = 0$ let p_0 be the zero morphism $p_0 : 0 \to B$.
(ii) Suppose that p_{n-1} is already constructed, then p_n is defined as the join map of $p_0 : 0 \to B$ and $p_{n-1} : G_{n-1}(B) \to B$. In particular, $G_n(B) = 0 *_B G_{n-1}(B)$.
We say that $Gcat(B) \leq n$ if and only if the n-th Ganea map $p_n : G_n(B) \to B$ admits a weak section. If no such n exists, then $Gcat(B) = \infty$.

Moreover, J.P. Doeraene introduced another version of LS-category, called Whitehead version.

Definition 3.7. [6, Definition 3.8 (i)] Let **C** be a pointed proper model category and let B be an e-fibrant object, we have that $Wcat(B) \leq n$ if and only if the diagonal morphism $\Delta : B \to B^{n+1}$ admits a weak lifting along the *n-th fat wedge* morphism $j_n : T^n(B) \to B^{n+1}$. In particular, we set $Wcat(B) = \infty$ if no such n exists.

In particular, J.P. Doeraene succeeded in proving that these definitions agree in a J-category (see [6, Theorem 3.10]). In this case we will call them simply *abstract LS-category* without any distinction and we will denote it with *cat*. Finally, he also proved that through modelization functors is possible to link the value of the abstract LS-category in different pointed proper model categories.

Theorem 3.8. [6, Corollary 6.3] *If $\mu : \boldsymbol{C} \to \boldsymbol{D}$ and $\nu : \boldsymbol{D} \to \boldsymbol{C}$ are a pair of pointed modelization functors between pointed proper model categories such that $\nu(\mu(B))$ is weakly equivalent to B in $\boldsymbol{C}$, then:*

$$Wcat^{\boldsymbol{D}}(\mu(B)) = Wcat^{\boldsymbol{C}}(B) \text{ and } Gcat^{\boldsymbol{D}}(\mu(B)) = Gcat^{\boldsymbol{C}}(B).$$

After having recalled some of the main results obtained by J.P. Doeraene in his work [6], we can prove that the abstract LS-category is invariant in the three different model structures on well-pointed topological spaces when we restrict to topological spaces with the homotopy type of a CW-complex.

Theorem 3.9. *Let B be a well-pointed topological space with the homotopy type of a CW-complex. Then, the join object obtained joining $p : 0 \to B$ with itself, $G_1(B)$, in the mixed structure is the same up to weak equivalence of the one obtained in the Strøm structure.*

Proof. Since B and 0 are both cofibrant objects in the mixed structure, thanks to Proposition 2.1 and Theorem 3.3 we can repeat the same arguments of Theorem 2.2. Hence, the thesis follows. □

Remark 3.10. Clearly, since the *join element* is always cofibrant in the mixed structure on well-pointed topological spaces with the homotopy type of a CW-complex, every Ganea group results to be the same in the two structures up to weak equivalence.

Arguing as in Proposition 2.4, it is easy to prove that the *n-th Ganea map* admits a weak section in the mixed structure if and only if it admits a weak section in the Strøm one. Hence, we obtain the following Theorem.

Theorem 3.11. *For a well-pointed space with the homotopy type of a CW-complex the following equality holds:*

$$cat^{Q}(X) = cat^{S}(X).$$

In particular, if X is also path-connected and normal, the following equality holds:

$$cat^Q(X) = \boldsymbol{cat}(X).$$

Proof. As in the proof of Theorem 3.1, we have to begin showing that $cat^S(X) = cat^M(X)$, where $cat^M(X)$ is the abstract sectional category computed in the mixed structure. Observe that since each $G_n(B)$ is a cofibrant object in the mixed structure, by Remark 3.10, we are working in the mixed model category on well-pointed topological spaces with the homotopy type of a CW-complex. Thus by the Remark above, the thesis holds, since we are working with J-category and so the Ganea version equals the abstract one.

Then, it is easy to observe that also the adjunction $Id : \mathbf{Top_*}^Q \leftrightarrows \mathbf{Top_*}^M : Id$ satisfies the hypotheses of Proposition 1.1. It follows that, these functors satisfy also the hypotheses of 3.8. Hence the following chain of equalities holds:

$$cat^Q(X) = cat^M(X) = cat^S(X).$$

The second one holds, since the hypotheses of [3, Theorem 1.55] are satisfied. □

As Corollary, we can show how the LS-category can be computed in a combinatorial way.

Corollary 3.12. *Let X be a normal path-connect well-pointed space with the homotopy type of a CW-complex. Then the following equality holds*

$$\boldsymbol{cat}(X) = cat^{\boldsymbol{sSet}}(\mathcal{S}(X)),$$

Proof. It is sufficient to observe that the singular functor and the geometric realization functor are also adjoint pointed modelization functors. □

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